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A SUFFICIENT CONDITION FOR ALL FRACTIONAL [a, b] -FACTORS IN GRAPHS

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Abstract. Let G a graph, and let $g(x)$ and $f(x)$ be two nonnegative integer-valuedfunctions defined on $V(G)$ with $g(x) \le f(x)$ for every $x \in V(G)$. We say that G has all fractional (g, f)-factors if G has a fractional r-factor for any $r: V(G) \to N^+$ with $g(x) \le r(x) \le f(x)$ for every $x \in V(G)$. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then all fractional (g, f)-factors are called all fractional $[a,b]$-factors. In this paper, we verify that $G - I$ admits all fractional $[a,b]$-factors for any independent set I of G if

$$\alpha(G) \le \frac{4a(\delta(G) - b + 1)}{(b+1)^2 + 4a}.$$

Furthermore, it is shown that the result is sharp.

Key words: graph, minimum degree, independence number, fractional factor, all fractional factors.

1. INTRODUCTION

All graphs considered in this paper will be finite undirected simple graphs. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. For every $x \in V(G)$, we use $d_G(x)$ to denote the degree of x in G, and write $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. For every $x \in V(G)$, we denote by $N_G(x)$ the set of vertices of G adjacent to x, and write $N_G[x] = N_G(x) \bigcup \{x\}$. For any vertex subset S of G, $G[S]$ denotes the subgraph of G induced by S, and we write $G - S = G[V(G) \setminus S]$ and $N_G(S) = \bigcup_{x \in S} N_G(x)$. For two disjoint vertex subsets S and T of G, we use $e_G(S,T)$ to denote the number of edges that join a vertex in S and a vertex in T. We denote by $\alpha(G)$ the independence number of G. Let r be a real number. Recall that $\lfloor r \rfloor$ is the greatest integer with $\lfloor r \rfloor \le r$.

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \le f(x)$ for every $x \in V(G)$. A spanning subgraph F of G is called a (g, f)-factor of G if F satisfies $g(x) \le d_F(x) \le f(x)$ for every $x \in V(F)$. If $g(x) = a$ and $f(x) = b$ for any $x \in V(F)$, then a (g, f)-factor is called an $[a,b]$-factor. A $[k,k]$-factor is simply called a k-factor.

If $g(x) \le \sum_{e \ni x} h(e) \le f(x)$ holds for every $x \in V(G)$, then we call graph F with vertex set $V(G)$ and edge set E_h a fractional (g, f)-factor of G with indicator function h, where $h: E(G) \to [0,1]$ be a function and $E_h = \{e : e \in E(G), h(e) > 0\}$. If $g(x) = a$ and $f(x) = b$ for every $x \in V(G)$, then a fractional (g, f)-factor is said to be a fractional $[a,b]$-factor. A fractional $[a,b]$-factor is called a fractional k-factor if $a = b = k$. If G has a fractional r-factor for any $r: V(G) \to N^+$ with $g(x) \le r(x) \le f(x)$ for every $x \in V(G)$, then we say that G has all fractional (g, f)-factors. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then all fractional (g, f)-factors are called all fractional $[a,b]$-factors.

Many authors studied factors [1–6], fractional factors [7–11] and all fractional factors [12, 13] of graphs. Cai and Liu [10] showed a minimum degree and independence number condition for a graph to have a fractional k-factor.

THEOREM 1 [10]. *Let k be an integer with $k \ge 1$, and let G be a graph. If*

$$\alpha(G) \le \frac{4k(\delta(G)-k)}{(k+1)^2},$$

then G admits a fractional k-factor.

Zhou, Xu and Sun [11] improved Theorem 1, and obtained the following result.

THEOREM 2 [11]. *Let k be an integer with $k \ge 1$, and let G be a graph. If*

$$\alpha(G) \le \frac{4k(\delta(G)-k+1)}{k^2+6k+1},$$

then G–I admits a fractional k-factor for any independent set I of G.

Lu [12] gave a necessary and sufficient condition for graphs to have all fractional $[a,b]$-factors,and obtained a neighborhood union condition for graphs to have all fractional $[a,b]$-factors.

THEOREM 3 [12]. *Let G be a graph and $a \le b$ be two positive integers. Then G has all fractional $[a,b]$-factors if and only if*

$$\theta_G(S,T) = a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| \ge 0$$

for any subset $S \subseteq V(G)$, where $T = \{x : x \in V(G)\setminus S, d_{G-S}(x) \le b-1\}$.

THEOREM 4 [12]. *Let a and b be two positive integers with $a \le b$, and let G be a graph of order n with $n \ge \frac{2(a+b)(a+b-1)}{a}$, $\delta(G) \ge \frac{(a+b-1)^2+4b}{4a}$.* If

$$N_G(x) \cup N_G(y) \ge \frac{bn}{a+b}$$

for any two nonadjacent vertices x and y in G, then G admits all fractional $[a,b]$-factors.

Zhou and Sun [13] posed a new neighborhood union condition for the existence of all fractional $[a,b]$-factors in graphs.

THEOREM 5 [13]. *Let a,b,r be three integers with $1 \le a \le b$ and $r \ge 2$. Let G be a graph of order n with $n > \frac{(a+b)(r(a+b)-2)}{a}$. If*

$$\delta(G) \ge \frac{(r-1)b^2}{a}$$

and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_r)| \ge \frac{bn}{a+b}$$

for any independent subset $\{x_1, x_2, \cdots, x_r\}$ in G, then G admits all fractional $[a,b]$-factors.

In this paper, we proceed to study the existence of all fractional $[a,b]$-factors in graphs. Motivated by Theorems 1 and 2, we obtain a sufficient condition for the existence of all fractional $[a,b]$-factors in vertices deleted graphs, which is shown in the following.

THEOREM 6. *Let a and b be two integers with $1 \le a \le b$, and let G be a graph. If*

$$\alpha(G) \le \frac{4a(\delta(G)-b+1)}{(b+1)^2+4a},$$

then $G-I$ *has all fractional* $[a,b]$*-factors for any independent set* I *of* G.

2. THE PROOF OF THEOREM 6

Proof of Theorem 6. We write $H = G - I$. Suppose that the result does not hold. By Theorem 3, there exists a vertex subset S of H satisfying

$$\theta_H(S,T) = a|S| + \sum_{x\in T} d_{H-S}(x) - b|T| < 0, \tag{1}$$

where $T = \{x : x \in V(H)\setminus S, d_{H-S}(x) \le b-1\}$.

Obviously, $T \ne \phi$ by (1). Let $h = \min\{d_{H-S}(x) : x \in T\}$. It follows from the definition of T that $0 \le h \le b-1$. We choose $x_1 \in T$ with $d_{H-S}(x_1) = h$. Thus, we have

$$\delta(H) \le d_H(x_1) \le d_{H-S}(x_1) + |S| = h + |S|. \tag{2}$$

Note that $H = G - I$. Hence, we obtain $\delta(H) \ge \delta(G) - |I|$. Combining this with (2), we get

$$|S| \ge \delta(G) - h - |I|. \tag{3}$$

Now we choose $y_1 \in T$ such that y_1 is the vertex with the least degree in $G[T]$. We write $N_1 = N_G[y_1] \bigcap T$ and $T_1 = T$. For $i \ge 2$, we write $T_i = T - \bigcup_{1\le j<i} N_j$ if $T - \bigcup_{1\le j<i} N_j \ne \phi$. Then choose $y_i \in T_i$ such that y_i is the vertex with the least degree in $G[T_i]$, and write $N_i = N_G[y_i] \bigcap T_i$. We continue these procedures until we reach the situation in which $T_i = \phi$ for some i, say for $i = r+1$. Then it follows from the above definition that $\{y_1, y_2, \cdots, y_r\}$ is an independent set of G. Note that $T \ne \phi$. Therefore, we have $r \ge 1$.

We write $|N_i| = n_i$. According to the definition of N_i, the following properties hold.

$$\alpha(G[T]) \ge r \tag{4}$$

and

$$|T| = \sum_{1\le i\le r} n_i. \tag{5}$$

It follows from the choice of y_i that all vertices in N_i have degree at least $n_i - 1$ in $G[T_i]$. Thus, we have

$$\sum_{1\le i\le r}\left(\sum_{x\in N_i} d_{G[T_i]}(x)\right) \ge \sum_{1\le i\le r} n_i(n_i-1). \tag{6}$$

Using (6), we obtain

$$\sum_{x\in T} d_{H-S}(x) = \sum_{x\in T} d_{G-I-S}(x) \ge \sum_{1\le i\le r} n_i(n_i-1) + \sum_{1\le i<j\le r} e_G(N_i, N_j) \ge \sum_{1\le i\le r} n_i(n_i-1). \tag{7}$$

Note that $\alpha(G[T]) \le \alpha(G)$. Combining this with (4), we get

$$\alpha(G) \geq \alpha(G[T]) \geq r . \tag{8}$$

In terms of (1), (3), (5) and (7), we have

$$0 > \theta_H(S,T) = a|S| + \sum_{x \in T} d_{H-S}(x) - b|T|$$

$$\geq a(\delta(G) - h - |I|) + \sum_{1 \leq i \leq r} n_i(n_i - 1) - b \sum_{1 \leq i \leq r} n_i$$

$$= a(\delta(G) - h - |I|) + \sum_{1 \leq i \leq r} n_i(n_i - b - 1) ,$$

that is

$$0 > a(\delta(G) - h - |I|) + \sum_{1 \leq i \leq r} n_i(n_i - b - 1) . \tag{9}$$

It is obvious that $n_i(n_i - b - 1) \geq -\frac{(b+1)^2}{4}$. Combining this with (8), (9), $0 \leq h \leq b-1$ and $|I| \leq \alpha(G)$, we have

$$0 > a(\delta(G) - h - |I|) + \sum_{1 \leq i \leq r} n_i(n_i - b - 1) \geq$$

$$\geq a(\delta(G) - h - \alpha(G)) - \frac{(b+1)^2}{4} r \geq$$

$$\geq a(\delta(G) - h - \alpha(G)) - \frac{(b+1)^2}{4} \alpha(G) =$$

$$= a(\delta(G) - h) - \frac{(b+1)^2 + 4a}{4} \alpha(G) \geq$$

$$\geq a(\delta(G) - b + 1) - \frac{(b+1)^2 + 4a}{4} \alpha(G) ,$$

which implies

$$\alpha(G) > \frac{4a(\delta(G) - b + 1)}{(b+1)^2 + 4a} ,$$

which contradicts that $\alpha(G) \leq \frac{4a(\delta(G) - b + 1)}{(b+1)^2 + 4a}$. Theorem 6 is proved.

3. REMARK

In Theorem 6, the bound in the hypothesis

$$\alpha(G) \leq \frac{4a(\delta(G) - b + 1)}{(b+1)^2 + 4a}$$

is best possible, which cannot be replaced by

$$\alpha(G) \le \frac{4a(\delta(G)-b+1)}{(b+1)^2+4a}+1.$$

We can show this by constructing a graph $G = K_m \vee (m+1)K_{a+1}$, where $\vee$ means "join" and $m = \left\lfloor \frac{4a(\delta(G)-b+1)}{(b+1)^2+4a} \right\rfloor$. It is obvious that

$$\frac{4a(\delta(G)-b+1)}{(b+1)^2+4a} < \alpha(G) = m+1 = \left\lfloor \frac{4a(\delta(G)-b+1)}{(b+1)^2+4a} \right\rfloor + 1 \le \frac{4a(\delta(G)-b+1)}{(b+1)^2+4a}+1 .$$

We choose a vertex in every K_{a+1}, say for y_i, $1 \le i \le m+1$. Write $I = \{y_1, y_2, \cdots, y_{m+1}\}$. It is easy to see that I is an independent set of G. Let $H = G - I = K_m \vee (m+1)K_a$, $S = V(K_m)$ and $T = V((m+1)K_a)$. Clearly, $|S| = m$, $|T| = (m+1)a$, $\sum_{x\in T} d_{H-S}(x) = (m+1)a(a-1)$, and so

$$\begin{aligned}\theta_H(S,T) &= a|S| + \sum_{x\in T} d_{H-S}(x) - b|T| \\ &= am + (m+1)a(a-1) - (m+1)ab \\ &= (m+1)a(a-b) - a \le -a < 0 .\end{aligned}$$

In terms of Theorem 3, $H = G - I$ does not admit all fractional $[a,b]$-factor.

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