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SCALE ANALYSIS AND ASYMPTOTIC SOLUTION FOR NATURAL CONVECTION OVER A HEATED FLAT PLATE AT HIGH PRANDTL NUMBERS

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Abstract. This study presents a free convection flow over a heated flat plate using Bejan's method of scale analysis for balancing forces. For Newtonian fluids of large Prandtl numbers, two different layers which are the thermal and velocity boundary layers exist. The thermal boundary layer is thinner than the velocity boundary layer. The method of matched asymptotic expansion is used to obtain the velocity and temperature within the two layers and these quantities are then matched at the interface. A 5–5 matching is used to obtain both inner and outer solutions for velocity and temperature. A natural small parameter in this problem is the inverse of the square root of the Prandtl number multiplying the highest derivative. In the Bejan formulation for large Prandtl number flows, the two dimensionless quantities that emerge are the Rayleigh and Prandtl numbers as opposed to the Grashof and Prandtl numbers obtained in previous works. The results of velocities, temperature, shear stress and Nusselt number presented in this study are for fluids that have Prandtl numbers ranging 10 to 100,000. The Nusselt number predicted as Prandtl number goes to infinity, approaches the same asymptote as in previous works, while there's about 30% difference in the skin friction predicted when the differences in scaling used are not taken into consideration.

***Key words*:** Large Prandtl number, Scale analysis, Asymptotic expansion, Boundary layer, Free convection.

1. INTRODUCTION

Considerable efforts have been expended on the solution of free convection flows of a steady, viscous, heat conducting, and incompressible fluid near a heated vertical surface. This type of flow has application in many industrial and engineering processes, such as shutdown of high temperature high pressure flows in the oil and gas industry, cooling of reactor walls, condensation processes, heat exchangers and other electronics components. On account of its technological implications, accurate prediction of the skin friction on the heated wall as well as heat transfer rates near and far from the wall is highly desirable. Starting with the work of Ostrach [1], boundary layer flows due to free convection have been studied both theoretically and experimentally by many researchers [1–7]. Most of these works considered moderate Prandtl numbers or moderate Grashof numbers. The boundary layer solutions considered flows only in a narrow zone, by using similarity solution to reduce the partial differential equations to a set of nonlinear ordinary differential equations, and were solved numerically. Some contributors used the method of matched asymptotic expansions to solve the similarity equations, some considered the inverse of Grashof's number as the small parameter while other investigators used inverse of square root of Prandtl number as the small parameter or the Prandtl number for small Prandtl number flows. Bejan [4] was able to compare the dominant forces for both low and high Prandtl number flows. Naturally, two dimensionless parameters arose and these are the Prandtl and the Rayleigh numbers. In this study, high Prandtl number flows are considered using the scaling laws of Bejan. Results obtained for Prandtl numbers as high as 100,000 are compared with the work of Kuiken [5], who rescaled Ostrach's equations, and also with the analytical investigation by Bachiri & Bouabdallah [6, 7] who used Bejan's scale analysis with a different solution method.

2. PHYSICAL MODEL

Consider a hot, semi-infinite vertical plate of temperature T_w immersed in a cold fluid of temperature T_∞, with the vertical plate aligned along the y-axis in a Cartesian coordinate system. The fluid is assumed to be heat conducting Newtonian, steady, viscous, and incompressible with constant thermo-physical properties. With the assumption of Boussinesq approximation, the two-dimensional governing continuity, momentum and energy equations for boundary layer fluid flow and heat transfer are equations (1) to (3), where u and v are the velocities of the fluid along the x- and y- axis, respectively, ϑ is the kinematic viscosity of the fluid, β is the volume expansion coefficient and b is the acceleration due to gravity. The boundary conditions at the wall ($x = 0$) are $u = v = 0$, $T = T_w$ while the boundary condition far away from the wall ($x \to \infty$), are $v = 0$ and $T = T_\infty$.

2.1. Inner layer equations and boundary conditions

The inner layer which is also the thermal boundary layer (δ_T), is ruled by friction-buoyancy at large Pr which can be expressed as,

$$\frac{\partial \mathrm{u}}{\partial \mathrm{x}}+\frac{\partial \mathrm{v}}{\partial \mathrm{y}}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\vartheta\frac{\partial^2 v}{\partial x^2}+b\beta(T-T_\infty), \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial x^2}, \tag{3}$$

$$\delta_T \sim (\mathrm{Ra}_H \,\mathrm{Pr})^{\frac{-1}{4}}. \tag{4}$$

Introducing a similarity variable $\eta=\dfrac{x}{\delta_T}$, and stream function $\psi_{inner}=\alpha \mathrm{Ra}_y^{\frac{1}{4}}\, g(\eta,\mathrm{Pr})$ which satisfies the continuity equation and dimensionless temperature expressed as $\theta(\eta)=\dfrac{T-T_\infty}{T_w-T_\infty}$, the partial differential boundary layer equations (1) to (3) are transformed to dimensionless ordinary differential equations (5) and (6) for similarity solutions $g(\theta)$ and $\eta(\theta)$ subject to boundary conditions in equation (7) as shown below [1],

$$\frac{3}{4}g\theta' = \theta'', \tag{5}$$

$$\varepsilon^2\left(\frac{1}{2}g'^2-\frac{3}{4}gg''\right)=-g'''+\theta, \tag{6}$$

$$g(0) = 0,\ g'(0) = 0,\ \theta(0) = 1, \tag{7}$$

where $\varepsilon=\left(\sqrt{\dfrac{1}{\mathrm{Pr}}}\right)$ is the ratio of the thicknesses of the two layers. From equations (5) to (7), it is seen that the energy equation is a second-order equation and the momentum equation is a third-order equation but there are only three boundary conditions. The missing boundary conditions will be obtained using the method of matched asymptotic expansion (MMAE) by equating the velocities at the interface of the two layers.

2.2. Outer layer equations and boundary conditions

The outer layer is ruled by inertia-friction balance and the thickness is expressed as equation (8),

$$\delta \sim H \, \mathrm{Pr}^{\frac{1}{2}} \, \mathrm{Ra}_H^{\frac{-1}{4}}, \tag{8}$$

$$\frac{3}{4} G\theta' = \varepsilon^2 \Theta'', \tag{9}$$

$$\frac{1}{2} G'^2 - \frac{3}{4} GG'' = -G''' + \varepsilon^{-2}\Theta, \tag{10}$$

$$G'(\delta \to \infty) = 0, \; \Theta(\varepsilon \to \infty) = 0. \tag{11}$$

Equations (8) to (10) are equivalent to a fifth-order equation, with only two boundary conditions (11). The matched asymptotic expansion method will be used to obtain the missing initial conditions. The velocity of flow and temperature at the edge of inner layer are same as the corresponding quantities at the beginning of the outer layer. The procedure outlined by Kuiken [5], where G is expanded in Taylors series near $\xi = 0$ while g is expanded algebraically as $\eta \to \infty$, is used to obtain equations (12) to (14) as follows:

$$\alpha \mathrm{Ra}_y^{\frac{1}{2}} g = \vartheta \, \mathrm{Pr}^{\frac{-1}{2}} \, \mathrm{Ra}_y^{\frac{1}{2}} G, \tag{12}$$

$\varepsilon g = \lim\limits_{\xi \to 0} G \; \lim\limits_{\eta \to \infty}$ becomes

$$\lim_{\eta \to \infty} 0 + \varepsilon\left[g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3\right] = \lim_{\xi \to 0} G_0(\xi) + \varepsilon G_1(\xi) + \varepsilon^2 G_2(\xi) + \varepsilon^3 G_3(\xi) + \varepsilon^4 G_4(\xi) \tag{13}$$

$$\begin{aligned} &0 + \varepsilon\left[g_{00} + g_{01}\eta + g_{02}\eta^2\right] + \varepsilon^2\left[g_{10} + g_{11}\eta + g_{12}\eta^2\right] + \varepsilon^3\left[g_{20} + g_{21}\eta + g_{22}\eta^2 + g_{23}\eta^3\right] + \\ &+ \varepsilon^4\left[g_{30} + g_{31}\eta + g_{32}\eta^2 + g_{33}\eta^3 + g_{34}\eta^4\right] = G_{00} + \varepsilon\left[\eta G_{01} + G_{10}\right] + \varepsilon^2\left[\eta^2 G_{02} + \eta G_{11} + G_{20}\right] + \\ &+ \varepsilon^3\left[\eta^3 G_{03} + \eta^2 G_{12} + \eta G_{21} + G_{30}\right] + \varepsilon^4\left[\eta^4 G_{04} + \eta^3 G_{13} + \eta^2 G_{22} + \eta G_{31} + G_{40}\right]. \end{aligned} \tag{14}$$

Matching gives the quantities that will serve as boundary conditions for the g equations to various orders and initial conditions for G equations as stated below,

O (1): $G_{00} = 0 \vee G_0(0) = 0$

O (ε): $g_{02} = 0, \; g_{01} = G_{01}, \; g_{00} = G_{10}$

O (ε^2): $g_{12} = G_{02}, g_{11} = G_{11}, g_{10} = G_{20}$.

3. RESULTS AND DISCUSSION

The scaling used for the previous work by Kuiken and results obtained were compared with the results of this study using Bejan's method of scale analysis. Table 1 presents the different scales used, and results obtained, for temperature and velocity in the inner and outer layer as well as the shear stress.

Table 1

Comparison of scales used in previous and present studies

Dimensionless quantities	Kuiken [5]	Present study
eta (η) (inner layer)	$\left(\frac{\mathrm{Ra}_y}{4}\right)^{\frac{1}{4}}\frac{x}{y}$	$\left(\mathrm{Ra}_y\right)^{\frac{1}{4}}\frac{x}{y}$
psi (ξ) (outer layer)	$\left(\frac{\mathrm{Ra}_y}{4}\right)^{\frac{1}{4}}\mathrm{Pr}^{\frac{-1}{2}}\frac{x}{y}$	$\left(\mathrm{Ra}_y\right)^{\frac{1}{4}}\mathrm{Pr}^{\frac{-1}{2}}\frac{x}{y}$
vertical velocity (g') (inner layer)	$v\frac{y}{2\alpha \mathrm{Ra}_y^{\frac{1}{2}}}$	$v\frac{y}{\alpha \mathrm{Ra}_y^{\frac{1}{2}}}$
Nusselt number	$-2^{\frac{-1}{2}}\left(\left.\theta'_0\right\|_{\eta=0}+\varepsilon\left.\theta'_1\right\|_{\eta=0}+\varepsilon^2\left.\theta''_2\right\|_{\eta=0}\right)$	$-\left(\left.\theta'_0\right\|_{\eta=0}+\varepsilon\left.\theta'_1\right\|_{\eta=0}+\varepsilon^2\left.\theta''_2\right\|_{\eta=0}\right)$
Shear stress ($g''(0)$)	$\tau_0\frac{y^2}{\sqrt{2}\mu_0\alpha \mathrm{Ra}_y^{\frac{3}{4}}}$	$\tau_0\frac{y^2}{\mu_0\alpha \mathrm{Ra}_y^{\frac{3}{4}}}$
Vertical velocity (G') (outer layer)	$v\frac{y\mathrm{Pr}}{2\vartheta \mathrm{Ra}_y^{\frac{1}{2}}}$	$v\frac{y\mathrm{Pr}}{\vartheta \mathrm{Ra}_y^{\frac{1}{2}}}$

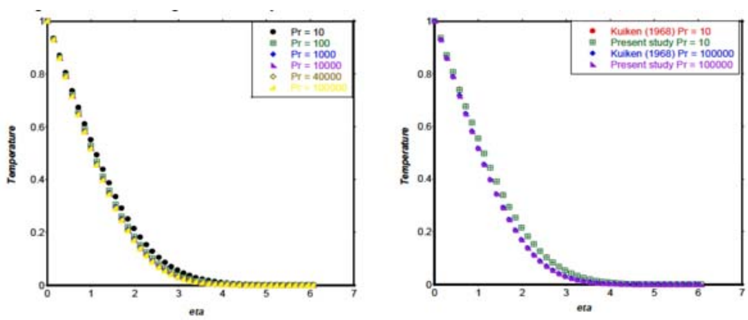
3.1. Temperature profiles and Nusselt number

Table 2 shows a comparison between the results obtained with MMAE and those of Bachiri & Bouabdallah. The results obtained using MMAE were more accurate than those obtained by Bachiri & Bouabdallah when compared with the theoretic work by Le Fevre. Results in Fig. 1 show that, as Prandtl number increases, the temperature approaches an asymptote. These results are similar to those of Kuiken. It is also shown that, as the Prandtl number increases, the Nusselt number increases but approaches an asymptotic value of about 0.502. This confirms the large Prandtl number limit stated by Bejan as Pr →∞.

Table 2

Comparison of Nusselt numbers obtained in this present study and previous works

Prandtl number	Present study	Kuiken [5]	Le Fevre	Bachiri & Bouabdallah [6, 7]
10	0.46581	0.46581	0.4650	0.4600
100	0.49004	0.49004	0.4900	0.4830
1000	0.49862	0.49863	0.4990	0.495
10 000	0.50143	0.50144	-	0.508
40 000	0.502095	-	-	-
100 000	0.502336	-	-	-

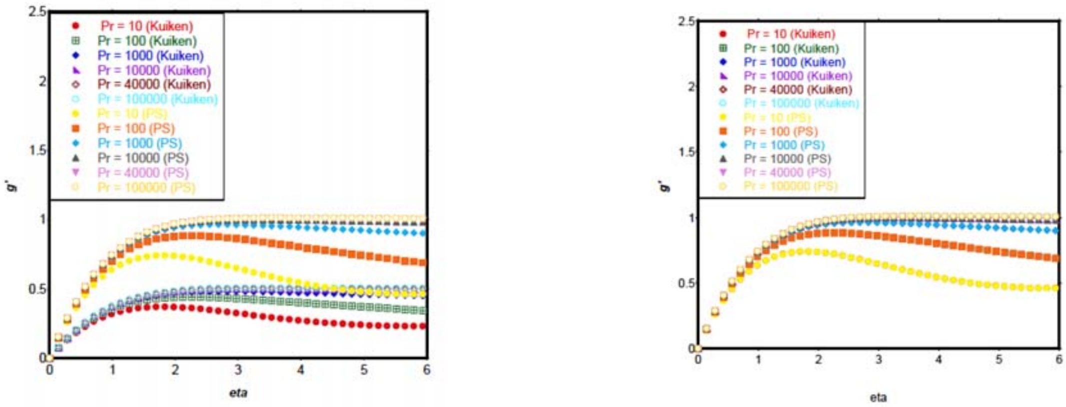


a) b)

Fig. 1 – Similarity temperature profiles: a) present study;
b) comparison between present study and Kuiken [5].

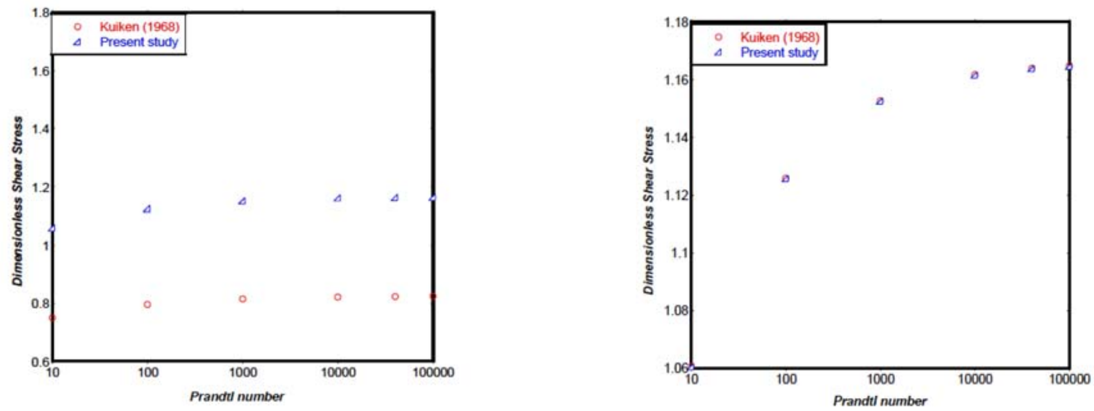
3.2. Velocity profiles and shear stress

Figure 2 shows results for dimensionless velocity and the dimensionless shear stress for the inner layer. The trend in the plots show that as the Prandtl number increases, the velocity increases but approaches an asymptote. The shear stress predicted by Kuiken is about 30% greater than those obtained using Bejan's method of scale analysis when Kuiken's scale as shown in Table 1 but with the use of the different scales, the results are similar. Results in Fig. 3 shows that in the outer viscous layer, the velocity decreases as Prandtl number increases. This is because as the Prandtl number increases, there is more resistance to flow as a result of increase in the viscosity of the fluid. Also, the outer viscous layer thickness is about 4.6 times thicker than the inner thermal layer. The results obtained showed that if the scales used by Kuiken is not taken into consideration, the results obtained for outer layer velocities in this present study will be 50% larger than those of Kuiken's.



a) b)

Fig. 2 – Inner layer: a) velocity profiles; b) dimensionless shear stress.



a) b)

Fig. 3 – Comparison of velocity profiles of present study with Kuiken [5]: a) using same scales; b) using different scales.

4. CONCLUSION

In this study, Bejan's method of scale analysis and the method of matched asymptotic expansion were used to determine the inner layer velocity, temperature, shear stress and outer layer velocity profiles for free convection flow over a heated plate at high Prandtl numbers of 10 to 100,000. Results obtained showed that the method of matched asymptotic expansion gave accurate predictions of large Prandtl number limits when combined with Bejan's method of scale analysis. It was also shown that for accurate comparison of results with previous works cited in literature, the scaling method used needs to be taken into consideration.

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