NOVEL STOCHASTIC DIFFERENTIAL MODEL FOR IMAGE RESTORATION

Tudor BARBU¹, Angelo FAVINI²

¹ Institute of Computer Science of the Romanian Academy ² University of Bologna, Department of Mathematics, Italy Corresponding author: Tudor BARBU, E-mail: tudor.barbu@iit.academiaromana-is.ro

Abstract. A stochastic diffusion-based image restoration technique is proposed in this paper. The stochastic differential model is described first. Then, the SDE-based denoising scheme is transformed into a parabolic PDE model. Next, a rigorous mathematical investigation is performed on it. A numerical approximation scheme is then developed for this PDE-based restoration approach. Our successfully restoration experiments and the performed method comparison are discussed next.

Key words: image denoising and restoration, stochastic differential equation, Kolmogorov equation, parabolic PDE model, well-posedness, numerical approximation scheme.

1. INTRODUCTION

The mathematical models have been increasingly and successfully used in several traditionally engineering domains like signal processing, image analysis and computer vision, during the past three decades. Although most of these models have been based on nonlinear partial differential equations (PDEs) [12, 24], the stochastic differential models (SDEs) are also increasingly used in the image processing field.

The SDE-based schemes represent probability models that could provide an effective image denoising and restoration, while preserving the boundaries and other image features. The SDE-based denoising models are more realistic than PDE-based smoothing methods because of the presence of stochastic perturbations in images.

Thus, numerous restoration techniques based on stochastic differential equations have been proposed in the last years [3, 4]. Some important SDE image denoising approaches are those based on modified diffusion [9], reflected stochastic equations [11], and stochastic relaxation and annealing [17].

A novel stochastic differential equation-based image restoration technique is proposed in this paper. Our stochastic diffusion model for image denoising is described in the next section. We developed numerous linear [1] and nonlinear PDE-based restoration models [2–5] in our past papers. They use second-order [1, 3–5] and fourth-order diffusions [2] and outperform some influential PDE filtering schemes, such as the anisotropic diffusion-based Perona-Malik models [22], the variational TV Denoising [23] and the You-Kaveh denoising scheme [25]. Now, we aim to obtain an effective second-order diffusion model for image denoising from a proper stochastic equation.

Thus, a parabolic PDE-based model is obtained from our SDE by using the associated Kolmogorov equation [8, 16]. Also, a rigorous mathematical treatment of this SDE-based restoration approach is provided in the third section, its well-posedness being treated.

A robust numerical approximation scheme is then developed for the discretization of the continuous denoising model. This finite-difference method-based discretization algorithm is described in the fourth section. We have performed numerical experiments that are described in the fifth section. Method, comparison are also discussed in that section. This article ends with a section of conclusions, acknowledgements and a list of references.

2. STOCHASTIC DIFFUSION-BASED FILTERING MODEL

Therefore, we propose the following stochastic differential equation-based model for image noise reduction:

$$\begin{cases} dX(t) + F(X(t)) dt = dW(t) \\ X(0, x, y) = X_0(x, y) \in R^2 \end{cases},$$
(1)

where the diffusion process $X(t) = \{X_1(t), X_2(t)\}$ and $W(t) = \mu\{\beta_1(t), \beta_2(t)\}, \mu \in (0,1)$ represents a 2D Brownian motion in a probability space $\{\Omega, F, P\}$ with the natural filtration $(F_t), t \ge 0$. We assume that the function $F : R^2 \to R^2$ is Lipschitzian and set $F(X(t)) = \{F_1(X_1(t), X_2(t)), F_2(X_1(t), X_2(t))\}$. While X(t) represents a random variable, $X_0(x, y)$ is a function on R^2 , being related to the initial image.

The solution to the SDE provided by (1) will be a stochastic process $X = X(t, X_0)$, adapted to the natural filtration $(F_t)_t \ge 0$, on the probability space, which satisfies the following equation:

$$X(t) + \int_{0}^{t} F(X(s)) ds = X_{0} + W(t), t \ge 0.$$
⁽²⁾

Such a solution exists and we refer to [16, 21] for demonstrating the existence of a unique solution to equation (2). Now, let the function $u_0: R^2 \to R$ represent the degraded image to be filtered of noise. Then, the restored image u(t) will be determined as following:

$$u(t, X_0(x, y)) = E \left[u_0(X(t), X_0(x, y)) \right], t \ge 0.$$
(3)

where *E* represents the expectation operator.

Then we consider the Kolmogorov equation [8, 13, 16, 21] associated with the SDE model (1). According to [13] (see section 4.4 at page 55, for more), the restored image represents the solution $u = u(t, \xi)$ of this equation, which is a parabolic PDE model having the next form:

$$\begin{cases} \frac{\partial u}{\partial t}(t,\xi) = \frac{\mu^2}{2} \Delta_{\xi} u(t,\xi) - F(\xi) \cdot \nabla_{\xi} u(t,\xi), t \ge 0\\ u(0,\xi) = u_0(\xi), \xi \in \mathbb{R}^2 \end{cases}$$
(4)

where $\xi = X_0(x, y) = \{(i, j)\}_{i=\overline{1,M}, j=\overline{1,N}} \in \mathbb{R}^2 \text{ and } \mu \in (0, 1].$

The parabolic equation (4) is defined on all of R^2 , but an image is defined on a given domain $K \subset R^2$. Therefore, to address this problem, one replaces (1) by the next reflection stochastic problem [11]

$$\begin{cases} dX(t) + F(X(t))dt + N_{K}(X(t))dt = dW(t) \\ X(0) = \xi \end{cases}$$
(5)

where K is a convex subset of R^{2} and N_{K} is the normal cone to ∂K . Thus, u given by (3) satisfies the equation (4) with Neumann boundary value conditions, which has the following form:

$$\begin{cases} \frac{\partial u}{\partial t}(t,\xi) - \frac{\mu^2}{2} \Delta_{\xi} u(t,\xi) + F(\xi) \cdot \nabla_{\xi} u(t,\xi) = 0, \forall t \ge 0, \xi \in K \\ \frac{\partial u}{\partial v} = 0, \quad \text{on } \partial K \\ u(0,\xi) = u_0(\xi), \forall \xi \in K. \end{cases}$$

$$(6)$$

We have performed several investigations on modelling the function F of this scheme, trying to determine that version of it that provides optimal image restoration results when used in (6). Thus, we have identified the optimal form for this function, which is expressed as following:

$$F(X_{1}(t), X_{2}(t)) = \left(e^{-\alpha_{1}\left(X_{1}(t)^{2} + X_{2}(t)^{2}\right)}, e^{-\alpha_{2}\left(X_{1}(t)^{2} + X_{2}(t)^{2}\right)}\right),$$
(7)

where $\alpha_1, \alpha_2 \ge 0$.

We note that in the special case where the drift term F is missing, it follows by (3) that $u(t) = u_0 (X_0 + W(t)), \forall X_0 \in \mathbb{R}^2$.

3. A ROBUST MATHEMATICAL INVESTIGATION OF THE MODEL

In this section we investigate the continuous mathematical model derived from the stochastic differential equation, analyzing its well-posedness. To properly investigate the existence and unicity of a weak solution, we consider a more general form of the parabolic equation provided by (4), namely:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\mu^2}{2} \Delta u + F \cdot \nabla u + g(u) = f, \text{ in } (0, T) \times R^d \\ u(0, \xi) = u_0(\xi), \xi \in R^d \end{cases}$$
(8)

where $u_0: \mathbb{R}^d \to \mathbb{R}$ and $f: [0, T] \times \mathbb{R}^d \to \mathbb{R}, d \ge 1$, $g: \mathbb{R} \to \mathbb{R}$ represent two functions that are important for the denoising process if they are suitable chosen. As regards the function $F: \mathbb{R}^d \to \mathbb{R}^d$, we assume that $|F(\xi) - F(\overline{\xi})|_d \le L |\xi - \overline{\xi}|_d$, $\forall \xi, \overline{\xi} \in \mathbb{R}^d$. Also g is monotonically nondecreasing, continuous and it satisfies

$$g(u) \leq C_1 |u| + C_2, \forall u \in R, L, C_1, C_2 > 0.$$
 (9)

We recall (see [6], page 150) that under assumption $|F(\xi) - F(\overline{\xi})|_d \le L |\xi - \overline{\xi}|_d$ there is a unique solution $\rho \in L^1(\mathbb{R}^d)$ to the elliptic equation

$$\frac{\mu^{2}}{2}\Delta\rho\left(\xi\right) + div\left(\rho\left(\xi\right), F\left(\xi\right)\right) = 0 , \xi \in \mathbb{R}^{d}, \qquad (10)$$

which satisfies the following:

$$\rho\left(\xi\right) > 0, \forall \xi \in \mathbb{R}^{d}, \int_{\mathbb{R}^{d}} \rho\left(\xi\right) d\xi = 1.$$
(11)

We consider the measure $v(d\xi) = \rho(\xi)d\xi$ and the space L_v^2 of all Lebesgue measurable functions $u: \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} u^2(\xi)\rho(\xi)d\xi < \infty$. If $X = X(t, X_0)$ represents the solution to the stochastic equation (1), then we set $(\rho_t u_0)(\xi) = Eu_0(X(t,\xi)), \xi \in \mathbb{R}^d$ (see [6], page 55). This is the transition semigroup associated with equation (1) [7]. It is well-known that v is the invariant measure of ρ_t , that is $\int_{\mathbb{R}^d} \rho_t u_0(\xi) v(d\xi) = \int_{\mathbb{R}^d} u_0(\xi) v(d\xi)$, $\forall u_0 \in L_v^2$ (see [13]). Moreover, ρ_t is a C_0 -semigroup of contractions on the space L_v^2 and its infinitesimal generator

 $M: D(M) \subset L_{\nu}^{2} \rightarrow L_{\nu}^{2}$ is the operator $Mu_{0} = \frac{\mu^{2}}{2} \Delta u_{0} - F \cdot \nabla u_{0}, \forall u_{0} \in D(M)$. It turns out that $D(M) = \left\{ u_0 \in H^2_{loc}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d), \sqrt{\rho} \nabla u_0 \in H^1(\mathbb{R}^d), \Delta u_0 \in L^2_{\nu} \right\}$, where $W^{1,\infty}(\mathbb{R}^d)$ is the Sobolev space $\left\{ u \in L^1(\mathbb{R}^d), \nabla u \in L^\infty(\mathbb{R}^d) \right\}$ and $C_b(\mathbb{R}^d)$ is the space of all continuous bounded functions on R^{d} . Now, the equation (8) can be re-written as following:

$$\begin{cases} \frac{\partial u}{\partial t}(t) - Mu(t) + g(u(t)) = f(t) \\ u(0) = u_0 \end{cases}$$
(12)

We assume that $u_0 \in D(M)$, $f, \frac{\partial f}{\partial t} \in C([0, T]; L_v^2)$, that is f is continuous from [0, T] to L_{ν}^{2} . One can demonstrate that operator $\Gamma u = -Mu + g(u), \forall u \in D(\Gamma) = D(M)$, is maccretive in the space L_{ν}^{2} . Then, by existence theory for the Cauchy problem in Banach spaces (see [6], page 127, for more), we get the following existence result for equation (8) or equivalently (12): there is a unique solution u to (8), which satisfies:

$$\begin{cases} u \in C\left(\left[0, T\right]; L_{v}^{2}\right), \frac{\partial u}{\partial t} \in L^{\infty}\left(0, T; L_{v}^{2}\right) \\ u(t) \in D\left(M\right), \forall t \in \left[0, T\right]. \end{cases}$$
(13)

It should be said also that by Crandall-Liggett exponential formula we have, for $f \equiv 0$, the following:

$$u(t) = \lim_{n \to \infty} \left(I - \frac{t}{n} (M - g) \right)^{-n} u_0 \text{ in } L_{\nu}^2, \forall t > 0.$$
 (14)

The function g provides an effective image restoration process if we consider the next form for it: $g(u) = \lambda (u - u_0)^p$, $p \ge 0$. We may take p = 0 and obtain the following PDE-based model that is discretized in the next section:

$$\frac{\partial u}{\partial t}(t,\xi) - \frac{\mu^2}{2} \Delta_{\xi} u(t,\xi) + F(\xi) \cdot \nabla_{\xi} u(t,\xi) + \lambda \left(u(t,\xi) - u(0,\xi) \right) = 0$$

$$u(0,\xi) = u_0(\xi), \xi \in \mathbb{R}^2.$$
(15)

FINITE-DIFFERENCE BASED NUMERICAL APPROXIMATION SCHEME

A consistent and fast-converging numerical approximation scheme is developed for the discretization of the parabolic model (15). The PDE-based restoration scheme is discretized by applying a finite-difference based method [19].

Thus, we consider a space grid size of h and a time step Δt . The space and time coordinates are quantized as: x = ih, y = jh, $t = n\Delta t$, $\forall i \in \{0, ..., I\}$, $j \in \{0, ..., J\}$, $n \in \{0, ..., N\}$, where N is the optimal number of iterations.

Therefore, the main equation of the continuous differential model provided by (15), $\frac{\partial u}{\partial t}(t,x,y) - \frac{\mu^2}{2}\Delta u(t,x,y) + F(x,y) \cdot \nabla u(t,x,y) + \lambda (u(t,x,y) - u_0(x,y)) = 0, \text{ is discretized by}$

using the finite differences [19]. The following discretization is obtained for the PDE-based model:

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$$\frac{u^{n+\Delta t}(i,j) - u^{n}(i,j)}{\Delta t} - \frac{\mu^{2}}{2} \frac{u^{n}(i+h,j) + u^{n}(i-h,j) + u^{n}(i,j+h) + u^{n}(i,j-h) - 4u^{n}(i,j)}{h^{2}} + \left\langle \left(e^{-\alpha_{i}(i^{2}+j^{2})}, e^{-\alpha_{2}(i^{2}+j^{2})}\right) \cdot \left(\frac{u^{n}(i+h,j) - u^{n}(i-h,j)}{2h}, \frac{u^{n}(i,j+h) - u^{n}(i,j-h)}{2h}\right) \right\rangle + \lambda \left(u^{n}(i,j) - u^{0}(i,j)\right) = 0.$$
(16)

We may consider the parameter values $\Delta t = 1$ and h = 1. Then, the relation (16) leads to the following explicit numerical approximation scheme:

$$u^{n+1}(i,j) = (\lambda - 2\mu^{2} + 1)u^{n}(i,j) + \frac{\mu^{2}}{2} (u^{n}(i+1,j) + u^{n}(i-1,j) + u^{n}(i,j+1) + u^{n}(i,j-1)) + e^{-\alpha_{1}(i^{2}+j^{2})} \frac{u^{n}(i+1,j) - u^{n}(i-1,j)}{2} - e^{-\alpha_{2}(i^{2}+j^{2})} \frac{u^{n}(i,j+1) - u^{n}(i,j-1)}{2} - \lambda u^{0}(i,j).$$
(17)

The iterative algorithm given by (17) receives the initial $[I \times J]$ noisy image as an input, then applies on the evolving $[I \times J]$ image u^n , for each n = 0, ..., N. The N value is quite low, since this iterative scheme is converging very fast to the optimal image restoration, which is u^{N+1} .

The developed numerical scheme is stable and consistent to the SDE-derived differential model (15), since it converges to its unique weak solution. The denoising results produced by this scheme are discussed in the next section.

5. EXPERIMENTS AND METHOD COMPARISON

We tested the proposed stochastic diffusion-based restoration framework on hundreds of images corrupted with various amounts of Gaussian noise. It provided satisfactory denoising results: a low execution time, of less than 1*s*, an effective noise removal and preservation of edges and other image features. The *Volume* 3 of the *USC-SIPI* database, which contains $[256 \times 256]$, $[512 \times 512]$ and $[1024 \times 1024]$ images, was the main image collection used in our experiments.

We identified the next set of parameters of this differential model that provide optimal restoration results: $\mu = 0.7$, $\alpha_1 = 2$, $\alpha_2 = 4$, $\lambda = 0.05$, N = 12. The low number of iterations, N, means also a low running time, the proposed restoration scheme executing very fast.

The performance of our image denoising technique was assessed by using the Peak Signal-to-Noise Ratio (PSNR) measure. From the performed method comparisons we found that our SDE-derived technique outperforms not only the conventional two-dimension image filters, but also some well-known PDE-based restoration approaches, getting higher PSNR values.

Thus, it provides better smoothing results than classic filters, such as 2D Gaussian, Average, 2D Wiener and Median [18]. Unlike these filters, our method overcomes the blurring effect and preserves the image details. Also, it provides a better denoising than some influential PDE-based schemes. Thus, it outperforms nonlinear second-order diffusion-based techniques, like both versions of the Perona-Malik scheme [22] and TV Denoising [23], removing a higher amount of Gaussian noise, converging much faster and overcoming the undesired staircasing effect [10].

The PSNR values obtained by our technique and other denoising models (2D Gaussian, Average, Wiener 2D, Perona-Malik 1, Perona-Malik 2, TV Denoising) are registered in Table 1. The denoising results provided by our approach and those filtering schemes on the $[512 \times 512]$ *Lena* image affected by Gaussian noise with parameters *average* = 0.21 and *var* = 0.02 are displayed in Fig. 1. These restoration results prove that our model provides a better image smoothing.

	26.94(dB)	22.43(dB)	23.29(dB)	24.23(dB)	25.6(dB)	25.83(dB)	24.96(dB)		
a) Original image				b) Noisy image			c) Our filter		
	d) 2D (Gaussian		e) Averagin	E	f) 21) Wiener		
	g) Pero	na-Malik 1	h)	Perona-Mal	ik 2	i) TV I	Denoising		

 Table 1

 PSNR values for several noise removal techniques

		-			
This model	Gaussian	Average	Wiener	P-M 1	P-M 2
26.04(dB)	22.43(dB)	23 20(dB)	24.23(dB)	25.6(dB)	25.83(dB)

Fig. 1. Lena image restored by using several techniques.

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6. CONCLUSIONS

An effective SDE-based image restoration technique has been proposed here. While we have an important experience in the PDE-based image denoising domain [1-5], this is the first time we derive an effective restoration model from a stochastic differential equation.

The proposed SDE-based model and the parabolic diffusion scheme obtained from it represent the main contributions of this paper. The mathematical investigation of our denoising model that have been provided here constitutes also a major contribution. We demonstrated that the derived parabolic PDE model could be well-posed, admitting a unique weak solution under some certain conditions.

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The fast-converging explicit numerical approximation scheme described here is another important result of this work. It is based on the finite-difference method and is consistent to the developed PDE scheme.

Our successfully image denoising tests and the performed method comparison prove the effectiveness of our restoration approach. We intend to further improve this SDE-based filtering model, by considering other versions of the drift term, as part of our future research. Also, we will focus on constructing more sophisticated SDE models, for example based on reflection stochastic problems [11], which produce more effective nonlinear diffusion-based denoising schemes.

ACNOWLEDGEMENTS

This work was mainly supported from the project PN-II-RU-TE-2014-4-0083-126/01.10.2015 financed by UEFSCDI Romania. It was supported also by the Institute of Computer Science of the Romanian Academy, Iasi branch.

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Received September 3, 2015