# ON A CLASS OF LINEAR OPERATORS ON $\ell^{p}$ AND ITS SCHUR MULTIPLIERS 

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> Abstract. In this paper we consider the spaces $B_{w}\left(\ell^{p}\right), 1 \leq p \leq \infty$, of infinite matrices $A$ defined by the norm $\|A\|_{B_{w}\left(\ell^{p}\right)}:=\sup _{\substack{\|x\|_{p} \leq 1 \\\left|x_{k}\right| \mid 00}}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} a_{j k} x_{k}\right|^{p}\right)^{\frac{1}{p}}$. We consider the Schur product of matrices and prove that $B_{w}\left(\ell^{p}\right)$ is not closed under this product. Moreover, we prove that linear and bounded operators on $\ell^{p}$ are Schur multipliers on $B_{w}\left(\ell^{p}\right)$, a result which is not obvious, since $B_{w}\left(\ell^{p}\right)$ is not a Schur algebra. Most of the results are sharp in the sense that they are given via necessary and sufficient conditions.

Key words: bounded linear operators, Hadamard product, Infinite matrices, matriceal harmonic analysis, norms in $\ell^{p}$ spaces, Schur product, Schur multipliers, Inequalities.

## 1. INTRODUCTION

For $A=\left(a_{j k}\right)$ and $B=\left(b_{j k}\right)$, two matrices of complex numbers having the same size (finite or infinite) their Schur product (or Hadamard product) is defined to be the matrix of elementwise products

$$
A^{*} B:=\left(a_{j k} b_{j k}\right) .
$$

This matrix product was first studied by Schur in his paper [17] and since then has appeared in several different areas of analysis for example in complex function theory [18], Banach spaces [5, 10, 22], operator theory [2, 13, 21, 20], matriceal harmonic analysis [3,14, 11] and multivariate analysis [23].

The space of Schur multipliers from $X$ to $Y$ is defined as

$$
(X, Y):=\{M: M * A \in Y \text { for every } A \in X\},
$$

where $X$ and $Y$ are two linear spaces of infinite matrices. If $X$ and $Y$ are Banach spaces, then we consider on the space $(X, Y)$ the natural norm

$$
\|M\|_{(X, Y)}:=\sup _{\|A\|_{X} \leq 1}\|M * A\|_{Y} .
$$

In particular, in [5], G. Bennett studied the $(p, q)$-multipliers i.e. matrices $M$ such that

$$
M * A \in B\left(\ell^{p}, \ell^{q}\right)
$$

for every $A \in B\left(\ell^{p}, \ell^{q}\right)$, where $B\left(\ell^{p}, \ell^{q}\right)$ denotes the set of bounded linear operators from $\ell^{p}$ to $\ell^{q}$ and $1 \leq p, q \leq \infty$.

Recently F. Sukochev and A. Tomskova in [22] studied $(E, F)$-multipliers, where $E$ and $F$ are two given symmetric sequence spaces.

Popa in [15] studied some special classes of infinite matrices. Among other results, the author constructed some subspaces of all Schur multipliers on $\ell^{2}$. Moreover, many authors studied Schur multipliers between some classes of infinite matrices.

In this paper we continue the study of the spaces $B_{w}\left(\ell^{p}\right)$ (introduced in [12]) defined by the norms $\|\cdot\|_{B_{w}\left(\ell^{p}\right)}, 1 \leq p \leq \infty$, where

$$
\|A\|_{B_{w}\left(\ell^{p}\right)}:=\sup _{\substack{\|x\|_{p} \leq 1 \\\left|x_{k}\right| \downarrow 0}}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} a_{j k} x_{k}\right|^{p}\right)^{\frac{1}{p}} .
$$

It is natural to consider and to study the spaces $B_{w}\left(\ell^{p}\right)$ for $1 \leq p \leq \infty$ and to compare with classical Banach spaces of infinite matrices $B\left(\ell^{p}\right)$. These spaces can be viewed as bounded operators between some classical spaces of sequences. We also consider some problems in connection with some particular classes of Schur multipliers. We denote by $d(p), 1 \leq p \leq \infty$ the space of all sequences $x=\left(x_{n}\right)_{n}$ for which $\|x\|_{d(p)}=\sum_{k=1}^{\infty}\left(\sup _{k \geq n}\left|x_{k}\right|\right)^{p}$ [7].

In the following we recall a notation which will be useful in the sequel. Let $A=\left(a_{i j}\right)_{i, j \in \mathrm{Z}_{+}}$be an infinite matrix. We denote by $A_{0}$ the matrix having on the main diagonal elements of $A$ and zero otherwise. More generally, we denote by $A_{k}=\left(a_{i j}^{\prime}\right)$ the matrix with elements

$$
a_{i j}^{\prime}=\left\{\begin{array}{cc}
a_{i j} & \text { if } j-i=k \\
0 & \text { otherwise }
\end{array}\right.
$$

This notion of "diagonals" has previously been used in the matricial analogue of Fejer theory and approximation problems [3, 4, 14]. A similarity between functions and matrices was first observed by Arazy in [1] and exploited further by Shields [19]. The historical development and a number of results concerning this area of "matricial harmonic analysis" was recently presented in the book [14] by L.E. Persson and N. Popa.

In this paper we derive some new results in this area more exactly: in Section 2 we state and prove some results in the case of diagonal matrices. In particular, we give a characterization for diagonal matrices to belong to $B_{w}\left(\ell^{p}\right)$. In Section 3 we study mainly Schur multipliers and Toeplitz matrices. Considering the Schur product of matrices we prove that $B_{w}\left(\ell^{p}\right)$ is not closed under this product. The main result is that linear and bounded operators on $\ell^{p}$ are Schur multipliers on $B_{w}\left(\ell^{p}\right)$, a result which is not obvious, since $B_{w}\left(\ell^{p}\right)$ is not a Schur algebra. Most of our results are sharp in the sense that they are given via necessary and sufficient conditions.

## 2. DIAGONAL MATRICES

In our first result we give a characterization for a diagonal matrix to be in the class $B_{w}\left(\ell^{p}\right)$ for $1 \leq p<\infty$.

THEOREM 2.1. Let $a=\left(a_{k}\right)_{k \geq 1}$ be a sequence of real numbers. We denote by $A=A_{0}$, the matrix with the sequence $a$ on the main diagonal. Then $A \in B_{w}\left(\ell^{p}\right)$ if and only if

$$
\sup _{n \in \mathcal{N}^{*}}\left(\frac{1}{n} \sum_{k=1}^{n}\left|a_{k}\right|^{p}\right)^{\frac{1}{p}}<\infty .
$$

Moreover,

$$
\|A\|_{B_{w}\left(\ell^{p}\right)}=\sup _{n \in N^{*}}\left(\frac{1}{n} \sum_{k=1}^{n}\left|a_{k}\right|^{p}\right)^{\frac{1}{p}}
$$

for $1 \leq p<\infty$.
Proof. For the necessity let us take $a=\left(a_{n}\right)_{n \geq 1}$ such that the matrix $A=A_{0}$ given by this sequence belongs to $B_{w}\left(\ell^{p}\right)$. Thus, $A x \in \ell^{p}$ for all $x=\left(x_{k}\right)_{n \geq 1} \in \ell^{p}$ with $\mid x_{k} \downarrow \downarrow$. We choose now the sequence $x^{(n)}=\left(x_{k}^{(n)}\right)_{k \geq 1}$ given by

$$
x_{k}^{(n)}=\left\{\begin{array}{cc}
n^{-\frac{1}{p}} & \text { if } k \leq n \\
0 & \text { otherwise }
\end{array}\right.
$$

Since $\left\|x^{(n)}\right\|_{p} \leq 1$ and $\left|x_{k}^{(n)}\right| \downarrow 0$ we have that

$$
\left\|A x^{(n)}\right\|_{p}=\left(\frac{1}{n} \sum_{k=1}^{n}\left|a_{k}\right|^{p}\right)^{\frac{1}{p}} \leq\|A\|_{B_{w}\left(\ell^{p}\right)} .
$$

It follows that

$$
\sup _{n \in \mathbb{N}^{*}}\left(\frac{1}{n} \sum_{k=1}^{n}\left|a_{k}\right|^{p}\right)^{\frac{1}{p}} \leq\|A\|_{B_{B_{w}}\left(\ell^{p}\right)} .
$$

For the sufficiency we use that

$$
\begin{equation*}
d(p) \cdot g(p) \subset \ell^{p} \tag{2.1}
\end{equation*}
$$

(see e.g. [7, p. 9]), where

$$
d(p)=\left\{y=\left(y_{k}\right)_{k \geq 1}: \sum_{n=1}^{\infty} \sup _{k \geq n}\left|y_{k}\right|^{p}<\infty\right\} \text { with }\|y\|_{d(p)}=\left(\sum_{n=1}^{\infty} \sup _{k \geq n}\left|y_{k}\right|^{p}\right)^{\frac{1}{p}}
$$

and

$$
g(p)=\left\{z=\left(z_{k}\right)_{z \geq 1}: \sum_{k=1}^{n}\left|z_{k}\right|^{p}=O(n)\right\} \text { with }\|z\|_{g(p)}=\sup _{n \geq 1}\left(\frac{1}{n} \sum_{k=1}^{n}\left|z_{k}\right|^{p}\right)^{\frac{1}{p}} .
$$

The product (2.1) is performed coordinatewise: $x_{k}=y_{k} z_{k}, k=1,2, \ldots$ and $0<p \leq \infty$.
Let us take an arbitrary $x=\left(x_{n}\right)_{n \geq 1} \in \ell^{p}$ with $\left|x_{k}\right| \downarrow 0$, which implies that $x \in d(p)$. Since

$$
\sup _{n \in \mathrm{~N}^{*}}\left(\frac{1}{n} \sum_{k=1}^{n}\left|a_{k}\right|^{p}\right)^{\frac{1}{p}}<\infty
$$

we have that $a=\left(a_{n}\right)_{n \geq 1} \in g(p)$ and it follows that $A x \in \ell^{p}$. The proof is complete.
Remark 2.2. One important consequence of previous result is that we can easily obtain examples of matrices from $B_{w}\left(\ell^{p}\right)$ which are not in $B\left(\ell^{p}\right)$. This shows us that the inclusion between these two spaces is proper. For example let us take the matrix $A=A_{0}$ given by the sequence $a=\left(a_{k}\right)_{k \geq 1}$, where

$$
a_{k}=\left\{\begin{array}{cc}
\sqrt[p]{2^{n}} & \text { if } k=2^{n}  \tag{2.2}\\
0 & \text { if } k \neq 2^{n}
\end{array}\right.
$$

From Theorem 2.1 it is clear that $A \in B_{w}\left(\ell^{p}\right)$. Since $a=\left(a_{k}\right)_{k \geq 1}$ is an unbounded sequence it follows that $A \notin B\left(\ell^{p}\right)$.

The next result gives us the behavior of a "diagonal" matrix. We omit the proof since we can use exactly the same arguments as in the proof of Theorem 2.1.

PROPOSITION 2.3. (a) Let $k>0$ and $A=A_{k}$ be given by the sequence $a=\left(a_{n}\right)_{n \geq 1}$. Then
$A \in B_{w}\left(\ell^{p}\right)$ if and only if $\sup _{n \in \mathrm{~N}^{*}}\left(\frac{1}{n+k} \sum_{l=1}^{n}\left|a_{l}\right|^{p}\right)^{\frac{1}{p}}<\infty$ and $\|A\|_{B_{w}\left(\ell^{p}\right)}=\sup _{n \in \mathrm{~N}^{*}}\left(\frac{1}{n+k} \sum_{l=1}^{n}\left|a_{l}\right|^{p}\right)^{\frac{1}{p}}$.
(b) Let $k<0$ and $A=A_{k}$ be given by the sequence $a=\left(a_{n}\right)_{n \geq 1}$. Then $A \in B_{w}\left(\ell^{p}\right)$ if and only if $\sup _{n \in \mathrm{~N}^{*}}\left(\frac{1}{n} \sum_{l=1}^{n}\left|a_{l}\right|^{p}\right)^{\frac{1}{p}}<\infty$ and $\|A\|_{B_{w}\left(\ell^{p}\right)}=\sup _{n \in \mathrm{~N}^{*}}\left(\frac{1}{n} \sum_{l=1}^{n}\left|a_{l}\right|^{p}\right)^{\frac{1}{p}}$.

It is well known (see e.g. [6]) that $B\left(\ell^{p}\right)$ is closed under Schur multiplication:

$$
\begin{equation*}
B\left(\ell^{p}\right) \subset \mathrm{M}\left(\ell^{p}\right) \tag{2.3}
\end{equation*}
$$

Our next remark shows that (2.3) is not true for $B_{w}\left(\ell^{p}\right)$.

## Remark 2.4.

1. $B_{w}\left(\ell^{p}\right), 1 \leq p<\infty$, is not closed under Schur multiplication. Indeed, for $A=A_{0}$ given by the sequence $a=\left(a_{k}\right)_{k \geq 1}$, with $a_{k}$ defined as in (2.2) we have that $A \in B_{w}\left(\ell^{p}\right)$ but $A * A \notin B_{w}\left(\ell^{p}\right)$.
2. $B_{w}\left(\ell^{p}\right)$ is not contained in $\mathrm{M}\left(\ell^{p}\right)$. We can take any matrix of the form $A=A_{0}$ given by $\left(a_{k}\right)_{k \geq 1}$ such that $\left(a_{k}\right)_{k \geq 1} \notin \ell^{\infty}$ and $A \in B_{w}\left(\ell^{p}\right)$ so that, in particular, $A \notin \mathrm{M}\left(\ell^{p}\right)$. Thus $B_{w}\left(\ell^{p}\right) \not \subset \mathrm{M}\left(\ell^{p}\right)$.
3. $B_{w}\left(\ell^{p}\right)$ does not contain $\mathrm{M}\left(\ell^{p}\right)$. If $1<p<\infty$ let us consider the matrix

$$
A=\left(\begin{array}{ccccc}
1 & 1 & \ldots & 1 & \ldots \\
0 & 0 & \ldots & 0 & \ldots \\
0 & 0 & \ldots & 0 & \ldots \\
0 & 0 & \ldots & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}\right)
$$

It is easy to see that $A \in \mathrm{M}\left(\ell^{p}\right)$ but $A \notin B_{w}\left(\ell^{p}\right)$ and, hence,

$$
\begin{equation*}
\mathrm{M}\left(\ell^{p}\right) \not \subset B_{w}\left(\ell^{p}\right) . \tag{2.4}
\end{equation*}
$$

In the case $p=1$ we can take the matrix

$$
B=\left(\begin{array}{ccccc}
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
\ldots & \cdots & \ldots & \ldots & \ldots
\end{array}\right) .
$$

In this case it is clear that $B \in \mathrm{M}\left(\ell^{1}\right)$ but $B \notin B_{w}\left(\ell^{1}\right)$ so that (2.4) holds also in this case.
In our next result we characterize the Schur multipliers for "diagonal" matrices:
THEOREM 2.5. Let $m=\left(m_{k}\right)_{k \geq 1}$ be a sequence of complex numbers. If $M=M_{0}$ is a matrix given by the sequence $m$ then we have that

$$
M \in M\left(B_{w}\left(\ell^{p}\right), B_{w}\left(\ell^{p}\right)\right), 1 \leq p<\infty
$$

if and only if $m=\left(m_{k}\right)_{k \geq 1} \in \ell^{\infty}$.
Proof. Let $A \in B_{w}\left(\ell^{p}\right)$ and $A_{0}$ be the diagonal matrix defined in the introduction. We claim that

$$
\begin{equation*}
\left\|A_{0}\right\|_{B_{w}\left(\ell^{p}\right)} \leq C\|A\|_{B_{w}\left(\ell^{p}\right)} \tag{2.5}
\end{equation*}
$$

where $C>0$ is a constant. Indeed,

$$
\left\|A_{0} x\right\|_{p}^{p}=\sum_{j}\left|a_{i j} x_{j}\right|^{p} \leq \sum_{j \geq 1}\left(\sum_{k \geq 1}\left|a_{j k} x_{k}\right|^{2}\right)^{\frac{p}{2}} .
$$

Now integrating and applying Khinchin inequality (see e.g. [9], p. 224) we obtain

$$
\sum_{j \geq 1}\left(\sum_{k \geq 1}\left|a_{j k} x_{k}\right|^{2}\right)^{\frac{p}{2}}=\sum_{j \geq 1}\left(\int_{0}^{1}\left|\sum_{k \geq 1} a_{j k} x_{k} r_{k}(t)\right|^{2} d t\right)^{\frac{p}{2}} \leq \operatorname{esssup}_{t \in[0,1]} \sum_{j \geq 1}\left|\sum_{k \geq 1} a_{j k} x_{k} r_{k}(t)\right|^{p},
$$

where $r_{k}, k \geq 1$, are the Rademacher functions (see e.g. [9], p. 221). Since $A \in B_{w}\left(\ell^{p}\right)$ then for any sequence $\left(x_{k}\right)$ with $\mid x_{k} \downarrow \downarrow 0$ we have

$$
\sum_{j \geq 1}\left|\sum_{k \geq 1} a_{j k} x_{k} r_{k}(t)\right|^{p} \leq\|A\|_{B_{w}\left(\ell^{p}\right)}^{p}\|x\|_{p}^{p}, \quad t \in[0,1] .
$$

Thus

$$
\left\|A_{0} x\right\|_{p}^{p} \leq\|A\|_{B_{w}\left(\ell^{p}\right)}^{p}\|x\|_{p}^{p}
$$

and taking the supremum over all sequences $\left(x_{k}\right)$ in $\ell^{p}$ with $\left|x_{k}\right| \downarrow 0$ one gets (2.5).
Let us assume that $m=\left(m_{k}\right)_{k \geq 1} \in \ell^{\infty}$. For $A \in B_{w}\left(\ell^{p}\right)$ and $x=\left(x_{k}\right)_{k \geq 1} \in \ell^{p}$, with $\left|x_{k}\right| \downarrow 0$ we have that

$$
\begin{aligned}
& \|(M * A) x\|_{p}^{p}=\left\|\left(M_{0} * A\right) x\right\|_{p}^{p}=\left\|\left(M_{0} * A_{0}\right) x\right\|_{p}^{p}=\sum_{k \geq 1}\left|m_{k} a_{k k} x_{k}\right|^{p} \leq\|m\|_{\infty}^{p} \sum_{k \geq 1}\left|a_{k k} x_{k}\right|^{p} \leq \\
& \quad \leq\|m\|_{\infty}^{p} \cdot\|A\|_{B_{w}\left(e^{p}\right)}^{p} \cdot\|x\|_{p}^{p} .
\end{aligned}
$$

It follows that $M \in M\left(B_{w}\left(\ell^{p}\right), B_{w}\left(\ell^{p}\right)\right)$. Conversely, if $M \in M\left(B_{w}\left(\ell^{p}\right), B_{w}\left(\ell^{p}\right)\right)$, then

$$
M * A \in B_{w}\left(\ell^{p}\right)
$$

for every matrix $A \in B_{w}\left(\ell^{p}\right)$, in particular for $A=A_{0} \in B_{w}\left(\ell^{p}\right)$. Applying Theorem 2.1 for the matrix $M^{*} A$ we have that

$$
\left(m_{k} a_{k}\right)_{k \geq 1} \in g(p) .
$$

Since $a=\left(a_{k}\right)_{k \geq 1} \in g(p)$ it follows that $m=\left(m_{k}\right)_{k \geq 1} \in \ell^{\infty}$ (see [7], p. 69). The proof is complete.

## 3. TOEPLITZ MATRICES AND SCHUR MULTIPLIERS

Our most important result of this section is that bounded and linear operators on $\ell^{p}$ are Schur multipliers on $B_{w}\left(\ell^{p}\right), p \geq 1$.

THEOREM 3.1. For $1 \leq p<\infty$ we have that $B\left(\ell^{p}, \ell^{\infty}\right) \subset M\left(B_{w}\left(\ell^{p}\right), B_{w}\left(\ell^{p}\right)\right)$. In particular, $B\left(\ell^{p}\right) \subset M\left(B_{w}\left(\ell^{p}\right), B_{w}\left(\ell^{p}\right)\right)$.

Proof. We show that for $A \in B\left(\ell^{p}\right), A^{*} B \in B_{w}\left(\ell^{p}\right)$ for any $B \in B_{w}\left(\ell^{p}\right)$. It yields that

$$
\begin{gather*}
\sum_{j}\left(\sum_{k}\left|a_{j k} b_{j k} x_{k}\right|\right)^{p}=\sum_{j}\left(\sum_{k}\left|a_{j k}\right| \cdot\left|b_{j k}\right| \cdot\left|x_{k}\right|\right)^{p} \leq \sum_{j}\left(\sum_{k}\left|a_{j k}\right|^{p^{*}}\right)^{p-1} \cdot \sum_{k}\left|b_{j k}\right|^{p} \cdot\left|x_{k}\right|^{p}  \tag{3.1}\\
\leq\left(\sup _{j} \sum_{k}\left|a_{j k}\right|^{p^{*}}\right)^{p-1} \sum_{j} \sum_{k}\left|b_{j k}\right|^{p}\left|x_{k}\right|^{p}=\|A\|_{p, \infty}^{p} \sum_{j} \sum_{k}\left|b_{j k}\right|^{p}\left|x_{k}\right|^{p},
\end{gather*}
$$

where $p^{-1}+p^{*(-1)}=1$. Using the same arguments as in the proof of Theorem 2.5 we have that

$$
\sum_{j} \sum_{k}\left|b_{j k} x_{k}\right|^{p} \leq\|B\|_{B_{w}\left(\ell^{p}\right)}^{p} \cdot\|x\|_{p}^{p},
$$

for every $x=\left(x_{k}\right)_{k \geq 1} \in \ell^{p}$ with $\mid x_{k} \downarrow \downarrow 0$. According to the well known fact that $\|\cdot\|_{p, \infty} \leq\|\cdot\|_{p}$ (see e.g. [6]) and taking the supremum over all sequences $x=\left(x_{k}\right)_{k \geq 1} \in \ell_{d e c}^{p}$ we obtain that

$$
\|A * B\|_{B_{w}\left(\ell^{p}\right)} \leq\|A\|_{B\left(\ell^{p}\right)} \cdot\|B\|_{B_{B_{w}}\left(\ell^{p}\right)} .
$$

This implies that $A \in M\left(B_{w}\left(\ell^{p}\right), B_{w}\left(\ell^{p}\right)\right)$. Since $B\left(\ell^{p}\right) \subset B\left(\ell^{p}, \ell^{\infty}\right)$ (see [6]), the proof is complete.For the proof of our next result we need to use the following Lemma proved in [12] (see also [8] and [16]). The notation $A \approx B$ means that there exist two positive constants $C_{1}$ and $C_{2}$ such that $A \leq C_{1} \cdot B$ and $B \leq C_{2} \cdot A$.

LEMMA 3.2. Let $a=\left(a_{n}\right)_{n \geq 1}$ be a sequence of complex numbers. Then we have that

$$
\left.\sup _{\left|x_{n}\right| \downarrow 0} \frac{\left|\sum_{n=1}^{\infty} a_{n} x_{n}\right|}{\left(\sum_{n=1}^{\infty}\left|x_{n}\right|^{p^{\prime}}\right)^{\frac{1}{p}}}=\sup _{\left|x_{n}\right| \downarrow 0}^{\left(\sum_{n=1}^{\infty}\left|x_{n}\right|^{p}\right)^{\frac{1}{p}}} \approx\left(\left.\sum_{n=1}^{\infty}\left|\frac{1}{n} \sum_{k=1}^{n}\right| a_{k} \right\rvert\,\right)^{p^{*}}\right)^{\frac{1}{p^{*}}},
$$

where $x=\left(x_{n}\right)_{n \geq 1}$ are sequences of complex numbers from $\ell^{p}$ with the property that $\mid x_{n} \downarrow 0, p>1$ and. $p^{-1}+p^{*(-1)}=1$.

Our last theorem characterize positive Toeplitz upper triangular matrices from the class $B_{w}\left(\ell^{p}\right)$ in terms of a special operator $T_{A}$ described as an average of some convolution of sequences.

THEOREM 3.3. Let $A=\left(a_{j-k}\right)_{j, k \geq 1}$, with $a_{j-k}=0$ for $j<k$ be a positive upper triangular Toeplitz matrix. Then $A \in B_{w}\left(\ell^{p}\right), p>1$, if and only if the sublinear operator $T_{A}$ is bounded from $\ell^{p^{*}}$ into $\ell^{p^{*}}$, where

$$
T_{A}(b)(j)=\frac{1}{j} \sum_{m=0}^{j}\left|\sum_{k+l=m} a_{k} b_{l}\right| \text {, for } j \geq 1
$$

and $T_{A}(b)(0)=\left|a_{0} b_{0}\right|$, where $a=\left(a_{k}\right)_{k \geq 0}, b=\left(b_{k}\right)_{k \geq 0} \in \ell^{p^{*}}, \frac{1}{p}+\frac{1}{p^{*}}=1$.
Proof. $A \in B_{w}\left(\ell^{p}\right)$ if and only if

$$
\sup _{\|\left. b\right|_{p^{p} \leq 1}}\left|\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{j} x_{j+k}\right) b_{k}\right|<\infty,
$$

for every $x=\left(x_{n}\right)_{n} \in \ell^{p}$ with $\left|x_{n}\right| \downarrow 0$. Let us denote by $c_{l}:=\sum_{k=0}^{l} a_{l-k} b_{k}$. Then

$$
\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{j} x_{j+k}\right) b_{k}=\sum_{l=0}^{\infty} x_{l} c_{l} .
$$

From Lemma 3.2 we have that

$$
\sup _{\left|x_{l}\right| \downarrow 0} \frac{\left|\sum_{l=1}^{\infty} x_{l} c_{l}\right|}{\left(\sum_{l=1}^{\infty}\left|x_{l}\right|^{p}\right)^{\frac{1}{p}}}=\sup _{\left|x_{l}\right| \downarrow 0} \frac{\sum_{l=1}^{\infty}\left|x_{l} \| c_{l}\right|}{\left(\sum_{l=1}^{\infty}\left|x_{l}\right|^{p}\right)^{\frac{1}{p}}} \approx\left|a_{0} b_{0}\right|^{p^{*}}+\sum_{j=1}^{\infty}\left(\frac{1}{j} \sum_{m=0}^{j}\left|\sum_{k=0}^{m} a_{m-k} b_{k}\right|\right)^{p^{*}} .
$$

The proof is complete.
Finally we give a necessary condition for an upper triangular Toeplitz matrix to belong to $B\left(\ell^{p}\right)$.
Remark 3.4. Let $A$ be a upper triangular Toeplitz matrix having as entries the sequence $a=\left(a_{n}\right)_{n \geq 0}$. If $A \in B\left(\ell^{p}\right)$, then $\tilde{A} \in B_{w}\left(\ell^{p}\right)$, where $\widetilde{A}=\widetilde{A}_{0}$ and is given by the sequence $\left(\widetilde{a}_{m}\right)_{m \geq 0}$ and $\widetilde{a}_{m}=\sum_{j=0}^{m} a_{j}$.

Indeed, if $A \in B\left(\ell^{p}\right)$ is a Toeplitz matrix, upper triangular given by the sequence $\left(a_{n}\right)_{n \geq 0}$, then

$$
\sum_{k=0}^{\infty}\left|\sum_{j=0}^{\infty} a_{j} x_{j+k}\right|^{p}<\infty, \text { for every }\left(x_{j}\right)_{j \geq 1} \in \ell^{p} .
$$

Let $e_{n}=(0,0, \ldots, 1,0, \ldots)$ so that $\left\|e_{1}+e_{2}+\ldots+e_{N}\right\|_{p}^{p}=N$. Then

$$
\begin{align*}
& \qquad A\left(\frac{1}{\sqrt[p]{N}}\left(e_{1}+e_{2}+\ldots \ldots+e_{N}\right)\right)=\frac{1}{\sqrt[p]{N}}\left(\tilde{a}_{N}, \tilde{a}_{N-1}, \ldots, \tilde{a}_{1}, \tilde{a}_{0}, 0, \ldots\right),  \tag{3.1}\\
& \text { which implies that } \sup _{N \geq 1} \frac{\sum_{m=0}^{N}\left|\tilde{a}_{m}\right|^{p}}{N}<\infty \text { and, by Theorem 2.1, it follows that } \tilde{A} \in B_{w}\left(\ell^{p}\right) .
\end{align*}
$$

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