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# FRACTIONAL ELECTROMAGNETIC WAVES IN PLASMA

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**Abstract.** In this paper, the fractional plasma oscillation model with Caputo derivative is presented. The order of the derivative is $0 < \gamma \leq 2$. The physical units of the system are preserved by introducing an auxiliary parameter $\sigma$. In the resulting equations we apply zero, static, and oscillating electric fields $E$. The obtained results show the displacement of the electrons in plasma. The resulting equations are given in terms of the Mittag-Leffler function and the numerical Laplace transform is used in the case of the oscillating electric field. The Markovian nature of the system is recovered when $\gamma = 1$.

***Key words***: fractional calculus; plasma oscillation; Caputo derivative; Mittag-Leffler function.

## 1. INTRODUCTION

A plasma is a ionized gas composed of roughly equal numbers of electrons and ions (quasi-neutral) and is characterized by collective effects. A plasma is sometimes defined as a gas that is sufficiently ionized to exhibit plasma-like behavior. In neutral gases the resulting plasmas generally contain equal numbers of positive and negative charge carriers. The plasma frequency is the most fundamental time-scale in plasma physics. This frequency corresponds to the typical electrostatic oscillation frequency of a given species in response to a small charge separation. Relative displacements between ions and electrons set up a restoring electric force (restoring force), which returns the electrons to equilibrium. The relatively heavy ions are considered as stationary. These displacements can be represented by Newton's law for an individual particle, and the electron motion is a simple harmonic one [1]. The *Fractional Calculus* (FC) represents a generalization of operations of differentiations and integrations to arbitrary non-integer order or imaginary, and is a generalization of the classical calculus [2–5]. FC has become an important tool in many areas of physics, mechanics, chemistry, engineering, finances, bioengineering and others fields, see Refs. [6–18]. FC describes the evolution of physical systems with loss, and this evolution is presented in the fractional exponent of the derivative and provides an excellent tool for the description of memory or hereditary properties of various materials and processes [19–22]; these properties are neglected in the classical models (integer-order models). The behavior of the fractional oscillator has attracted much attention of the physicists since many years ago. In Ref. [23] was discussed the fractional oscillator equation involving fractional time derivatives of the Riemann-Liouville type. Naber [24] studies the linearly damped oscillator equation, written as a fractional derivative of the Caputo type. In this work, it is presented an analytical solution and a comparison with the ordinary linearly damped oscillator. In Ref. [25] the authors consider a fractional oscillator which is interpreted as an ensemble of ordinary harmonic oscillators governed by a stochastic time arrow. Tarasov [26] considers the fractional oscillator as an open (non-isolated) system with memory; the environment is defined as an infinite set of independent harmonic oscillators coupled to a system. In this work the equations of motion were obtained from the interaction between the system and the environment with power-law spectral density. The Numerical Laplace Transform (NLT) is essentially a modified discrete Fourier transform through a windowing function (Gibbs phenomenon) and a stability factor (aliasing) [27].

In the frequency domain, several methods have been developed applying FC [28–29]. Sheng [30] has investigated the validity of applying numerical inverse Laplace transform algorithms in FC. In Ref. [31] the authors proposed a systematic way to construct fractional differential equations for the physical systems, in particular, the mass-spring and spring-damper systems without source term; the representation consists in analyzing the dimensionality of the ordinary derivative operator and setting it as the fractional derivative's operator. In the present work the idea proposed by Gómez in Ref. [31] is applied to the study of the fractional plasma oscillation model. This paper is organized as follows. In Sec. 2, basic definitions of FC are presented. In Sec. 3, the model of the plasma oscillations is presented and simulations applying different electric fields are shown. The Conclusions are given in Sec. 4.

## 2. BASIC DEFINITIONS

The definitions of the fractional order derivative are not unique and there exist several definitions, including: Riemann-Liouville, Grünwald-Letnikov, and the Caputo representation for Fractional Derivative (CFD). In the Caputo case, the derivative of a constant is zero and the initial conditions for the fractional differential equations have a known physical interpretation, which is important from the physical and engineering point of view. For a function $f(t)$, CFD is defined as [5]

$$ {}_{o}^{C}D_{t}^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_{0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\gamma-n+1}} \mathrm{d}\tau. \tag{1} $$

In this case, $0 < \gamma \leq 1$, is the order of the fractional derivative, $n = 1, 2, \ldots, \varepsilon\ N$ and $n-1 < \gamma \leq n$.

The Laplace transform of CFD is given by [5]

$$ L\left[ {}_{o}^{C}D_{t}^{\gamma} f(t) \right] = S^{\gamma} F(s) - \sum_{k=0}^{m-1} S^{\gamma-k-1} f^{(k)}(0). \tag{2} $$

The Mittag-Leffler function is present in the solutions of fractional differential equations. The Mittag-Leffler function of order $a$, defined in the complex plane by the series expansion is given by [32]

$$ E_{a}(t) = \sum_{m=0}^{\infty} \frac{t^{m}}{\Gamma(am+1)}, \quad a > 0, \tag{3} $$

where $\Gamma(.)$ is the gamma function.

Some common Mittag-Leffler functions are [32]

$$ E_{1}(\alpha) = e^{\alpha}, \tag{4} $$

$$ E_{2}(-\alpha^{2}) = \cos(\alpha), \tag{5} $$

$$ E_{3}(\alpha) = \frac{1}{2}\left[ e^{\alpha^{1/3}} + 2e^{-(1/2)\alpha^{1/3}} \cos\left( \frac{\sqrt{3}}{2} \alpha^{1/3} \right) \right], \tag{6} $$

$$ E_{4}(\alpha) = \frac{1}{2}\left[ \cos\left(\alpha^{1/4}\right) + \cosh\left(\alpha^{1/4}\right) \right]. \tag{7} $$

## 3. PLASMA OSCILLATIONS

For this research a fully ionized gas is considered; this gas is a hydrogen plasma, which has the same quantity of electrons and protons. The hydrogen plasma is considered as a uniform slab of plasma of thickness $L$ in the $x$ direction and having very large dimensions in the $y$ and $z$ dimensions. We take the proton mass to be effectively infinite compared to the electron mass and the positive charges are therefore effectively fixed in a place. Suppose that we displaced the electrons from the protons by a distance $x << L$. An electric field is set up that would exert a force on the electrons, pulling them back to the protons. Letting the electrons go, they would rush back toward the protons, overshoot and an oscillation would be set up with a characteristic frequency [1]. We will develop a simple model, *i.e.* we regard the medium as an assembly of molecular oscillators. If the electrons are displaced in the plasma an electric force (restoring force), $-kx(t)=$ $=-4\pi ne^2x(t)$ is produced in the direction of this equilibrium position. The equation of motion of each electron is therefore

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+\frac{4\pi ne^2}{m_e}x(t)=E(t), \tag{8}$$

where the charge per unit area is $-neL$, the force per unit area is $F=-4\pi n^2e^2Lx$, and the mass per unit area is $nm_eL$. The equation (8) is the equation of a harmonic oscillator with a frequency

$$\omega_0=\sqrt{\frac{4\pi ne^2}{m_e}}. \tag{9}$$

The equation (9) gives the expression for the electron plasma frequency; this plasma frequency is the most fundamental time-scale in plasma physics and plays a significant role in the propagation of electromagnetic waves in plasma. There is a different plasma frequency for each species.

### 3.1. FRACTIONAL CASE

Now we can write a fractional differential equation corresponding to the equation (8) in the following way

$$\frac{1}{\sigma^{2(1-\gamma)}}\frac{\mathrm{d}^{2\gamma}x(t)}{\mathrm{d}t^{2\gamma}}+\frac{4\pi ne^2}{m_e}x(t)=E(t),\quad 0<\gamma\le 1,\quad 1<\gamma\le 2. \tag{10}$$

The auxiliary parameter $\sigma^{2(1-\gamma)}$ is introduced with the finality to be consistent with dimensionality of the fractional differential equation; here $\sigma$ has dimensions of time (seconds) [31].

Consider $E(t)=0,\ x(0)=x_0,\ (x_0>0),\ \dot{x}(0)=0.$ The equation (10) may be written as follows

$$\frac{\mathrm{d}^{2\gamma}x(t)}{\mathrm{d}t^{2\gamma}}+\omega^2x(t)=0, \tag{11}$$

where

$$\omega^2=\frac{4\pi ne^2}{m_e}\sigma^{2(1-\gamma)}=\omega_0^2\sigma^{2(1-\gamma)}, \tag{12}$$

is the fractional electron plasma frequency for different values of $\gamma$ and $\omega_0^2=\dfrac{4\pi ne^2}{m_e}$ is the electron plasma frequency (9). When $\gamma = 1$, we recover the classical case.

The solution of equation (11) is given by

$$x(t)=x_0E_{2\gamma}\left\{-\omega^2t^{2\gamma}\right\}, \tag{13}$$

where $E_{2\gamma}$ is the Mittag-Leffler function (3). Now will be analyzed the cases when $\gamma$ take different values.

**First case**. In the case $\gamma = 2$, from (12), we have $\omega^2=\omega_0^2\sigma^{-2}$, and the solution of the equation (13) is

$$x(t)=\frac{x_0}{2}\left[\cos\left(-\omega^{1/2}t\right)+\cosh\left(-\omega^{1/2}t\right)\right]. \tag{14}$$

**Second case**. In the case $\gamma$ =3/2, from (12), we have $\omega^2=\omega_0^2\sigma^{-1}$, and the solution of the equation (13) is

$$x(t)=\frac{x_0}{2}\left[e^{-\omega^{2/3}t}+2e^{\frac{\omega^{2/3}}{2}t}\cos\left(-\frac{\sqrt{3}}{2}\omega^{2/3}t\right)\right]. \tag{15}$$

**Third case**. In the case $\gamma$ =1, from (12), we have $\omega=\omega_0$, and the solution of the equation (13) is

$$x(t)=x_0\cos(\omega_0 t). \tag{16}$$

It can be noted that the expression (16) represents the classical case.

**Fourth case**. In the case $\gamma$ =1/2, from (12), we have $\omega^2=\omega_0^2\sigma$, and the solution of the equation (13) is

$$x(t)=x_0e^{-\omega^2 t}. \tag{17}$$

In this case there exists a physical relation between the auxiliary parameter $\sigma$ and the fractional order time derivative $\gamma$. For the system described by the fractional equation (11), we can write down the relation

$$\gamma=\frac{\sigma}{\sqrt{\frac{m_e}{4\pi ne^2}}}=\sigma\omega_0, \qquad 0<\sigma\leq\sqrt{\frac{m_e}{4\pi ne^2}}. \tag{18}$$

Taking into account the expression (18), the solution (13) of the equation (10) for $E(t)$ = 0 can be rewritten through $\gamma$ by

$$x(\tilde{t})=x_0E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}\tilde{t}^{2\gamma}\right\}, \tag{19}$$

where $\tilde{t}=\omega_0 t$ is a dimensionless parameter. Plots for different values of $\gamma$ arbitrarily chosen are shown in Figs. 1 and 2.

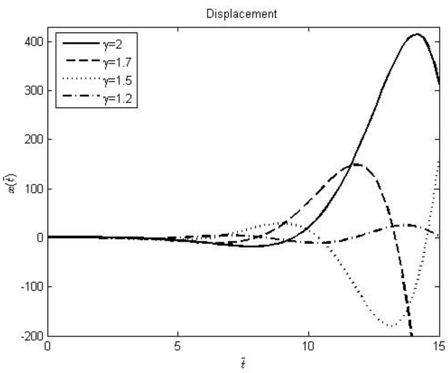


Fig. 1 – Plasma oscillations for $\gamma$ = 2, $\gamma$ = 1.7, $\gamma$ = 1.5, and $\gamma$ = 1.2.

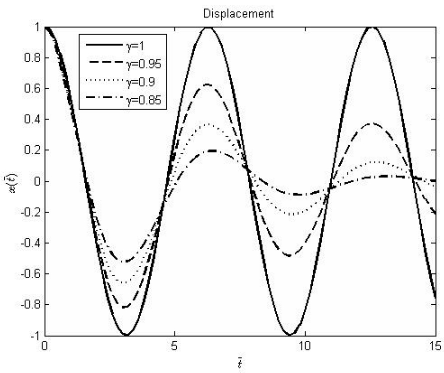


Fig. 2 – Plasma oscillations for $\gamma$ = 1, $\gamma$ = 0.95, $\gamma$ = 0.9, and $\gamma$ = 0.85.

Now we apply a static electric field, $E(t)=E_0$, $x(0)=x_0$, $(x_0>0)$, $\dot{x}(0)=0$. The equation (10) may be written as follows

$$\frac{\mathrm{d}^{2\gamma}x(t)}{\mathrm{d}t^{2\gamma}} = \frac{E_0\omega^2}{4\pi ne} - \omega^2 x(t), \tag{20}$$

where $\omega^2$ is given by (12). The solution for the equation (20) is given by

$$x(t) = \left(x_0 - \frac{E_0}{4\pi ne}\right) E_{2\gamma}\left\{-\omega^2 t^{2\gamma}\right\} + \frac{E_0}{4\pi ne}, \tag{21}$$

where $E_{2\gamma}$ is the Mittag-Leffler function (3). Now will be analyzed the cases when $\gamma$ take different values.

**First case**. In the case $\gamma$ = 2, from (12), we have $\omega^2 = \omega_0^2\sigma^{-2}$. The solution (21) becomes

$$x(t) = \left(x_0 - \frac{E_0}{4\pi ne}\right)\left(\frac{1}{2}\right)\left[\cos\left(-\omega^{1/2}t\right) + \cosh\left(-\omega^{1/2}t\right)\right] + \frac{E_0}{4\pi ne}. \tag{22}$$

**Second case**. In the case $\gamma$ = 3/2, from (12), we have $\omega^2 = \omega_0^2\sigma^{-1}$. The solution (21) becomes

$$x(t) = \left(x_0 - \frac{E_0}{4\pi ne}\right)\left(\frac{x_0}{2}\right)\left[e^{-\omega^{2/3}t} + 2e^{\frac{\omega^{2/3}}{2}t}\cos\left(-\frac{\sqrt{3}}{2}\omega^{2/3}t\right)\right] + \frac{E_0}{4\pi ne}. \tag{23}$$

**Third case**. In the case $\gamma$ = 1, from (12), we have $\omega = \omega_0$. The solution (21) becomes

$$x(t) = \left(x_0 - \frac{E_0}{4\pi ne}\right)\cos(\omega_0 t) + \frac{E_0}{4\pi ne}. \tag{24}$$

Notice that the expression (24) represents the classical case.

**Fourth case**. In the case $\gamma$ =1/2, from (12), we have $\omega^2 = \omega_0^2\sigma$. The solution (21) becomes

$$x(t) = \left(x_0 - \frac{E_0}{4\pi ne}\right)\left(e^{-\omega^2 t}\right) + \frac{E_0}{4\pi ne}. \tag{25}$$

Taking into account the physical relation (18), the solution (21) of the equation (10) for $E(t) = E_0$ can be rewritten through $\gamma$ by

$$x(\hat{t}) = \left(x_0 - \frac{E_0}{4\pi ne}\right) E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}\hat{t}^{2\gamma}\right\} + \frac{E_0}{4\pi ne}, \tag{26}$$

where $\hat{t} = \omega_0 t$ is a dimensionless parameter. Plots for different values of $\gamma$ arbitrarily chosen are shown in Figs. 3 and 4.

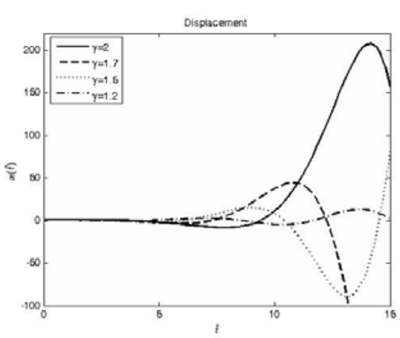


Fig. 3 – Plasma oscillations for $\gamma$ = 2, 1.7,1.5, and $\gamma$ =1.2.

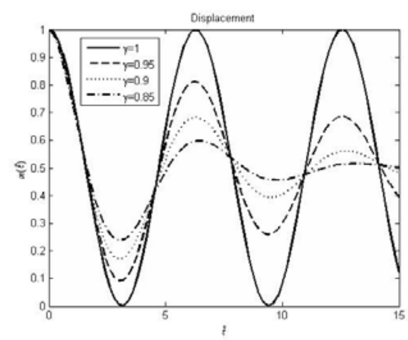


Fig. 4 – Plasma oscillations for $\gamma$ =1, 0.95, 0.9, $\gamma$ = 0.85.

As can be seen from (19) and (26) the displacement of the plasma oscillations is described by the Mittag-Leffler function $E_{2\gamma}$. For values of $\gamma$ between $1 < \gamma \leq 2$ the displacement of each electron exhibits an increment in the amplitude; in the range of $5 < \tilde{t}, \hat{t} < 15$, the behavior is anomalous dispersive (the displacement increases algebraically with increasing order of the derivative), see Figs. 1 and 3. If $\gamma$ is less than 1 the displacement of the electron shows the behavior of a damped harmonic oscillator, see Figs. 2 and 4; in this case, the displacement decreases algebraically in time and the damping of fractional oscillator is intrinsic to the equation of motion and not by introducing an additional force as in the case of an ordinary damped harmonic oscillator.

If the electric field is that of an electromagnetic wave (an oscillating electric field), then $E(t) = E_0 e^{j\eta t}$, $x(0) = x_0$, $(x_0 > 0)$, $\dot{x}(0) = 0$. This situation may arise, for example, from a plane wave passing a fixed point $x$=0. The electric field will result in a force driving oscillations of the electrons in the medium; the natural frequency ($\alpha_0$) of oscillations of the electron will be different from the frequency ($\eta$) of the electric field. The magnetic field in the electromagnetic wave will also contribute to the force on an electron in the medium. However, for a plane wave we have $B \approx \frac{E}{c}$ and for electrons in atoms $v << c$. Thus, the magnetic contribution to the force will be much less than the electric contribution.

Assuming a damping force $m_e \chi \frac{\mathrm{d}x}{\mathrm{d}t}$ that is proportional to the velocity of the electron, the fractional full equation of motion is then

$$\frac{\mathrm{d}^{2\gamma} x(t)}{\mathrm{d}t^{2\gamma}} + \beta \frac{\mathrm{d}^{\gamma} x(t)}{\mathrm{d}t^{\gamma}} = \left(\frac{e\alpha^2}{k}\right) E_0 e^{j\eta t} - \alpha^2 x(t), \tag{27}$$

where $\alpha^2 = \omega^2$ is given by (12), $k = 4\pi ne$, and $\beta = \chi\sigma^{1-\gamma}$. These constants that characterize the medium must be determined experimentally.

The Laplace transform (2) of the equation (27) is given by

$$\begin{aligned} x(s) = &\left(\frac{E_0 e}{k}\right)\left[\frac{\alpha^2 s}{\left(s^{2\gamma} + \beta s^{\gamma} + \alpha^2\right)\left(s^2 + \eta^2\right)}\right] + \left(x_0\right)\left[\frac{s+\beta}{s^{2\gamma} + \beta s^{\gamma} + \alpha^2}\right] + \\ &+ \mathrm{i}\left(\frac{E_0 e}{k}\right)\left[\frac{\alpha^2 \eta}{\left(s^{2\gamma} + \beta s^{\gamma} + \alpha^2\right)\left(s^2 + \eta^2\right)}\right], \quad 0 < \gamma \le 1, \quad 1 < \gamma \le 2. \end{aligned} \tag{28}$$

If we apply the NLT algorithm of fractional order and if we simulate the equation (28), we obtain the results plotted in Figs. 5 and 6 for different values of $\gamma$, which are arbitrarily chosen.

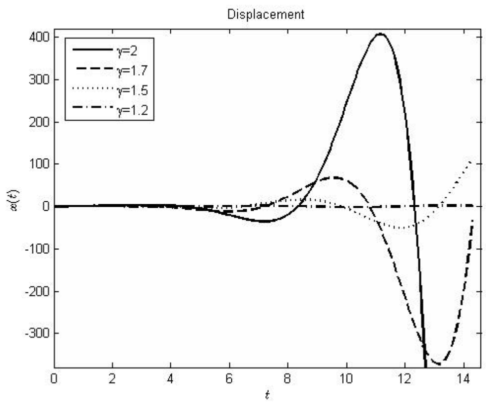


Fig. 5 – Plasma oscillations for an oscillating electric field, $\gamma = 2$, $\gamma = 1.7$, $\gamma = 1.5$, and $\gamma = 1.2$.

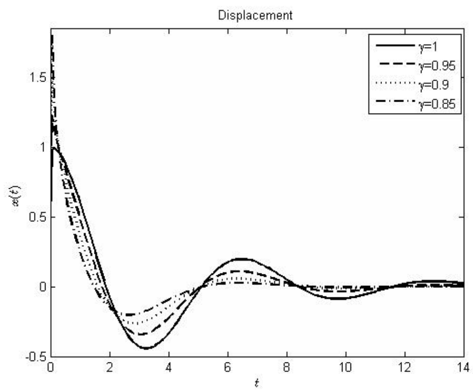


Fig. 6 – Plasma oscillations with an oscillating electric field, $\gamma = 1$, $\gamma = 0.95$, $\gamma = 0.9$, and $\gamma = 0.85$.

## 4. CONCLUSIONS

In this paper we have proposed a new fractional differential equation of order $0 < \gamma \leq 1$ and $1 < \gamma \leq 2$ to describe the fractional plasma oscillation model. With the purpose of maintaining the physical units of the system, an auxiliary parameter $\sigma$ is introduced. For zero and static electric fields the analytical solutions are given in terms of the Mittag-Leffler function depending on the parameter $\gamma$ and in the case of the oscillating electric fields the numerical Laplace transform algorithm is used in the simulation. The classical cases are recovered when $\gamma$=1 and the displacement of each electron shows its natural behavior.

For the equation (10), as we can see from (19) and (26) the displacement of the plasma oscillations is essentially described by the Mittag-Leffler function $E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}t^{2\gamma}\right\}$. Figures 1, 3, and 5 show the displacement of the electron. In the range $1 < \gamma \leq 2$, the displacement exhibit an increment in the amplitude in the range of $5 < \tilde{t},\hat{t} < 15$, and the behavior is anomalous dispersive (the displacement increases with increasing order of $\gamma$). If $\gamma$ is less than 1 the displacement shows the behavior of a damped harmonic oscillator with temporal-decaying amplitude with respect to time $t$, see Figs. 2, 4, and 6.

In the cases when $\gamma$ is in the range $1 < \gamma \leq 2$ the oscillations increase with increasing order of derivative and the vibrational frequency increases; this frequency is dependent on the properties of the system itself. In the cases when $\gamma$ is in the range $0 < \gamma \leq ½$ the displacement exhibits an exponential behavior and in the range $½ < \gamma \leq 1$ the displacement represents a damping oscillatory motion (fractional oscillator). For these cases the fractional exponent shows that the time constant tends to move forward in time, that is, the stability occurs in more time than it would take the integer-order exponent (the displacement is slower) [31]. Fractional oscillator contains simultaneously the oscillatory motion and the relaxation; this phenomenon indicates the existence of another time-scale (plasma frequency), different from the ideal plasma frequency described in equation (9). The damping is due to internal causes and the auxiliary parameter $\sigma$ represents the fractional structures (components that show an intermediate behavior between a conservative system and a dissipative one). For example, in the case $\gamma = 1$, the fractional oscillator is a conservative one (the total energy is conserved) and in the case dissipative, for example, when $\gamma$ is less to 1, the fractional oscillator shows its dissipative nature related to the viscosity of the medium (irreversible dissipative effects such as friction). Due to these dissipative effects the time symmetry is broken and the non-conservative behavior is present.

The fractional differentiation with respect to time represents a non-local displacement interpreted as an existence of memory effects (the order $\gamma$ of the fractional derivative represents a quantitative measure of memory effects), which corresponds to intrinsic dissipation of the system and is related to displacement in a fractal space-time geometry giving an entire new family of solutions for the displacement of each electron. The fractional oscillator (see equation (10)) should be considered as an ensemble average of harmonic oscillators.

We consider that the analysis above provides a method of measuring the electron number density of plasma using the model of the fractional electromagnetic waves as a probe. The fractional angular frequency of the transmitted wave is varied until propagation no longer occurs and a reflected wave is detected.

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