THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 17, Number 1/2016, pp. 16–23

GLOBAL ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS WITH BLOW-UP AT THE BOUNDARY FOR FRACTIONAL NONLINEAR PROBLEMS

Imed BACHAR [1], Habib MÂAGLI [2]

[1] King Saud University, College of Science, Mathematics Department, P.O.Box 2455, Riyadh 11451, Saudi Arabia
[2] King Abdulaziz University, College of Sciences and Arts, Rabigh Campus, Department of Mathematics, P.O.Box 344, Rabigh 21911, Saudi Arabia
Corresponding author: Imed BACHAR, E-mail: abachar@ksu.edu.sa

Abstract. Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ $(n \geq 2)$ and $0 < \alpha < 2.$ We prove the existence and global asymptotic behavior of positive continuous solutions to the following nonlinear fractional problem $\left(-\Delta_{|D}\right)^{\frac{\alpha}{2}} u = f(.,u)$ in $D,$ subject to some boundary conditions. In particular, we obtain solutions which blow-up at the boundary. Here, the nonlinearity f is required to satisfy some appropriate conditions related to a Kato class $K_\alpha(D)$. Our approach is based on the Schauder's fixed point theorem.

Key words: fractional nonlinear problems, Green's function, global asymptotic behavior, boundary blow-up, Schauder's fixed point theorem.

1. INTRODUCTION

Let D be a bounded $C^{1,1}$– domain in $\mathbb{R}^n$ $(n \geq 2)$ and $0 < \alpha < 2.$ In this paper we are concerned with the existence and global asymptotic behavior of positive continuous solutions to the following nonlinear fractional problem:

$$\begin{cases} \left(-\Delta_{|D}\right)^{\frac{\alpha}{2}} u = f(.,u) & \text{in } D \text{ (in the sense of distributions)}, \\ u > 0 & \text{in } D, \\ \lim\limits_{x \to z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \lambda \varphi(z), \end{cases} \tag{1}$$

where λ is a positive number, φ is a fixed non-trivial nonnegative continuous function on ∂D and f satisfies some convenient conditions related to the Kato class $K_\alpha(D)$ (see Definition 1.1 below).

Here the fractional power $\left(-\Delta_{|D}\right)^{\frac{\alpha}{2}}$ of the negative Dirichlet Laplacian in $D,$ is the infinitesimal generator of the subordinate killed Brownian motion process Z_α^D. For more description of the process, Z_α^D we refer to [10, 11, 21].

The nonnegative function $M_\alpha^D 1$ is defined by the formula

$$M_\alpha^D 1(x) = \frac{1-\frac{\alpha}{2}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{-2+\frac{\alpha}{2}} (1 - P_t^D 1(x)) \mathrm{d}t, \tag{2}$$

where $(P_t^D)_{t>0}$ is the semi-group corresponding to the killed Brownian motion upon exiting D.

We recall that from [10, Theorem 3.1], the function $M_\alpha^D 1$ is harmonic with respect to Z_α^D and by [21, Remark 3.3], there exists a constant $c>0$ such that

$$\frac{1}{c}\left(\delta(x)\right)^{\alpha-2} \le M_\alpha^D 1(x) \le c\left(\delta(x)\right)^{\alpha-2}, \text{ for all } x\in D, \tag{3}$$

where $\delta(x)$ denotes the Euclidian distance from x to the boundary of D.

In the classical case (*i.e.* $\alpha=2$), there exist a lot of works related to problem (1); see for example, the papers of Alves, Carriao and Faria [1], Barile and Salvatore [3], de Figueiredo, Girardi and Matzeu [9], Cîrstea, Ghergu and Rădulescu [4], Ghergu and Rădulescu [12–15], Lair and Wood [16], Zhang [22] and references therein. In all these papers, the main tools used are Galerkin method, sub-supersolution method, symmetric mountain pass theorem and variational techniques.

In [24], Zhang and Zhao studied the following problem

$$\begin{cases} \Delta u + V(x)u^p = 0, & \text{in } \Omega\setminus\{0\} \text{ (in the sense of distributions)},\\ u>0, & \text{in } \Omega\setminus\{0\},\\ u_{|\partial\Omega}=0, & \\ u(x)\sim \frac{c}{|x|^{n-2}}, \text{ near } x=0, & \text{for any sufficiently small } c>0, \end{cases} \tag{4}$$

where Ω is a bounded $C^{1,1}-$ domain in $\mathbb{R}^n$ $(n\ge 3)$ containing 0, $p>1$ and V is a measurable function such that

$$x\to \frac{V(x)}{|x|^{(n-2)(p-1)}} \text{ is in the classical Kato class } K^n(\Omega).$$

Definition and properties of the classical class $K^n(\Omega)$ can be found in [2, 6]. Then, they showed the existence of infinitely many solutions of (4). On the other hand, in [17, 18], the authors proved that the existence of infinitely many singular solutions is valid for the following nonlinear problem

$$\begin{cases} \Delta u + g(x,u) = 0, & \text{in } \Omega\setminus\{0\} \text{ (in the sense of distributions)},\\ u>0, & \text{in } \Omega\setminus\{0\},\\ u_{|\partial\Omega}=0, & \end{cases} \tag{5}$$

where Ω is a bounded $C^{1,1}-$ domain in $\mathbb{R}^n$ containing 0 and $g(x,t)$ is a measurable function in $\Omega\times(0,\infty)$ satisfying some appropriate conditions related to a Kato class $K(\Omega)$ which properly contains the classical Kato class $K^n(\Omega)$. More precisely, they showed that there exists a number $b_0>0$ such that for each $b\in(0,b_0]$, there exists a positive continuous solution u in $\Omega\setminus\{0\}$ of (5) satisfying

$$\lim_{|x|\to 0}\frac{u(x)}{G(x,0)} = b,$$

where $G(x,y)$ is the Green's function of the Laplacian in D. In particular they have extended the result of [24].

The fractional Kato class $K_\alpha(D)$ is defined by means of the Green function $G_\alpha^D(x,y)$ of Z_α^D as follows.

Definition 1.1 [7]. A Borel measurable function ρ in D belongs to the Kato class $K_\alpha(D)$ if

$$\lim_{r\to 0}(\sup_{x\in D}\int_{(|x-y|\le r)\cap D}\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,\ y)\,|\,\rho(y)\,|\,\mathrm{d}y)=0.$$

It has been shown in [7], that the function

$$x \to (\delta(x))^{-\lambda} \text{ belongs to } K_\alpha(D), \text{ for } \lambda < \alpha. \tag{6}$$

For two nonnegative functions θ and ψ defined on a set S, the notation $\theta(x) \approx \psi(x), \quad x \in S$, means that there exists $c > 0$ such that $\frac{1}{c}\psi(x) \le \theta(x) \le c\psi(x)$, for all $x \in S$.

Throughout this paper, for $\varphi \in C^+(\partial D)$, we denote by $M_\alpha^D \varphi$ (see [10]) the unique positive continuous solution of

$$\begin{cases} (-\Delta_{|D})^{\frac{\alpha}{2}} u = 0 & \text{in } D \text{ (in the sense of distributions)}, \\ u > 0 & \text{in } D \\ \lim\limits_{x \to z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z). & \end{cases} \tag{7}$$

Note that there exists $c > 0$ such that for all $x \in D$,

$$M_\alpha^D \varphi(x) \le \|\varphi\|_\infty M_\alpha^D 1(x) \le c(\delta(x))^{\alpha-2}. \tag{8}$$

Observe that problem (1) is in fact a perturbation of problem (7) with the nonlinear term $f(.,u)$. The purpose of this paper is to prove that for λ sufficiently small parameter and under some adequate assumptions on f, we obtain a positive continuous solution for (1) which behaves like $M_\alpha^D \varphi$ (9).

The following hypotheses on f are adopted.

- $(\mathbf{H}_1)$ f is a Borel measurable function in $D \times (0,\infty)$, continuous with respect to the second variable.
- $(\mathbf{H}_2)$ $|f(x,t)| \le tq(x,t)$, where q is a nonnegative Borel measurable function in $D \times (0,\infty)$, nondecreasing with respect to the second variable such that $\lim\limits_{t \to 0} q(x,t) = 0$.
- $(\mathbf{H}_3)$ $\forall c > 0,\ x \to q(x, c(\delta(x))^{\alpha-2})$ is in $K_\alpha(D)$.

Our main result is the following.

THEOREM 1.2. *Assume that hypotheses $(H_1)-(H_3)$ are fulfilled. Then problem (1) has infinitely many solutions. More precisely, there exists $\lambda_0 > 0$ such that for each $\lambda \in (0,\lambda_0]$, there exists a positive continuous solution u of (1) satisfying for each $x \in D$*

$$\frac{\lambda}{2} M_\alpha^D \varphi(x) \le u(x) \le \frac{3\lambda}{2} M_\alpha^D \varphi(x). \tag{9}$$

Observe that, since the function $M_\alpha^D \varphi(x)$ blows-up at the boundary, we deduce from the global asymptotic behavior (9) that also u blows-up at the boundary.

We point out that in the case $\alpha = 2$, the existence of positive solutions blowing-up on ∂D has been studied by many authors (see for instance [5, 8, 19] and the references therein).

Using (6), we can verify that hypotheses $(H_1)-(H_3)$ are satisfied for the special nonlinearity

$$f(x,t) = p(x)t^\mu,\ \mu > 1,$$

where p is a Borel measurable function satisfying: There exists a constant $c > 0$, such that for each $x \in D$,

$$|p(x)| \le \frac{c}{(\delta(x))^\tau}, \text{ with } \tau + (2-\alpha)(\mu-1) < \alpha.$$

Our paper is organized as follows. In Section 2, we collect some properties of functions belonging to the Kato class $K_\alpha(D)$, which are useful to establish our main result. In Section 3, we prove Theorem 1.2.
As usual, let $C_0(D)$ be set of continuous functions in D vanishing continuously on ∂D. Note that $C_0(D)$ is a Banach space with respect to the uniform norm:

$$\|u\|_\infty = \sup_{x\in D}|u(x)|.$$

2. THE KATO CLASS $K_\alpha(D)$

In this section, we give some properties of functions belonging to the Kato class $K_\alpha(D)$, which are useful to establish our main result.

PROPOSITION 2.1. [20]. *For* $(x,y)\in D\times D$, *we have*

$$G_\alpha^D(x,y)\approx |x-y|^{\alpha-n}\min\left(1,\frac{\delta(x)\delta(y)}{|x-y|^2}\right). \tag{10}$$

PROPOSITION 2.2. [7]. *Let* ρ *be a function in* $K_\alpha(D)$, *then we have*

(i)
$$a_\alpha(\rho):=\sup_{x,\ y\in D}\int_D \frac{G_\alpha^D(x,\ z)G_\alpha^D(z,\ y)}{G_\alpha^D(x,\ y)}|\rho(z)|\mathrm{d}z<\infty. \tag{11}$$

(ii) *Let* h *be a positive excessive function on* D *with respect to* Z_α^D. *Then we have*

$$\int_D G_\alpha^D(x,\ y)h(y)\,|\,\rho(y)\,|\,\mathrm{d}y\le a_\alpha(\rho)h(x). \tag{12}$$

Furthermore, for each $x_0\in\overline{D}$, *we have*

$$\lim_{r\to 0}(\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,\ r)\cap D}G_\alpha^D(x,\ y)h(y)\,|\,\rho(y)\,|\,\mathrm{d}y)=0. \tag{13}$$

(iii) *The function* $x\to(\delta(x))^{\alpha-1}\rho(x)$ *is in* $L^1(D)$.
The next Lemma is crucial in the proof of Theorem 1.2.

LEMMA 2.3. *Let* φ *be a non-trivial nonnegative continuous function on* ∂D *and* ρ *be a nonnegative function in* $K_\alpha(D)$, *then the family of functions*

$$\Lambda_\rho=\{x\to\frac{1}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,\ y)M_\alpha^D\varphi(y)g(y)\mathrm{d}y,\ \ |g|\le\rho\}$$

is uniformly bounded and equicontinuous in $\overline{D}$. *Consequently* Λ_ρ *is relatively compact in* $C_0(D)$.

Proof. By taking $h\equiv M_\alpha^D\varphi$ in (12), we deduce that for $|g|\le\rho$ and $x\in D$, we have

$$\left|\int_D\frac{G_\alpha^D(x,\ y)}{M_\alpha^D\varphi(x)}M_\alpha^D\varphi(y)g(y)\mathrm{d}y\right|\le\int_D\frac{G_\alpha^D(x,\ y)}{M_\alpha^D\varphi(x)}M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y\le a_\alpha(\rho)<\infty. \tag{14}$$

So the family Λ_ρ is uniformly bounded.

Next, we aim at proving that the family Λ_ρ is equicontinuous in $\overline{D}$.

Let $x_0 \in \overline{D}$ and $\varepsilon > 0$. By (13), there exists $r > 0$ such that

$$\sup_{z\in D} \frac{1}{M_\alpha^D \varphi(z)} \int_{B(x_0,\,2r)\cap D} G_\alpha^D(z,\ y) M_\alpha^D \varphi(y)\rho(y)\mathrm{d}y \le \frac{\varepsilon}{2}.$$

If $x_0 \in D$ and $x, x' \in B(x_0, r) \cap D$, then for $|g| \le \rho$, we have

$$\left|\int_D \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y) g(y)\mathrm{d}y - \int_D \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')} M_\alpha^D\varphi(y) g(y)\mathrm{d}y\right| \le$$

$$\le \int_D \left|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\right| M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y \le$$

$$\le 2\sup_{z\in D} \int_{B(x_0,\,2r)\cap D} \frac{1}{M_\alpha^D\varphi(z)} G_\alpha^D(z,\ y) M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y +$$

$$+\int_{(|x_0-y|\ge 2r)\cap D} \left|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\right| M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y \le$$

$$\le \varepsilon + \int_{(|x_0-y|\ge 2r)\cap D} \left|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\right| M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y.$$

On the other hand, for every $y \in B^c(x_0, 2r) \cap D$ and $x, x' \in B(x_0, r) \cap D$, by using (10) and (3),

$$\left|\frac{1}{M_\alpha^D\varphi(x)} G_\alpha^D(x,y) - \frac{1}{M_\alpha^D\varphi(x')} G_\alpha^D(x',y)\right| M_\alpha^D\varphi(y) \le c(\delta(y))^{\alpha-1}.$$

Now since $x \to \frac{1}{M_\alpha^D\varphi(x)} G_\alpha^D(x,y)$ is continuous outside the diagonal of D and $\rho \in K_\alpha(D)$, we deduce by *Proposition* 2.2 (iii) and the dominated convergence theorem, that

$$\int_{(|x_0-y|\ge 2r)\cap D} \left|\frac{G_\alpha^D(x,\ y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',\ y)}{M_\alpha^D\varphi(x')}\right| M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y \to 0 \text{ as } |x - x'| \to 0.$$

If $x_0 \in \partial D$ and $x \in B(x_0, r) \cap D$, then we have

$$\left|\int_D \frac{G_\alpha^D(x,\ y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y) g(y)\mathrm{d}y\right| \le \frac{\varepsilon}{2} + \int_{(|x_0-y|\ge 2r)\cap D} \frac{G_\alpha^D(x,\ y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y.$$

Now, since $\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} \to 0$ as $|x - x_0| \to 0$, for $|x_0 - y| \ge 2r$, then by same argument as above, we get

$$\int_{(|x_0-y|\ge 2r)\cap D} \frac{G_\alpha^D(x,\ y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y \to 0 \text{ as } |x - x_0| \to 0.$$

Consequently, by Ascoli's theorem, we deduce that Λ_ρ is relatively compact in $C_0(D)$.

3. PROOF OF THEOREM 1.2

Assume that hypotheses (H_1)–(H_3) are fulfilled. We aim at proving the existence of a constant $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$, there exists a positive continuous function u in D satisfying the following integral equation

$$u(x)=\lambda M_\alpha^D\varphi(x)+\int_D G_\alpha^D(x,\ y)f(y,\ u(y))\mathrm{d}y,\quad x\in D.$$

Let $\beta\in(0,1)$. Using (8), (H_2) and (H_3) we deduce that the function $x\to q(y,\beta M_\alpha^D\varphi(y))$ is in $K_\alpha(D)$. So by Lemma 2.3, the function

$$T_\beta(x)=\frac{1}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,\ y)M_\alpha^D\varphi(y)q(y,\ \beta M_\alpha^D\varphi(y))\mathrm{d}y,$$

is continuous in $\overline{D}$. Moreover, by using again hypotheses (H_2), (H_3), and Proposition 2.2, we deduce by the dominated convergence theorem that

$$\forall x\in\overline{D},\ \lim_{\beta\to0}T_\beta(x)=0.$$

Since the function $\beta\to T_\beta(x)$ is nondecreasing in $(0,1)$, we deduce by Dini Lemma that

$$\lim_{\beta\to0}\left(\sup_{x\in D}\frac{1}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,\ y)M_\alpha^D\varphi(y)q(y,\ \beta M_\alpha^D\varphi(y))\mathrm{d}y\right)=0.$$

Hence there exists $\beta\in(0,1)$ such that for each $x\in\overline{D}$,

$$\frac{1}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,\ y)M_\alpha^D\varphi(y)q(y,\ \beta M_\alpha^D\varphi(y))\mathrm{d}y\le\frac{1}{3}.$$

Let $\lambda_0=\frac{2}{3}\beta$ and $\lambda\in(0,\lambda_0]$. Let S be the nonemty closed bounded and convex set in $C(\overline{D})$ given by

$$S=\{\omega\in C(\overline{D}):\ \frac{\lambda}{2}\le\omega(x)\le\frac{3\lambda}{2}\}.$$

Define the operator Γ on S by

$$\Gamma\omega(x)=\lambda+\frac{1}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,\ y)f(y,\ \omega(y)M_\alpha^D\varphi(y))\mathrm{d}y,\quad x\in D.$$

By (H_2), (3), (H_3) and Lemma 2.3, $\Gamma S\subset C(\overline{D})$. Moreover, for each $\omega\in S$ and any $x\in D$, we have

$$\left|\Gamma\omega(x)-\lambda\right|\le\frac{3\lambda}{2}\frac{1}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,y)M_\alpha^D\varphi(y)q(y,\beta M_\alpha^D\varphi(y))\mathrm{d}y\le\frac{\lambda}{2}.$$

It follows that $\frac{\lambda}{2}\le\Gamma\omega\le\frac{3\lambda}{2}$ and so $\Gamma(S)\subset S$.

Next, we prove the continuity of the operator Γ in S in the supremum norm. Let $(\omega_k)_k$ be a sequence in S which converges uniformly to a function ω in S. Then we have

$$\left|\Gamma\omega_k(x)-\Gamma\omega(x)\right|\le\int_D\frac{G_\alpha^D(x,\ y)}{M_\alpha^D\varphi(x)}\left|f\left(y,\ \omega_k(y)M_\alpha^D\varphi(y)\right)-f\left(y,\ \omega(y)M_\alpha^D\varphi(y)\right)\right|\mathrm{d}y.$$

Using (H_2) and (3), there exists $c>0$, such that for each $y\in D$

$$\left|f\left(y,\ \omega_k(y)M_\alpha^D\varphi(y)\right)-f\left(y,\ \omega(y)M_\alpha^D\varphi(y)\right)\right|\le2M_\alpha^D\varphi(y)q(y,\ \omega(y)M_\alpha^D\varphi(y))\le$$

$$\le2M_\alpha^D\varphi(y)q(y,\ c\left(\delta(y)\right)^{\alpha-2}).$$

So we conclude by (H_1), (H_3), *Proposition* 2.2 and the dominated convergence theorem that

$$\forall x \in D,\ \Gamma\omega_k(x) \to \Gamma\omega(x) \text{ as } k \to \infty.$$

Using the fact that ΓS is relatively compact in $C(\overline{D})$, we obtain the uniform convergence, namely

$$\|\Gamma\omega_k - \Gamma\omega\|_\infty \to 0 \text{ as } k \to \infty.$$

Thus we have proved that Γ is a compact mapping from S to itself. Hence by the Schauder's fixed point theorem, there exists $\omega \in S$ such that

$$\omega(x) = \lambda + \frac{1}{M_\alpha^D\varphi(x)} \int_D G_\alpha^D(x,y) f(y, \omega(y) M_\alpha^D\varphi(y)) \mathrm{d}y.$$

Let $u(x) = M_\alpha^D\varphi(x)\omega(x)$. So u is a continuous function in D satisfying for each $x \in D$

$$u(x) = \lambda M_\alpha^D\varphi(x) + \int_D G_\alpha^D(x,y) f(y, u(y)) \mathrm{d}y. \tag{15}$$

In addition, since for each $y \in D$,

$$|f(y,u(y))| \le M_\alpha^D\varphi(y) q(y, c(\delta(y))^{\alpha-2}) \le c(\delta(y))^{\alpha-2} q(y, c(\delta(y))^{\alpha-2}),$$

we deduce by *Proposition* 2.2 (iii) that the map $y \to f(y,u(y)) \in L^1_{loc}(D)$ and by (15), that $x \to \int_D G_\alpha^D(x,y) f(y,u(y)) \mathrm{d}y \in L^1_{loc}(D)$. Hence, applying $(-\Delta_{|D})^{\frac{\alpha}{2}}$ on both sides of (15), we conclude by [11, p.230] that u is the required solution. □

Example. Let $\mu > 1$, $0 < \alpha < 2$, φ be a non-trivial nonnegative continuous function on ∂D and p be a Borel measurable function satisfying

$$|p(x)| \le \frac{c}{(\delta(x))^\tau}, \text{ with } \tau + (2-\alpha)(\mu - 1) < \alpha.$$

Then by *Theorem* 1.2, there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$, the problem

$$\begin{cases} (-\Delta_{|D})^{\frac{\alpha}{2}} u = p(x)u^\mu & \text{in } D \text{ (in the sense of distributions)}, \\ u > 0 & \text{in } D, \\ \lim\limits_{x \to z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \lambda\varphi(z), & \end{cases}$$

has a positive continuous solution u satisfying for each $x \in D$

$$\frac{\lambda}{2} M_\alpha^D\varphi(x) \le u(x) \le \frac{3\lambda}{2} M_\alpha^D\varphi(x).$$

ACKNOWLEDGEMENTS

This Project was funded by the National Plan for Science, Technology and Innovation (MAARIFAH), King Abdulaziz City for Science and Technology, Kingdom of Saudi Arabia, Award Number (13-MAT2137-02).

The authors want to thank the anonymous referee for the careful reading of the manuscript.

REFERENCES

1. ALVES, C. O., CARRIAO, P. C., FARIA, L. O., *Existence of Solution to Singular Elliptic Equations with Convection Terms via the Galerkin Method*, Electronic Journal of Differential Equations, **2010**, *12*, pp. 1–12, 2010.
2. AIZENMAN, M., SIMON, B., *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math, **XXXV**, pp. 209–273, 1982.
3. BARILE, S., SALVATOR, A., *Existence and multiplicity results for some Lane-Emden elliptic systems: subquadratic case*, Adv. Nonlinear Anal., **4**, *1*, pp. 25–35, 2015.
4. CÎRSTA, F., GHERGU, M., RĂDULESCU, V.D., *Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane-Emden-Fowler type*, J. Math. Pures Appliquées, **84**, pp. 493–508, 2005.
5. CÎRSTA, F., RĂDULESCU, V.D., *Blow-up boundary solutions of semilinear elliptic problems*, Nonlinear Anal., **48**, pp. 521–534, 2002.
6. CHUNG, K.L., ZHAO, Z., *From Brownian Motion to Schrodinger's Equation*, Springer Verlag, 1995.
7. DHIFLI, A., MÂAGLI, H., ZRIBI, M., *On the subordinate killed B.M in bounded domains and existence results for nonlinear fractional Dirichlet problems,* Math. Ann., **352**, *2*, pp. 259–291, 2012.
8. DUMONT, S., DUPAIGNE, L., GOUBET, O., RĂDULESCU, V.D., *Back to the Keller-Osserman condition for boundary blow-up solutions*, Advanced Nonlinear Studies, **7**, pp. 271–298, 2007.
9. DE FIGUEIREDO, D. G., GIRARDI, M., MATZEU, M., *Semilinear elliptic equations with dependence on the gradient via mountain pass techniques*, Diff. and Integral Eqns., **17**, pp. 119–126, 2004.
10. GLOVER, J., POP-STOJANOVIC, Z., RAO, M., ŠIKIĆ, H., SONG, R., VONDRAČEK, Z., *Harmonic functions of subordinate killed Brownian motion*, J. Funct. Anal., **215**, pp. 399–426, 2004.
11. GLOVER, J., RAO, M., ŠIKIĆ, H., SONG, R., Γ*-potentials*, In: GowriSankaran K. et al. (eds.), *Classical and modern potential theory and applications* (Chateau de Bonas, 1993), Kluwer Acad.Publ. Dordrecht, 1994, pp. 217–232.
12. GHERGU, M., RĂDULESCU, V.D., *On a class of sublinear singular elliptic problems with convection term*, J. Math. Anal. Appl., **311**, pp. 635–646, 2005.
13. GHERGU, M., RĂDULESCU, V.D., *Ground state solutions for the singular Lane-Emden-Fowler equation with sublinear convection term*, J. Math. Anal. Appl., **333**, pp. 265–273, 2007.
14. GHERGU, M., RĂDULESCU, V.D., *Singular elliptic problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Applications, **37**, Oxford University Press, 2008.
15. GHERGU, M., RĂDULESCU, V.D., *PDEs Mathematical Models in Biology, Chemistry and Population Genetics*, Springer Monographs in Mathematics, Springer Verlag, 2012.
16. LAIR, A. V., WOOD, A. W., *Large solutions of semilinear elliptic equations with nonlinear gradient terms*, Int. J. Math. Sci., **22**, pp. 869–883, 1999.
17. MÂAGLI, H., MÂATOUG, L., *Singular solutions of a nonlinear equation in bounded domains of* $\mathbb{R}^2$, J. Math. Anal. Appl., **270**, pp. 230–246, 2002.
18. MÂAGLI, H., ZRIBI, M., *On a new Kato class and singular solutions of a nonlinear elliptic equation in bounded domains of* $\mathbb{R}^n$, Positivity, **9**, pp. 667–686, 2005.
19. RĂDULESCU, V.D., *Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities*, in *Handbook of Differential Equations: Stationary Partial Differential Equations,* **4** *(*Michel Chipot, Editor), North-Holland Elsevier Science, Amsterdam, 2007, pp. 483–591.
20. SONG, R., Sharp bounds on the density, *Green function and jumping function of subordinate killed BM*, Probab. Theory Relat. Fields, **128**, pp. 606–628, 2004.
21. SONG, R., VONDRAČEK, Z., *Potential theory of subordinate killed Brownian motion in a domain*, Probab.Theory Relat. Fields, **125**, pp. 578–592, 2003.
22. ZHANG, Z., *Nonexistence of positive classical solutions of a singular nonlinear Dirichlet problem with a convection term*, Nonlinear Analysis, *8*, pp. 957–961, 1996.
23. ZHANG, Z., YUAN, R., *Rong Infinitely-many solutions for subquadratic fractional Hamiltonian systems with potential changing sign*, Adv. Nonlinear Anal, **4,** *1*, pp. 59–72, 2015.
24. ZHANG, Qi. S., ZHAO, Z., *Singular solutions of semilinear elliptic and parabolic equations*, Math. Ann, **310**, pp. 777–794, 1998.

Received June 26, 2015