THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 17, Number 1/2016, pp. 11–15

OPERATORS WHICH COMMUTE WITH THE CONJUGATE CONVOLUTION OPERATIONS

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Abstract. Let G be a locally compact group with a fixed left Haar measure λ. Let $L^\infty(G)$, $L^1(G)$ be the usual Lebesgue spaces with respect to λ. In this paper, we show that there is an isometric homomorphism from $U^\infty(G)^*$ into the set of all bounded linear operators on $L^\infty(G)$ which commute with conjugate convolution, where $U^\infty(G)$ is the subspace of $L^\infty(G)$ consisting of all $f \in L^\infty(G)$ for which the mapping $y \to {}_{y^{-1}}f_y$ from G into $L^\infty(G)$ is continuous.

Key words: locally compact group, conjugate convolution, isometric homomorphism.

1. INTRODUCTION

Let G be a locally compact group with a fixed left Haar measure λ. Let $L^\infty(G)$, $L^1(G)$ be the usual Lebesguse spaces with respect to λ as defined in [3]. For each $y \in G$ and $f \in L^p(G)$ $(1 \le p \le \infty)$, we define

$$\rho_y(f)(x) = f(y^{-1}xy) \qquad (x \in G).$$

For every $\varphi,\psi \in L^1(G)$, define the conjugate convolution $R_\varphi(\psi)$ by

$$R_\varphi(\psi)(x) = \int_G \varphi(y)\Delta(y)\psi(y^{-1}xy)\mathrm{d}y,$$

Where $x, y \in G$ and Δ is the modular function of G. It is easy to see that for any $\varphi,\psi \in L^1(G)$

$$\int_G R_\varphi(\psi)(x)\mathrm{d}x = (\int_G \varphi(y)\mathrm{d}y)(\int_G \psi(x)\mathrm{d}x).$$

This implies that $R_\varphi(\psi) \in L^1(G)$ and $\|R_\varphi(\psi)\|_1 \le \|\varphi\|_1 \|\psi\|_1$ for any $\varphi,\psi \in L^1(G)$ (see [5]).

Let $\varphi,\psi,\nu \in L^1(G)$. Then we have $R_\varphi(R_\psi(\nu)) = (R_{(\varphi*\psi)}(\nu))$, where $*$ denotes that convolution multiplication of $L^1(G)$. This implies that $L^1(G)$ with the conjugate convolution is not a Banach algebra.

For each $\varphi \in L^1(G)$ and $f \in L^\infty(G)$, define the complex-valued function $R_\varphi(f)$ on G by

$$R_\varphi(f)(x) := \int_G \varphi(y) f\left(y^{-1}xy\right)\mathrm{d}y \quad (x \in G).$$

We note that $\left\|R_\varphi(f)\right\|_\infty \leq \|f\|_\infty \|\varphi\|_1$.

A bounded linear operator T on $L^\infty(G)$ commutes with conjugate translations if

$$T\left(\rho_y(f)\right) = \rho_y\left(T(f)\right)$$

for all $y \in G$ and $f \in L^\infty(G)$. Also, T commutes with conjugate convolution if

$$T\left(R_\varphi(f)\right) = R_\varphi\left(T(f)\right)$$

for all $\varphi \in L^1(G)$ and $f \in L^\infty(G)$.

The bounded linear operator on $L^\infty(G)$ that commutes with conjugate translations and conjugate convolution have been introduced and studied by A. Gaffari in [1].

In this paper, we show that there is an isometric homomorphism from $U^\infty(G)^*$ into the set of all bounded linear operators on $L^\infty(G)$ which commute with conjugate convolution.

2. THE RESULTS

We note that $L^1(G)$ with the conjugate convolution operation does not have right bounded approximate identities, in general. Indeed, let G is an abelian group, then for each $\varphi, \psi \in L^1(G)$, we have

$$R_\varphi(\psi)(x) = \int_G \varphi(y)\psi\left(y^{-1}xy\right)\mathrm{d}y = \psi(x)\int_G \varphi(y)\mathrm{d}y$$

and so $L^1(G)$ with the conjugate convolution operation has not a right bounded approximate identity.

Definition 2.1. A net (φ_α) in $L^1(G)$ is called a conjugate left bounded approximate identity for $L^1(G)$ if

$$\left\|R_{(\varphi_\alpha)}(\varphi) - \varphi\right\|_1 \to 0 \text{ for all } \varphi \in L^1(G).$$

THEOREM 2.2. *Let G be a locally compact group. Then for any $\varphi \in L^1(G)$ and $\varepsilon > 0$, there is a neighbourhood U of $\varepsilon > 0$ in G such that*

$$\left\|R_\mu(\varphi) - \varphi\right\|_1 < \varepsilon$$

for all $\mu \in M^+(G)$, such that $\mu(G) = 1$ and $\mu\left(U^c\right) = 0$, where $R_\mu(\varphi)$ is given by

$$R_\mu(\varphi)(x) = \int_G \varphi\left(y^{-1}xy\right)\mathrm{d}\mu(y) \qquad (x \in G).$$

Proof. Let U be a neighbourhood U of e in G such that $\left\| {}_{y^{-1}}\varphi_y - \varphi \right\|_1 < \frac{\varepsilon}{2}$ if $y \in G$ ([3], Theorem 20.4). Then if $g \in C_{00}(G)$ (the set of all bounded continuous functions on G with compact support) and $\mu(U^c) = 0,$ the function

$$(x, y) \to \left| {}_{y^{-1}}\varphi_y(x) - \varphi(x) \right| \left| g(x) \right|$$

satisfies the Fubini theorem. Thus we have

$$\begin{aligned}
&\int_G \left| R_\mu(\varphi)(x) - \varphi(x) \right| \left| g(x) \right| \mathrm{d}x = \int_G \left| \int_U \varphi(y^{-1}xy)\mathrm{d}\mu(y) - \int_U \varphi(x)\mathrm{d}\mu(y) \right| \left| g(x) \right| \mathrm{d}x \\
&\le \int_G \int_U | \varphi(y^{-1}xy) - \varphi(x) | \mathrm{d}\mu(y) \, | g(x) | \mathrm{d}x = \int_U \int_G | \varphi(y^{-1}xy) - \varphi(x) \| g(x) | \mathrm{d}x\mathrm{d}\mu(y) \\
&\le \int_G \int_U | \varphi(y^{-1}xy) - \varphi(x) | \mathrm{d}\mu(y) \, | g(x) | \mathrm{d}x.
\end{aligned}$$

Theorem 12.13 of [3] implies that $\| R_\mu(\varphi) - \varphi \| < \varepsilon$.

COROLLARY 2.3. *Let G be a locally compact group. The group algebra* $L^1(G)$ *contains a conjugate left bounded approximate identity*.

Proof. Let $\mathfrak{A}$ denote the family of compact neighbourhoods of e and regard $\mathfrak{A}$ as a directed set in the usual way: $U \ge V$ if $U \subseteq V$. For each $U \in \mathfrak{A}$ choose a measure $\mu \in M^+(G)$ such that $\mu(U) = 1$ and $\mu(U^c) = 0$. Indeed; we can put $\mu_U := 1_U \big/ |U|$, where 1_U is the characteristic function on U. Then $(\mu_U)_{u \in \mathfrak{A}}$ is a conjugate left bounded approximate identity for $L^1(G)$ by Theorem 2.3.

PROPOSITION 2.4. *Let T is a bounded linear operators on* $L^\infty(G)$ *which commute with conjugate convolution. Then T commutes with conjugate translations.*

Proof. For every $f \in L^\infty(G)$ and $\varphi, \psi \in L^1(G)$, we have

$$\begin{aligned}
\langle f, R_\varphi(\psi) \rangle &= \int_G f(y) R_\varphi(\psi)(y)\mathrm{d}y = \\
&= \int_G \int_G f(y)\Delta(s)\psi(s^{-1}ys)\varphi(s)\mathrm{d}s\mathrm{d}y = \\
&= \int_G \int_G f(sy)\Delta(s)\varphi(s)\psi(ys)\mathrm{d}s\mathrm{d}y = \\
&= \int_G \int_G f(sys^{-1})\Delta(s)\varphi(s)\Delta(s^{-1})\psi(y)\mathrm{d}s\mathrm{d}y = \\
&= \int_G \int_G f(sys^{-1})\varphi(s)\psi(y)\mathrm{d}s\mathrm{d}y = \\
&= \int_G R_\varphi(f)(y)\psi(y)\mathrm{d}y = \\
&= \langle R_\varphi(f), \psi \rangle.
\end{aligned}$$

Now, let (φ_α) be a conjugate left bounded approximate identity for $L^1(G)$. Then for every $y \in G$, $f \in L^\infty(G)$ and $\varphi \in L^1(G)$ we have

$$\begin{aligned}\langle T(\rho_y(f)),\varphi\rangle &= \lim\langle T(\rho_y(f)), R_{\varphi_\alpha}(\varphi)\rangle = \\ &= \lim\langle R_{\varphi_\alpha}(T(\rho_y(f))),\varphi\rangle = \\ &= \lim\langle T(R_{\varphi_\alpha}(\rho_y(f))),\varphi\rangle = \\ &= \lim\langle T(R_{\varphi_\alpha*\delta_y}(f)),\varphi\rangle = \\ &= \lim\langle R_{\varphi_\alpha*\delta_y}(T(f)),\varphi\rangle = \\ &= \lim\langle R_{\varphi_\alpha}(\rho_y(T(f))),\varphi\rangle = \\ &= \lim\langle \rho_y(T(f)), R_{\varphi_\alpha}(\varphi)\rangle = \\ &= \langle \rho_y(T(f)),\varphi\rangle.\end{aligned}$$

That is $T(\rho_y(f)) = \rho_y(T(f))$.

For each , define $\varphi \in L^1(G)$ a seminorm ρ_φ on the space $L^\infty(G)$ by

$$\rho_\varphi(f) = \left\| R_\varphi(f)\right\|_\infty \qquad (f \in L^\infty(G)).$$

Note that $P = \{\rho_\varphi, \varphi \in L^1(G)\}$ separates the points of $L^\infty(G)$. The locally convex topology on $L^\infty(G)$ determined by this seminorm is denoted by τ_c (see, [2]). In Lemma 1.7 in [4] it is shown that $U^\infty(G) = R_{L^1(G)}(L^\infty(G))$. So for every $m \in U^\infty(G)^*$ we can define the operator m_C on $L^\infty(G)$ by

$$\langle m_C(f),\varphi\rangle = \langle m, R_\varphi(f)\rangle \quad (f \in L^\infty(G), \varphi \in L^1(G)).$$

THEOREM 2.5. *Let* G *be a locally compact group and* m *be an operator on* $L^\infty(G)$. *Then* m_C *commute with conjugate convolution*.

Proof. First we show that m_C is τ_c-continuous. Let

$$f_\alpha \to f$$

in the τ_c-topology of $L^\infty(G)$. By Lemma 3.3 of [2], for each $\varphi \in L^1(G)$,

$$R_\varphi(f_\alpha) \to R_\varphi(f)$$

in the norm topology. This implies that

$$\lim\langle m_C(f_\alpha),\varphi\rangle = \lim\langle m, R_\varphi(f_\alpha)\rangle = \lim\langle m, R_\varphi(f)\rangle = \langle m_C(f),\varphi\rangle.$$

Hence m_C is τ_c-continuous. Theorem 2.7 from [2] implies that m_C commute with conjugate convolution.

PROPOSITION 2.6. *Let G be a locally compact group and T a bounded linear operator on* $L^\infty(G)$ *which commute with conjugate convolution. Then* $T(h) \in U^\infty(G)$ *for any* $h \in U^\infty(G)$.

Proof. Let $h \in U^\infty(G)$, Then $y \to \rho_y(h)$ from G into $L^\infty(G)$ is continuous. By Proposition 2.4 we have $\rho_y(T(h)) = T(\rho_y(h))$ for each $y \in G$ and so $T(h) \in U^\infty(G)$.

For every $m, n \in U^\infty(G)^*$ we define $m.n$ on $U^\infty(G)$ by

$$(m.n)(h) = m(n_C(h)) \quad (h \in U^{\infty}(G)).$$

The following Theorem is the main result of this paper.

THEOREM 2.7. *Let* G *be a locally compact group. There exists an isometric homomorphism from* $U^{\infty}(G)^*$ *into the set of all bounded linear operator on* $L^{\infty}(G)$ *which commute with conjugate convolution.*

Proof. Define Ψ on $U^{\infty}(G)^*$ by

$$\Psi(m) = m_C \quad (m \in U^{\infty}(G)^*).$$

For any $\varphi \in L^1(G)$ with $\|\varphi\|_1 \le 1$ and $f \in L^{\infty}(G)$ we have

$$|\langle m_C(f), \varphi\rangle| = |\langle m, R_{\varphi}(f)\rangle| \le \|m\| \|R_{\varphi}(f)\|_{\infty} \le \|m\| \|\varphi\|_1 \|f\|_{\infty}.$$

Therefore, $\|m_C\| \le \|m\|$.

To see that equality holds, let (e_{α}) be a bounded approximate identity for $L^1(G)$ bounded by one and $h \in U^{\infty}(G)$. Then there is $\varphi \in L^1(G)$ and $f \in L^{\infty}(G)$ such that $h = R_{\varphi}(f)$. This implies that

$$\|R_{e_{\alpha}}(h) - h\|_{\infty} = \|R_{e_{\alpha}}(R_{\varphi}(f)) - R_{\varphi}(f)\|_{\infty} = \|R_{e_{\alpha}*\varphi}(f) - R_{\varphi}(f)\|_{\infty} \le \|e_{\alpha} * \varphi - \varphi\|_1 \|f\|_{\infty} \to 0.$$

Hence

$$\|m_C(h)\| \ge |\langle m_C(h), e_{\alpha}\rangle| = |\langle m, R_{e_{\alpha}}(h)\rangle|$$

which converges to $|\langle m, h\rangle|$. Hence $\|m_C\| \ge \|m\|$ and so Ψ is an isometry.

Now, let $m, n \in U^{\infty}(G)^*$, we claim that $(m.n)_C = m_C(n_C)$. For any $\varphi \in L^1(G)$ and $f \in L^{\infty}(G)$,

$$\begin{aligned}\langle (m.n)_C(f), \varphi\rangle &= \langle (m.n), R_{\varphi}(f)\rangle = \langle m, n_C(R_{\varphi}(f))\rangle = \langle m, R_{\varphi}(n_C(f))\rangle = \langle m_C(n_C(f)), \varphi\rangle \\ &= \langle (m_C(n_C))(f), \varphi\rangle\end{aligned}$$

That is, $(m.n)_C = m_C(n_C)$. This implies that Ψ is homomorphism.

Remark 2.8. We believe, but have been unable to prove, that the isometry of the final theorem is surjective.

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Received July 2, 2012