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THIRRING OPTICAL SOLITONS WITH SPATIO-TEMPORAL DISPERSION

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The dynamics of Thirring optical solitons, with spatio-temporal dispersion, is studied in this paper. Bright, dark, and singular optical solitons are obtained by the ansatz method. There are constraint conditions that naturally emerge from the mathematical analysis.

Key words: solitons, birefringence, integrability.

1. INTRODUCTION

The study of optical solitons is a growing area of research in the field of nonlinear fiber optics. There are a lot of interesting results that are being constantly reported over the past years [1-31]. While most of the results are for polarization-preserving fibers, it is important to divert the focus of attention to birefringent fibers, where the governing equation is the vector nonlinear Schrödinger’s equation (NLSE). It is therefore imperative to shine some light on the integrability aspect of such models. For birefringent fibers, when the self-phase modulation (SPM) is negligible, it is the cross-phase modulation (XPM) that dominates. Therefore, upon neglecting the SPM terms, one recovers Thirring NLSE whose solutions are Thirring solitons [1]. This paper will extract bright, dark, and singular Thirring 1-soliton solutions for the governing equation.

In addition to group velocity dispersion (GVD), this paper includes the spatio-temporal dispersion (STD) terms, so that the governing equation is well-posed. This fact was first pointed out during 2012 [6, 7]. This paper will subsequently apply the ansatz method of integration to recover bright, dark, and singular 1-soliton solution to the governing coupled system of NLSEs. There are constraint conditions that will fall out naturally and must remain valid for these solitons to exist. It is important to note that Thirring solitons were studied earlier from an integrability perspective [1]. However, STD was not included in that paper. Therefore the results of this work generalize those reported in the earlier published paper [1].

2. GOVERNING EQUATIONS

The dynamics of Thirring solitons is governed by coupled nonlinear Schrödinger equations (CNLSEs). The corresponding coupled system in its dimensionless form is given by [1]

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 |r|^2 q = 0\,, \tag{1}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 |q|^2 r = 0\,. \tag{2}$$

In (1) and (2), the first terms are the evolution terms. The coefficients of a_l, for $l = 1, 2$, represent the GVD while the coefficients of b_l are the STD terms. Finally, the coefficients of c_l are the XPM terms. To integrate the CNLSEs (1–2) the initial hypothesis adopted has the form [1, 2, 12]:

$$q(x,t) = P_1(x,t)e^{i\phi_1(x,t)}, \tag{3}$$

$$r(x,t) = P_2(x,t)e^{i\phi_2(x,t)}, \tag{4}$$

where $P_l(x,t)$, $l = 1, 2$, represents the amplitude components of the soliton solution, while the phase factor is

$$\phi_l(x,\ t) = -\kappa_l x + \omega_l t + \theta_l. \tag{5}$$

Here κ_l is the frequency of the solitons while ω_l represents the wave number and θ_l stands for the phase constant. Substituting (3–5) into (1) and (2) and then decomposing into real and imaginary parts gives

$$a_l \frac{\partial^2 P_l}{\partial x^2} + b_l \frac{\partial^2 P_l}{\partial x \partial t} + (b_l \kappa_l \omega_l - \omega_l - a_l \kappa^2) P_l + c_l P_l P_{\bar{l}}^2 = 0 \tag{6}$$

for the real parts, where $\bar{l} = 3 - l$, and for imaginary components we have

$$(1 - \kappa_l b_l)\frac{\partial P_l}{\partial t} + (\omega_l b_l - 2a_l \kappa_l)\frac{\partial P_l}{\partial x} = 0. \tag{7}$$

The imaginary parts (7) lead to the speed of the solitons

$$v = \frac{\omega_l b_l - 2a_l \kappa_l}{1 - b_l \kappa_l} \tag{8}$$

as long as the constraints

$$\kappa_l b_l \neq 1 \tag{9}$$

are satisfied. It is worth noting that $P(x,\ t)$ can be represented as $g(x - vt)$, where the function g is the soliton wave profile, and v is the speed of the soliton. Now, equating the two expressions for the soliton speed in (8) leads to a constraint relation between the soliton parameters as

$$(1 - b_2\kappa_2)(\omega_1 b_1 - 2a_1\kappa_1) = (1 - b_1\kappa_1)(\omega_2 b_2 - 2a_2\kappa_2). \tag{10}$$

This constraint identity holds for bright, dark, and singular solitons. The real part equations given in (6) will now be analyzed separately in the next three sections in order to obtain bright, dark, and singular solitons.

2.1. Bright solitons

In this section, the case where both components in (6) simultaneously support bright solitons is considered. In this case, the initial assumption for the wave profile is [1, 2]:

$$P_l = A_l \operatorname{sech}^{p_l} \tau, \tag{11}$$

with

$$\tau = B(x - vt), \tag{12}$$

where A_l and B represent the amplitude and inverse width of the soliton, respectively. As was stated previously, v stands for the soliton speed. Substitution of (11) and (12) reduce (6) to

$$\left\{(b_l\omega_l\kappa_l-\omega_l-a_l\kappa_l^2)+p_l^2(a_l-b_lv)B^2\right\}\ \operatorname{sech}^{p_l}\tau-$$
$$-p_l(1+p_l)(a_l-b_lv)B^2\operatorname{sech}^{p_l+2}\tau+c_lA_{\bar{l}}^2\operatorname{sech}^{p_l+2p_{\bar{l}}}\tau=0, \tag{13}$$

where l = 1, 2 and $\bar{l}=3-l$. Next, from (13), equating pairwise the exponent pairs $(p_1+2p_2,\ p_1+2)$ and $(p_2+2p_1,\ p_2+2)$ leads to

$$p_1=p_2=1 \tag{14}$$

by means of the balancing principle. Setting the coefficients of the linearly independent functions, $\operatorname{sech}^{p_\alpha+\beta}\tau$, where α = 1, 2 and β = 0, 2, to zero in each component of (13) leads to the wave numbers of the solitons being given by

$$\omega_l=\frac{c_lA_{\bar{l}}^2-2a_l\kappa_l^2}{2(1-b_l\kappa_l)} \tag{15}$$

subject to the constraints given in (9). Similarly, with the aid of (8) the width of the solitons can be retrieved as

$$B=A_{\bar{l}}\left[\frac{c_l(1-b_l\kappa_l)}{2(a_l+a_lb_l\kappa_l-b_l^2\omega_l)}\right]^{\frac{1}{2}}, \tag{16}$$

which in turn introduce the constraints

$$c_l(1-b_l\kappa_l)\times(a_l+a_lb_l\kappa_l-b_l^2\omega_l)>0\,. \tag{17}$$

By equating the two values of the width B given in (16) one have

$$\frac{A_1}{A_2}=\left[\frac{c_1(1-b_1\kappa_1)(a_2+a_2b_2\kappa_2-b_2^2\omega_2)}{c_2(1-b_2\kappa_2)(a_1+a_1b_1\kappa_1-b_1^2\omega_1)}\right]^{\frac{1}{2}}, \tag{18}$$

which is valid whenever

$$[c_1(1-b_1\kappa_1)(a_2+a_2b_2\kappa_2-b_2^2\omega_2)]\times[c_2(1-b_2\kappa_2)(a_1+a_1b_1\kappa_1-b_1^2\omega_1)]>0\,. \tag{19}$$

Thus, Thirring bright 1-soliton solutions to the system (1–2) are given by

$$q(x,t)=A_1\operatorname{sech}[B(x-vt)]e^{i(-\kappa_1x+\omega_1t+\theta_1)} \tag{20}$$

and

$$r(x,t)=A_2\operatorname{sech}[B(x-vt)]e^{i(-\kappa_2x+\omega_2t+\theta_2)}, \tag{21}$$

where the ratio of the amplitudes satisfies (18), the relation between the amplitudes and soliton widths are given in (16), and the expressions for the soliton speed were given earlier in (8). All the mentioned expressions hold and maintain the existence of the bright solitons as long as the constraints given by (9), (17) and (19) remain valid.

2.2. Dark solitons

For dark soliton solutions the assumption for the wave profile of the two components is [1, 2]:

$$P_l=A_l\tanh^{p_l}\tau \tag{22}$$

where τ is being defined as in (12). However, for dark solitons the speed parameters A_l and B are considered herein as free parameters and v correspond to the dark soliton speed as stated in (8). Substitution of (22) and (12) into (6) lead to

$$\begin{aligned}&\left\{(b_l\omega_l\kappa_l-\omega_l-a_l\kappa_l^2)-2p_l^2(a_l-b_lv)B^2\right\}\tanh^{p_l}\tau+p_l(1+p_l)(a_l-b_lv)B^2\tanh^{p_l+2}\tau+\\&+p_l(p_l-1)(a_l-b_lv)B^2\tanh^{p_l-2}\tau+c_lA_{\bar{l}}^2\tanh^{p_l+2p_{\bar{l}}}\tau=0,\end{aligned} \tag{23}$$

where l = 1, 2 and $\bar{l}=3-l$. From (23), equating the exponent pair $(p_l+2p_{\bar{l}},\ p_l+2)$ in each component one retrieve the values of p_1 and p_2 as in (14). The stand alone linearly independent functions are $\tanh^{p_\alpha-2}\tau$ for α = 1, 2, whose coefficients, when set to zero, also lead to (14). Thus, from (23), setting the coefficients of the linearly independent $\tanh^{p_\alpha+\beta}\tau$ functions for α = 1, 2 and β = 0, 2 to zero, gives the wave number of the dark solitons as

$$\omega_l=\frac{c_lA_{\bar{l}}^2-a_l\kappa_l^2}{2(1-b_l\kappa_l)} \tag{24}$$

subject to the constraints given by (9). Similarly, with the aid of (8) the width of the solitons can be retrieved as

$$B=A_{\bar{l}}\left[-\frac{c_l(1-b_l\kappa_l)}{2(a_l+a_lb_l\kappa_l-b_l^2\omega_l)}\right]^{\frac{1}{2}}, \tag{25}$$

that in turn introduce the constraints

$$c_l(1-b_l\kappa_l)\times(a_l+a_lb_l\kappa_l-b_l^2\omega_l)<0\,. \tag{26}$$

By equating the two values of the width B from (25) one retrieves again the relation (18) with corresponding constraint (19). Therefore the Thirring dark 1-soliton solution to the CNLSE system (1–2) with STD is given by

$$q(x,t)=A_1\tanh[B(x-vt)]e^{i(-\kappa_1x+\omega_1t+\theta_1)} \tag{27}$$

and

$$r(x,t)=A_2\tanh[B(x-vt)]e^{i(-\kappa_2x+\omega_2t+\theta_2)}, \tag{28}$$

where the ratio of the amplitudes satisfies (18), the relation between the amplitudes and soliton widths are given in (25), and the expressions for the soliton speed were provided in (8). All the mentioned expressions hold and maintain the existence of the dark solitons whenever the constraints given by (9), (19) and (26) remain valid.

2.3. Singular solitons

For singular solitons the assumption for the wave profile of the two components is [1]:

$$P_l=A_l\operatorname{csch}^{p_l}\tau\,, \tag{29}$$

where τ is being defined as in (12), while the parameters A_l and B are considered free parameters as in the previous subsection. Upon substituting (12) and (29) into (6) one get

$$\begin{aligned}&\left\{(b_l\omega_l\kappa_l-\omega_l-a_l\kappa_l^2)+p_l^2(a_l-b_lv)B^2\right\}\operatorname{csch}^{p_l}\tau-\\&-p_l(1+p_l)(a_l-b_lv)B^2\operatorname{csch}^{p_l+2}\tau+c_lA_{\bar{l}}^2\operatorname{csch}^{p_l+2p_{\bar{l}}}\tau=0,\end{aligned} \tag{30}$$

where $l = 1, 2$ and $\bar{l} = 3 - l$. Now, from (30), equating the exponent pair $(p_l + 2p_{\bar{l}},\ p_l + 2)$ leads again to the same value for p_1 and p_2 as in (14). Next, setting the coefficients of the linearly independent functions, $\operatorname{sech}^{p_\alpha - 2}\tau$, for $\alpha = 1, 2$, whose coefficients, when set to zero, also lead to (14). Thus, from (23), setting the coefficients of the linearly independent $\tanh^{p_\alpha + \beta}\tau$ functions for $\alpha = 1, 2$ and $\beta = 0, 2$ to zero in (30) led to the wave numbers of the solitons being given by

$$\omega_l = -\frac{c_l A_{\bar{l}}^2 + a_l \kappa_l^2}{2(1 - b_l \kappa_l)} \tag{31}$$

subject to the constraints given by (9). Similarly, with the aid of (8) the width of the solitons can be retrieved as

$$B = A_{\bar{l}} \left[-\frac{c_l (1 - b_l \kappa_l)}{(a_l + a_l b_l \kappa_l - b_l^2 \omega_l)} \right]^{\frac{1}{2}}, \tag{32}$$

that impose again the constraints provided in (26). Consequently one can retrieve again the relation (18) with corresponding constraint (19). Finally, the Thirring singular 1-soliton solution to the system (1) - (2) is given by

$$q(x,t) = A_1 \operatorname{csch}[B(x - vt)] e^{i(-\kappa_1 x + \omega_1 t + \theta_1)} \tag{33}$$

and

$$r(x,t) = A_2 \operatorname{csch}[B(x - vt)] e^{i(-\kappa_2 x + \omega_2 t + \theta_2)}, \tag{34}$$

where the ratio of the amplitudes satisfies (18), the relation between the amplitudes and soliton widths are given in (32), and the expressions for the soliton speed were provided in (8). These results impose the constraints given by (9), (19) and (26) in order to keep the existence of the singular solitons.

3. CONCLUSIONS

This paper addressed Thirring soliton solutions for coupled NLSEs with STD. Bright, dark, and singular 1-soliton solutions are retrieved from the governing equations. There are several constraint conditions that are obtained. These integrability conditions must hold in place for the solitons to exist. The results obtained in this work can be extended in many different directions. There are several perturbation terms that will be included. Families of soliton solutions will be obtained with the perturbation terms taken into consideration. Later, Thirring solitons for parabolic law nonlinearity will also be addressed. These results will be reported elsewhere.

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