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ON THE SOLUTIONS OF TIME-FRACTIO3NAL GENERALIZED HIROTA-SATSUMA COUPLED-KDV EQUATION WITH MODIFIED RIEMANN-LIOUVILLE DERIVATIVE BY AN ANALYTICAL TECHNIQUE

Mehmet MERDAN

Gümüşhane University, Facuty of Engineering, Mathematical Engineering, Gümüşhane, Turkey
E-mail: mmerdan@gumushane.edu.tr

In this paper, the fractional variational iteration method (FVIM) was implemented to solve time-fractional generalized Hirota-Satsuma coupled-KDV equation with modified Riemann-Liouville derivative. A new application of fractional variational iteration method (FVIM) was extended to derive analytical solutions in the form of a series for these equations. It is indicated that the solutions obtained by the FVIM are reliable and effective method for strongly nonlinear couple partial equations with modified Riemann-Liouville derivative. The effects of different fractional derivative for the coupled system are investigated.

Key words: fractional variational iteration method, time-fractional generalized Hirota-Satsuma coupled-KDV equation, Riemann-Liouville derivative,

1. INTRODUCTION

In recent years, significantly interest in fractional calculus used in many areas such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, engineering, acoustics, material science and signal processing can be successfully modeled by linear or nonlinear fractional order differential equations [1–10].

The variational iteration method (VIM), which first proposed by Ji-Huan He [11–13], was successfully applied to autonomous ordinary and partial differential equations and other fields. Ji-Huan He [12] was the first to apply the variational iteration method to fractional differential equations. In recent years, a new modified Riemann-Liouville left derivative is suggested by G. Jumarie [14–16].

In this paper, we extend the application of the VIM in order to derive analytical approximate solutions to nonlinear time-fractional generalized Hirota-Satsuma coupled-KDV equation

$$\frac{\partial^{q_1} u(x,t)}{\partial t^{q_1}} + 3uu_x - 3vw_x - 3wv_x - \frac{1}{2}u_{xxx} = 0, \tag{1}$$

$$\frac{\partial^{q_2} v(x,t)}{\partial t^{q_2}} - 3uv_x + v_{xxx} = 0, \tag{2}$$

$$\frac{\partial^{q_3} w(x,t)}{\partial t^{q_3}} - 3uw_x + w_{xxx} = 0, \quad 0 < q_1,\ q_2,\ q_3 \le 1, \quad t > 0, \quad x \in R. \tag{3}$$

Subject to the initial conditions:

$$u(x,0) = \frac{\beta - 2k^2}{3} + 2k^2 \tanh^2(kx), \tag{4}$$

$$v(x,0) = -\frac{4k^2 c_0\left(\beta + k^2\right)}{3c_1^2} + \frac{4k^2\left(\beta + k^2\right)\tanh(kx)}{3c_1}, \tag{5}$$

$$w(x,0) = c_0 + c_1 \tanh(kx), \tag{6}$$

where $c_0, k, c_1 \neq 0$, and β are arbitrary constants. The Hirota-Satsuma system was proposed by Hirota and Satsuma [17] to describe the interaction of two long waves with different dispersion relations. Recently a numerical method is proposed for a time-fraction generalized Hirota-Satsuma coupled KDV equation by Ganji et.al [18], they are used the homotopy perturbation method. A new iterative technique [19] is employed to solve a system of nonlinear fractional partial differential equations.

The goal of this paper is to extend the application of the variational iteration method to solve time-fractional generalized Hirota-Satsuma coupled-KDV equation with modified Riemann-Liouville derivative.

2. BASIC DEFINITIONS

Here, some basic definitions and properties of the fractional calculus theory which can be found in [1, 6, 14–16, 20]

Definition 2.1. The left-sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad \text{for } \alpha > 0, x > 0 \text{ and } J_a^0 f(x) = f(x). \tag{7}$$

$$J_a^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau,t) \mathrm{d}\tau, \quad \alpha > 0, x > 0 \tag{8}$$

The properties of the operator J^α can be found in [1, 6, 20].

Definition 2.2. The modified Riemann-Liouville derivative[14–16] is defined as

$${}_0D_\alpha^x f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^0 (x-\tau)^{n-\alpha} \left(f(\tau) - f(0) \right) \mathrm{d}\tau, \tag{9}$$

where $x \in [0,1]$, $n-1 \leq \alpha < n$ and $\mathrm{n} \geq 1$.

Definition 2.3. Fractional derivative of compounded functions [1–416] is defined as

$$d^\alpha f \cong \Gamma(1+\alpha) \mathrm{d}f, \; 0 < \alpha < 1. \tag{10}$$

Definition 2.4. The integral with respect to $(dx)^\alpha$ [14-16] is defined as the solution of the fractional differential equation

$$\mathrm{d}y \cong f(x)(\mathrm{d}x)^\alpha, \quad x \geq 0, \quad y(0) = 0, \quad 0 < \alpha < 1. \tag{11}$$

LEMMA 2.1. Let $f(x)$ denote a continuous function [14–16] ,then the solution of the Eq. (11) is defined as

$$y = \int_0^x f(\tau)(\mathrm{d}\tau)^\alpha = \alpha \int_0^x (x-\tau)^\alpha f(\tau) \mathrm{d}\tau, \quad 0 < \alpha < 1. \tag{12}$$

For example $f\left(x\right)=x^{\beta}$ in Eq. (11) one obtains

$$\int_0^x \tau^{\beta}\left(d\tau\right)^{\alpha}=\frac{\Gamma\left(\beta+1\right)\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha+\beta+1\right)}x^{\beta+\alpha},\quad 0<\alpha<1. \tag{13}$$

Firstly, let's discuss VIM for the differential equation in follows.

3. VARIATIONAL ITERATION METHOD

According to the variational iteration method [12], we consider the following the partial differential differential equation:

$$\mathrm{L}u\left(x,t\right)+\mathrm{N}u\left(x,t\right)=g(x,t), \tag{14}$$

where L is a linear operator, N is a non-linear operator, and $g(x,t)$ is the source term. Then, we can construct a correct functional as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left\{\mathrm{L}u_n(x,s)+\mathrm{N}\tilde{u}_n(x,s)-g(x,s)\right\}\mathrm{d}s, \tag{15}$$

where λ is a general Lagrangian multiplier [11–13], which can be identified optimally *via* variational theory. The second term on the right is called the correction and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n=0$.

4. FRACTIONAL VARIATIONAL ITERATION METHOD

To describe the solution procedure of the fractional variational iteration method, we consider the following fractional differential equation [14–16, 21]:

$$\frac{\partial^q u\left(x,t\right)}{\partial t^q}=L\left[x\right]\left[u\left(x,t\right)v(x,t)w(x,t)\right],\quad u(x,0)=f(x),\quad 0<q,\quad t>0,\quad x\in R, \tag{16}$$

where $\mathrm{L}\left[x\right]$ is the differential operator in x, $f(x)$ is continuous function. According to the VIM, we can construct a correct functional for Eq. (16) as follows

$$\begin{aligned} u_{n+1}\left(x,t\right)&=u_n(x,t)+I^q\left[\lambda_1\left(x,t\right)\left(\frac{\partial^q u_n(x,t)}{\partial t^q}-\mathrm{L}[x]\left[u\left(x,t\right)v(x,t)w(x,t)\right]\right)\right],\\ u_{n+1}\left(x,t\right)&=u_n(x,t)+\frac{1}{\Gamma(q)}\int_0^t\left(t-\tau\right)^{q-1}\lambda_1\left(x,\tau\right)\left(\frac{\partial^q u_n(x,\tau)}{\partial\tau^q}-\mathrm{L}[x]\left[u\left(x,\tau\right)v(x,\tau)w(x,\tau)\right]\right)\mathrm{d}\tau, \end{aligned} \tag{17}$$

Using Eq. (12), we obtain a new correction functional

$$u_{n+1}\left(x,t\right)=u_n\left(x,t\right)+\frac{1}{\Gamma\left(q_1+1\right)}\int_0^t\lambda_1\left(x,\tau\right)\left(\frac{\partial^{q_1}u_n(x,\tau)}{\partial\tau^{q_1}}-\left[\mathrm{L}\left[x\right]u\left(x,\tau\right)\right]v(x,\tau)w(x,\tau)\right)\left(\mathrm{d}\tau\right)^{q_1}. \tag{18}$$

It is obvious that the sequential approximations $u_k, k\geq 0$ can be established by determining λ, a general Lagrange's multiplier, which can be identified optimally with the variational theory. The initial values are usually used for choosing the zeroth approximation u_0. With λ determined, then several approximations $u_k, k\geq 0$ follows immediately[21]. Consequently, the exact solution may be procured by using

$$u(x,t)=\lim_{n\to\infty}u_n(x,t). \tag{19}$$

5. APPROXIMATE SOLUTIONS OF THE TIME-FRACTIONAL GENERALIZED HIROTA-SATSUMA COUPLED-KDV EQUATION

In this section, we present the solution of time-fractional generalized Hirota-Satsuma coupled-KDV equation as the applicability of FVIM.

According to the FVIM, we construct a correction functional for the Eqs. (1–3). We have

$$u_{n+1}(x,t)=u_n(x,t)+\frac{1}{\Gamma(q_1+1)}\int_0^t\lambda_1(x,\tau)\left(\begin{array}{l}\frac{\partial^{q_1}u_n(x,t)}{\partial t^{q_1}}+3u_n(x,t)\frac{\partial u_n(x,t)}{\partial x}-3v_n(x,t)\frac{\partial w_n(x,t)}{\partial x}\\ -3w_n(x,t)\frac{\partial v_n(x,t)}{\partial x}-\frac{\partial^3 u_n(x,t)}{2\partial x^3}\end{array}\right)(\mathrm{d}\tau)^{q_1}, \tag{20}$$

$$v_{n+1}(x,t)=v_n(x,t)+\frac{1}{\Gamma(q_2+1)}\int_0^t\lambda_2(x,\tau)\left\{\frac{\partial^{q_2}v_n(x,\tau)}{\partial\tau^{q_2}}-3u_n(x,t)\frac{\partial v_n(x,t)}{\partial x}+\frac{\partial^3 v_n(x,t)}{\partial x^3}\right\}(\mathrm{d}\tau)^{q_2}, \tag{21}$$

$$w_{n+1}(x,t)=w_n(x,t)+\frac{1}{\Gamma(q_3+1)}\int_0^t\lambda_3(x,\tau)\left\{\frac{\partial^{q_2}{}_n(x,\tau)}{\partial\tau^{q_2}}-3u_n(x,t)\frac{\partial w_n(x,t)}{\partial x}+\frac{\partial^3 w_n(x,t)}{\partial x^3}\right\}(\mathrm{d}\tau)^{q_3}, \tag{22}$$

we have

$$\begin{aligned}\delta u_{n+1}(x,t)&=\delta u_n(x,t)+\frac{1}{\Gamma(q_1+1)}\delta\int_0^t\lambda_1(x,\tau)\left(\begin{array}{l}\frac{\partial^{q_1}u_n(x,t)}{\partial t^{q_1}}+3u_n(x,t)\frac{\partial u_n(x,t)}{\partial x}-3v_n(x,t)\frac{\partial w_n(x,t)}{\partial x}\\ -3w_n(x,t)\frac{\partial v_n(x,t)}{\partial x}-\frac{\partial^3 u_n(x,t)}{2\partial x^3}\end{array}\right)(\mathrm{d}\tau)^{q_1}=\\ &=\delta u_n+\lambda_1\delta u_n\big|_{\tau=t}-\frac{1}{\Gamma(q_1+1)}\int_0^t\frac{\partial^{q_1}\lambda_1(x,\tau)}{\partial\tau^{q_1}}\delta u_n(x,\tau)(\mathrm{d}\tau)^{q_1},\end{aligned} \tag{23}$$

$$\begin{aligned}\delta v_{n+1}(x,t)&=\delta v_n(x,t)+\frac{1}{\Gamma(q_2+1)}\delta\int_0^t\lambda_2(x,\tau)\left\{\frac{\partial^{q_2}v_n(x,\tau)}{\partial\tau^{q_2}}-3u_n(x,t)\frac{\partial v_n(x,t)}{\partial x}+\frac{\partial^3 v_n(x,t)}{\partial x^3}\right\}(\mathrm{d}\tau)^{q_2}=\\ &=\delta v_n+\lambda_2\delta v_n\big|_{\tau=t}-\frac{1}{\Gamma(q_2+1)}\int_0^t\frac{\partial^{q_2}\lambda_2(x,\tau)}{\partial\tau^{q_2}}\delta v_n(x,\tau)(\mathrm{d}\tau)^{q_2}\end{aligned} \tag{24}$$

$$\begin{aligned}\delta w_{n+1}(x,t)&=\delta w_n(x,t)+\frac{1}{\Gamma(q_3+1)}\delta\int_0^t\lambda_3(x,\tau)\left\{\frac{\partial^{q_2}w_n(x,\tau)}{\partial\tau^{q_2}}-3u_n(x,t)\frac{\partial w_n(x,t)}{\partial x}+\frac{\partial^3 w_n(x,t)}{\partial x^3}\right\}(\mathrm{d}\tau)^{q_3}=\\ &=\delta w_n+\lambda_2\delta w_n\big|_{\tau=t}-\frac{1}{\Gamma(q_3+1)}\int_0^t\frac{\partial^{q_3}\lambda_3(x,\tau)}{\partial\tau^{q_3}}\delta w_n(x,\tau)(\mathrm{d}\tau)^{q_3},\end{aligned} \tag{25}$$

where λ_1, λ_2 and λ_3 are general Lagrange multipliers, n denotes the n^{th} approximation and u_n, v_n, w_n denotes restricted variations, i.e. $\delta u_n=\delta v_n=\delta w_n=0$, and

$$\begin{aligned}&1+\lambda_1(x,\tau)\big|_{\tau=t}=0,\quad 1+\lambda_2(x,\tau)\big|_{\tau=t}=0,\quad 1+\lambda_3(x,\tau)\big|_{\tau=t}=0,\\ &\frac{\partial^{q_1}\lambda_1(x,\tau)}{\partial\tau^{q_1}}=0,\quad \frac{\partial^{q_2}\lambda_2(x,\tau)}{\partial\tau^{q_2}}=0,\quad \frac{\partial^{q_3}\lambda_3(x,\tau)}{\partial\tau^{q_3}}=0.\end{aligned} \tag{26}$$

The generalized Lagrange multiplier can be identified by the above equations

$$\lambda_1(x,t)=-1,\quad \lambda_2(x,t)=-1,\quad \lambda_3(x,t)=-1. \tag{27}$$

Substituting Eq. (27) into Eqs. (20–22) produces the iteration formulation as follows

$$u_{n+1}(x,t)=u_n(x,t)-\frac{1}{\Gamma(q_1+1)}\int_0^t\left\{\begin{array}{l}\dfrac{\partial^{q_1}u_n(x,t)}{\partial t^{q_1}}+3u_n(x,t)\dfrac{\partial u_n(x,t)}{\partial x}-3v_n(x,t)\dfrac{\partial w_n(x,t)}{\partial x}\\ -3w_n(x,t)\dfrac{\partial v_n(x,t)}{\partial x}-\dfrac{\partial^3 u_n(x,t)}{2\partial x^3}\end{array}\right\}(\mathrm{d}\tau)^{q_1}, \tag{28}$$

$$v_{n+1}(x,t)=v_n(x,t)-\frac{1}{\Gamma(q_2+1)}\int_0^t\left\{\frac{\partial^{q_2}v_n(x,\tau)}{\partial\tau^{q_2}}-3u_n(x,t)\frac{\partial v_n(x,t)}{\partial x}+\frac{\partial^3 v_n(x,t)}{\partial x^3}\right\}(\mathrm{d}\tau)^{q_2}, \tag{29}$$

$$w_{n+1}(x,t)=w_n(x,t)-\frac{1}{\Gamma(q_3+1)}\int_0^t\left\{\frac{\partial^{q_2}{}_n(x,\tau)}{\partial\tau^{q_2}}-3u_n(x,t)\frac{\partial w_n(x,t)}{\partial x}+\frac{\partial^3 w_n(x,t)}{\partial x^3}\right\}(\mathrm{d}\tau)^{q_3}, \tag{30}$$

Taking the initial value

$$\begin{aligned}
u_0(x,t)&=u_0(x,0)=\frac{\beta-2k^2}{3}+2k^2\tanh^2(kx),\\
v_0(x,t)&=v_0(x,0)=-\frac{4k^2c_0(\beta+k^2)}{3c_1^2}+\frac{4k^2(\beta+k^2)\tanh(kx)}{3c_1},\\
w_0(x,t)&=w_0(x,0)=w(x,0)=c_0+c_1\tanh(kx),
\end{aligned}$$

we can derive and so on, in the same manner the rest of the components of the iteration formulae (28–30) can be calculated by Maple. Then, the approximate solutions in a series form are

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\frac{\beta-2k^2}{3}+2k^2\tanh^2(kx)+\left[4k^3\beta\tanh(kx)\operatorname{sech}^2(kx)\right]\frac{t^{q_1}}{\Gamma(q_1+1)}+\ldots, \tag{31}$$

$$v(x,t)=\lim_{n\to\infty}v_n(x,t)=-\frac{4k^2c_0(\beta+k^2)}{3c_1^2}+\frac{4k^2(\beta+k^2)\tanh(kx)}{3c_1}+\left[\frac{4k^3\beta(\beta+k^2)\operatorname{sech}^2(kx)}{3c_1}\right]\frac{t^{q_2}}{\Gamma(q_2+1)}+\cdots, \tag{32}$$

$$w(x,t)=\lim_{n\to\infty}w_n(x,t)=c_0+c_1\tanh(kx)+\left[c_1k\beta\operatorname{sech}^2(kx)\right]\frac{t^{q_3}}{\Gamma(q_3+1)}+\cdots. \tag{33}$$

For the special case $q_1=q_2=q_3=1, c=-\beta$ is

$$u(x,t)=\frac{\beta-2k^2}{3}+2k^2\tanh^2\left(k(x-ct)\right), \tag{34}$$

$$v(x,t)=-\frac{4k^2c_0(\beta+k^2)}{3c_1^2}+\frac{4k^2(\beta+k^2)\tanh\left(k(x-ct)\right)}{3c_1}, \tag{35}$$

$$w(x,t)=c_0+c_1\tanh\left(k(x-ct)\right), \tag{36}$$

which is an exact solution to the coupled-KDV (1–3) equation.

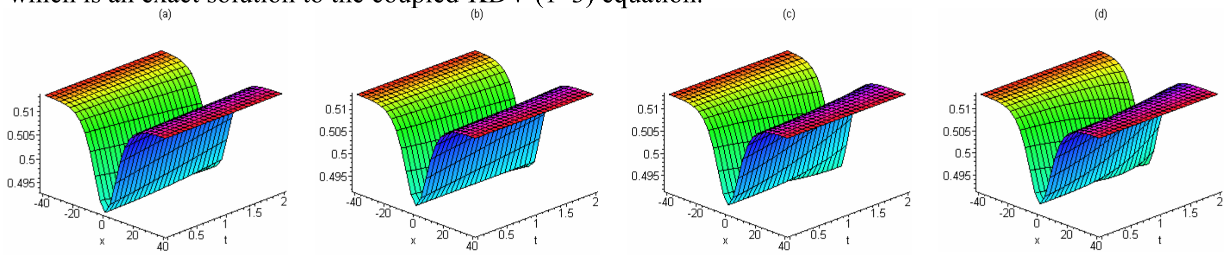

Fig. 1 – The FVIM results for $u(x,t)$ for the approximation, show: a) exact solution; b) $u_1(x,t)$; c) $u_2(x,t)$; d) $u_3(x,t)$, for $q_1 = q_2 = q_3 = 1$, when $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$, for the solitary wave solutions with the initial conditions (4) of Eq. (1), respectively.

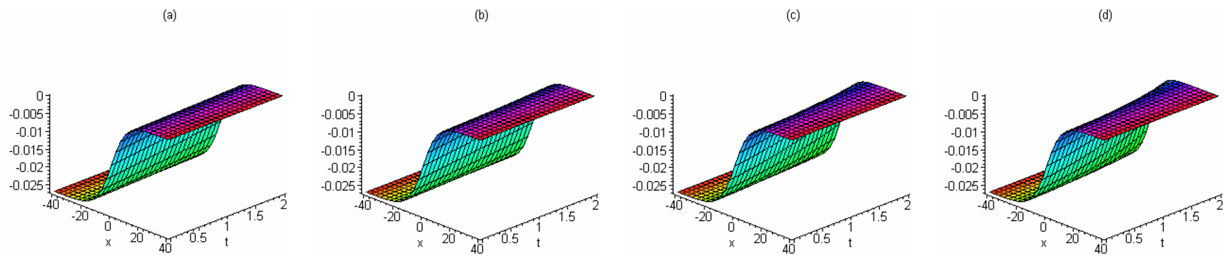

Fig. 2 – The FVIM results for $v(x,t)$ for the approximation, show: a) exact solution; b) $v_1(x,t)$; c) $v_2(x,t)$; d) $v_3(x,t)$, for $q_1 = q_2 = q_3 = 1$, when $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$, for the solitary wave solutions with the initial conditions (5) of Eq. (2), respectively.

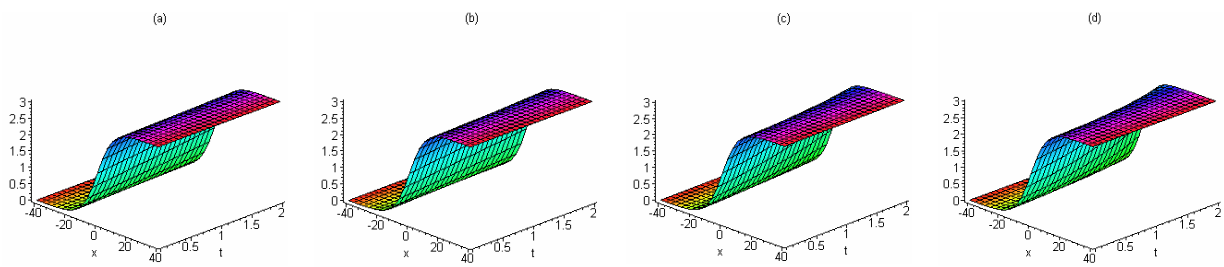

Fig. 3 – The FVIM results for $w(x,t)$ for the approximation, show a) exact solution; b) $w_1(x,t)$; c) $w_2(x,t)$; d) $w_3(x,t)$, for $q_1 = q_2 = q_3 = 1$, when $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$, for the solitary wave solutions with the initial conditions (6) of Eq. (3), respectively.

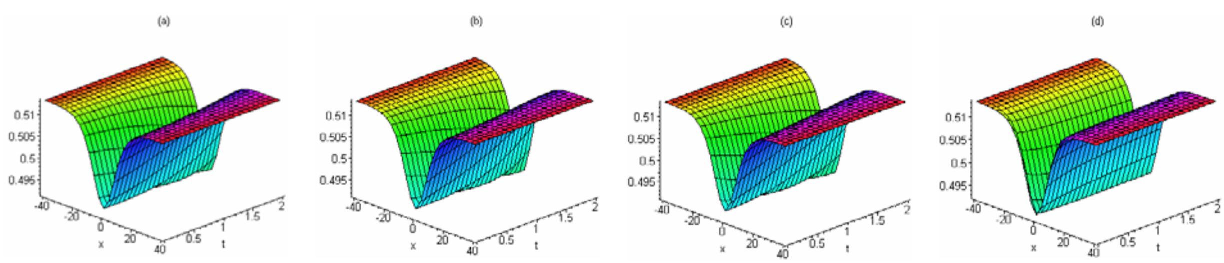

Fig. 4 – The surface indicates the solution $u(x,t)$ for Eq.(1), when $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$: a) approximate solution $u_3(x,t)$ for $q_1 = 0.9, q_2 = 0.8, q_3 = 0.7$; b) approximate solution $u_3(x,t)$ for $q_1 = 0.7, q_2 = 0.8, q_3 = 0.9$; c) approximate solution $u_3(x,t)$ for $q_1 = 0.6, q_2 = 0.6, q_3 = 0.6$; d) approximate solution $u_3(x,t)$ for $q_1 = 0.1$, $q_2 = 0.1$, $q_3 = 0.1$.

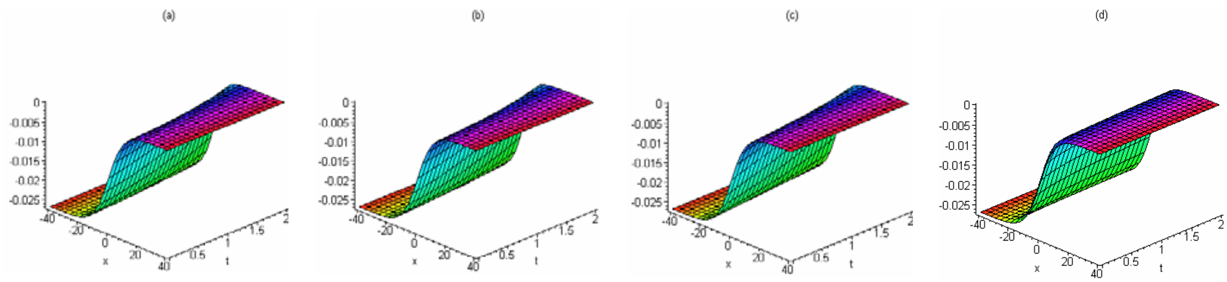

Fig. 5 – The surface indicates the solution $v(x,t)$ for Eq.(2), when $k=0.1, \beta=1.5, c_1=1.5, c_0=1.5$: a) approximate solution $v_3(x,t)$ for $q_1=0.9, q_2=0.8, q_3=0.7$; b) approximate solution $v_3(x,t)$ for $q_1=0.7, q_2=0.8, q_3=0.9$; c) approximate solution $v_3(x,t)$ for $q_1=0.6, q_2=0.6, q_3=0.6$; d) approximate solution $v_3(x,t)$ for $q_1=0.1, q_2=0.1, q_3=0.1$.

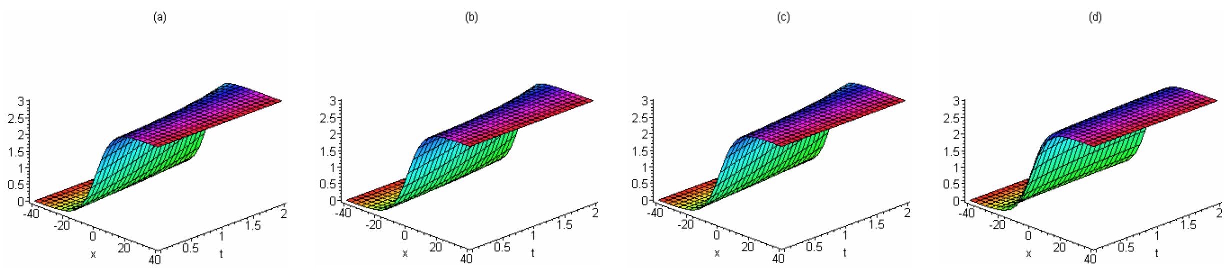

Fig. 6 – The surface indicates the solution $w(x,t)$ for Eq.(3), when $k=0.1, \beta=1.5, c_1=1.5, c_0=1.5$: a) approximate solution $w_3(x,t)$ for $q_1=0.9, q_2=0.8, q_3=0.7$; b) approximate solution $w_3(x,t)$ for $q_1=0.7, q_2=0.8, q_3=0.9$; c) approximate solution $w_3(x,t)$ for $q_1=0.6, q_2=0.6, q_3=0.6$; d) approximate solution $w_3(x,t)$ for $q_1=0.1, q_2=0.1, q_3=0.1$.

6. CONCLUSIONS

In this paper, the He's variational iteration method was used for finding soliton solutions of a generalized time fractional Hirota-Satsuma coupled KdV equation with modified Riemann-Liouville derivative. In this paper, we have discussed modified variational iteration method having integral w.r.t. $(\mathrm{d}\tau)^{\alpha}$ used for the first time successfully by Jumarie. The obtained results indicate that this method is powerful and meaningful for solving the nonlinear couple fractional differential equations. The solutions obtained by using FVIM having integral w.r.t. $(\mathrm{d}\tau)^{\alpha}$ are in agreement with those obtained by HPM[18] which is available in the literature.

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