THE QUASI-STATIC GENERALIZED EQUATION OF THE VIBRATIONS OF THE ELASTIC BARS WITH DISCONTINUITIES

Wilhelm W. KECS¹, Antonela TOMA²

¹ University of Petrosani ² "Politehnica" University of Bucharest E-mail: wwkecs@yahoo.com

In this paper the quasi-static equation in the distributions space $\mathcal{D}'(\mathbb{R}^2)$, is established with respect to the quantity EIv = rv, called deflection stiffness, for the vibrations of the elastic bars with geometrical, material and external discontinuities. This equation incorporates both the distributed loads and the concentrated ones, whether given or constraint, as well as the influence of the discontinuities due to the jumps of the quantity rv and its derivative. It is shown that the quasi-static equation also describes the bending of the elastic bar with discontinuities. Finally the quasi-static equation is given for a double-embedded bar with two material discontinuities and subjected to concentrated loads, varying in time.

Key words: distributions theory, rods with discontinuities, bar bending.

1. INTRODUCTION

In the study of the boundary-value problems regarding the bending and the transverse vibrations of the elastic bars difficulties are encountering due to the discontinuity points of the bar. These points of discontinuity can be divide into three classes, namely:

a) External points of discontinuity, which are the action points of the concentrated forces and moments;

b) Internal discontinuity points, which are the points in which the mechanical properties of the bar, i.e. the stiffness *EI* and the mass density ρ , change;

c) Geometric discontinuity points which are due to the joints.

In the study of the boundary-value problem occur the following physical quantities: $(EI)(x), \rho(x), \rho(x)$

 $v(x,t), T(x,t), M(x,t), (x,t) \in [a,b] \times \mathbb{R}_+$, representing the bar stiffness, the mass density, the deflection, the shear force and the bending moment, respectively.

These quantities and their derivatives of a certain order may have discontinuities of the first order at the points of discontinuity of the classes **a**, **b** or **c**. This means that, in a such point, for the physical quantity, the lateral limits exist and are finite. The discontinuity at a point of a physical quantity is expressed by using the jump of that quantity. Thus, if $\alpha \in (a,b)$ is a point of discontinuity of the first order for the physical quantity $f(x,t), (x,t) \in [a,b] \times \mathbb{R}_+$, then its jump at α is $[f(x,t)]_{\alpha} = f(\alpha + 0,t) - f(\alpha - 0,t)$, where $f(\alpha + 0,t) = \lim_{\substack{x \to \alpha \\ x > \alpha}} f(x,t), f(\alpha - 0,t) = \lim_{\substack{x \to \alpha \\ x < \alpha}} f(x,t)$ are the right and left limit at α , respectively.

We shall show that physical quantity with which can express the bar discontinuities is $(EIv)(x,t) = = (EI)(x)v(x,t), (x,t) \in [a,b] \times \mathbb{R}_+$. For this reason the quantity (*EIv*) we shall call the rigidity of the deflection v.

This quantity plays a similar role as the mechanical quantity mv, representing the impulse of a material point of mass m and speed v.

We note that the classical method of solving the boundary-value problem can be applied only to the portions of the bar where the loads, as well as the mechanical properties of the bar, have not discontinuities. This leads to the study of several equations (one equation for each segment) with certain conditions in the joints of the segments. This method proves to be laborious and generally does not allow writing a single equation that incorporates the influence of geometrical, material and external discontinuities.

These deficiencies can be overcome by using the distributions theory that erases the boundary between continuous and discrete. Thus, concentrated forces and moments can be represented in a unified form using Dirac distribution [1] i.e. $P\delta(x-a), m\delta'(x-\alpha)$. Thus, the problems of elasticity is approach in this manner in the papers: [2, 3, 4, 6].

In this paper by consistently using distribution theory, the quasi-static equation is established, in the distributions space $\mathcal{D}'(\mathbb{R}^2)$, for bars with discontinuities. Note that this equation incorporates both the distributed loads and the concentrated ones, whether given or constraint. Also, the equation highlights the influence of the discontinuities by the rigidity jumps (EIv)(x,t) of the deflection and its derivative.

These jumps can be interpreted as the charge densities due to the discontinuity points of the bar. It is shown that the quasi-static equation describes the bending of the elastic bars with discontinuities as a particular case.

2. GENERAL RESULTS. GENERALIZED EQUATION FOR QUASI-STATIC ELASTIC BARS WITH DISCONTINUITIES

We consider a straight homogeneous elastic rod of finite length $x \in [a,b]$ and the points $c_1 = a, c_2, ..., c_{i-1}, c_i, c_j, c_{i+1}, ..., c_n = b$ (Fig. 2.1). We suppose that the point $c_j \in (a,b)$ is an articulation point and that at the points $c_i, i = \overline{1,n}$ act concentrated forces and moments. These loads can be given or constraint, depending on the bar attachment. Consequently, the considered bar is a system of two bars $[a, c_j]$ and $[c_j, b]$ connected by the joint c_j . Some of the points $c_i \neq c_j$ can be attachment points of the system of two bars, by the supports and the embedding.





We denote by $P_i(t), m_i(t) \in L^1_{loc}(\mathbb{R}_+)$, locally integrable functions, representing the intensity of concentrated force and moment, which vary in time, and act at the point c_i , $i = \overline{1, n}$, $c_1 = a$, $c_n = b$. The forces P_i act perpendicular to bar in the vertical plane Oxv, and the moments m_i cause the bending of the bar in the same plane Oxv. We will also denote by $v(x,t), q(x,t), T(x,t), M(x,t), (x,t) \in [a,b] \times \mathbb{R}_+$, respectively, the deflection of the bar, the intensity of the distributed load, the shear force and the bending moment. We admit that the intensity q(x,t) of the distributed loads is continuous in each of the intervals

 $(c_i, c_{i+1}), i = \overline{1, n-1}$, having discontinuities of the first order at the ends of intervals. Denoting $J = \bigcup_{i=1}^{n-1} (c_i, c_{i+1})$,

means that $q(x,t) \in C^0(J \times \mathbb{R}_+)$, so q is integrable on [a,b] with respect to the variable $x \in [a,b]$. Let E_1, E_2 be the elasticity modulus on the segments $[a, c_j]$ and $[c_j, b]$, respectively, and I_1, I_2 the moments of inertia of the cross section to the neutral axis to the two segments. Consequently, the rigidity EI of the bar

on each of the two segments is constant, but has different values in general, i.e., $r_1 = E_1 I_1 = \text{const.}, x \in [a, c_j), r_2 = E_2 I_2 = \text{const.}, x \in [c_j, b]$. It follows that the bar rigidity r = EI is a function of $x \in [a, b]$ and has the expression $r(x) = (EI)(x) = \begin{cases} r_1 = E_1 I_1, & x \in [a, c_j), \\ r_2 = E_2 I_2, & x \in [c_j, b], \end{cases}$ having at the joint point

 $c_i \in (a,b)$ a discontinuity of the first order.

We note that the deflection $v(x,t), (x,t) \in [a,b] \times \mathbb{R}_+$ is a continuous function on $[a,b] \times \mathbb{R}_+$. As regards the ordinary derivative $\tilde{\partial}_x v(x,t)$, it is continuous on [a,b], except the point $c_j \in (a,b)$, where it has a discontinuity of the first order. This means that the rotation angles of the sections on either side of the joint, i.e. $\tilde{\partial}_x v(c_j + 0, t)$ and $\tilde{\partial}_x v(c_j - 0, t)$ will generally be different.

Therefore, the equation of the deformed axis of the bar will be different for the portions separated by joints. From the above, it follows that the rigidity r = EI of the bar has a discontinuity of the first order in the joint c_j , while the deflection v is continuous at c_j . Consequently, the quantity rv = EIv, which we call deflection rigidity, has a discontinuity of the first order in the joint c_j . Thus, for the deflection rigidity we have

$$(rv)(x,t) = r(x)v(x,t) = \begin{cases} r_1v_1(x,t) = E_1I_1v_1(x,t), & (x,t) \in [a,c_j) \times \mathbb{R}_+, \\ r_2v_2(x,t) = E_2I_2v_2(x,t), & (x,t) \in [c_j,b) \times \mathbb{R}_+, \end{cases}$$

where v_1 and v_2 are the bar deflections on the segments $[a, c_i]$ and $[c_i, b]$.

In connection with the considered functions we assume the assumptions

$$v(x,t) \in C^{4,2}(J \times \mathbb{R}_{+}) \cap C^{1,2}([a,c_{j}) \cup (c_{j},b] \times \mathbb{R}_{+}), q(x,t) \in C^{0}(J \times \mathbb{R}_{+}),$$

$$T(x,t) \in C^{1,0}(J \times \mathbb{R}_{+}), M(x,t) \in C^{2,0}(J \times \mathbb{R}_{+}), J = \bigcup_{i=1}^{n-1} (c_{i},c_{i+1}), c_{1} = a, c_{n} = b.$$
(2.1)

These assumptions, taking into account [6], [1], allow the writing of the complete system of equations of transverse vibrations of elastic bars [a,b], namely

$$\tilde{\partial}_{x}T(x,t) + q(x,t) - \rho\tilde{\partial}_{t}^{2}v(x,t) = 0, \quad T(x,t) = \tilde{\partial}_{x}M(x,t), \quad (x,t) \in J \times \mathbb{R}_{+}, \quad (2.2)$$

$$M(x,t) = -EI\tilde{\partial}_{x}^{2}v(x,t) = -\tilde{\partial}_{x}^{2}(EIv)(x,t),$$

where ρ is the mass density per unit length, and $\tilde{\partial}_x, \tilde{\partial}_t$ are the partial derivatives in the ordinary sense.

To these equations, valid for each of the intervals (c_i, c_{i+1}) , $i = \overline{1, n-1}$, we must add the appropriate boundary conditions of the ends intervals, as well as the initial conditions

$$v(x,t)\Big|_{t=+0} = u_0(x), \ \tilde{\partial}_t v(x,t)\Big|_{t=+0} = u_1(x), \ x \in [a,b].$$

An important case of equations (2.2) is that where the distributed loads and the concentrated $P_i(t), m_i(t)$ (forces and moments) ones, whether given or constraint, vary slowly with respect to time. Then, in the system (2.2) we can neglect the inertial force $\rho \tilde{\partial}_t^2 v(x,t)$ and thus we obtain the system of quasi-static equations

$$\tilde{\partial}_x T(x,t) + q(x,t) = 0, \ T(x,t) = \tilde{\partial}_x M(x,t), \ M(x,t) = -\tilde{\partial}_x^2 (rv)(x,t), \ (x,t) \in J \times \mathbb{R}_+.$$
(2.3)

These equations describe the quasi-static transverse vibrations of the elastic bars. We shall admit that the initial conditions corresponding to quasi-static equations (2.3) are zero.

To consider the quantities appearing in equations (2.3) as function type distributions from $\mathcal{D}'(\mathbb{R}^2)$, we will prolong them with null values outside their domain of definition $[a,b] \times \mathbb{R}_+$. We mention that $\mathcal{D}(\mathbb{R}^2)$ is

the test space of indefinitely differentiable functions with compact support and $\mathcal{D}'(\mathbb{R}^2)$, [1] the set of linear and continuous functionals defined on $\mathcal{D}(\mathbb{R}^2)$.

Let $\chi(x) = \begin{cases} 1, & x \in [a,b], \\ 0, & x \notin [a,b], \end{cases}$ be the characteristic function corresponding to the interval [a,b] and

 $H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \ge 0, \end{cases}$ Heaviside's function. We define the following function type distributions from $\mathcal{D}'(\mathbb{R}^2)$

$$\hat{v}(x,t) = \begin{cases} v(x,t), & (x,t) \in [a,b] \times \mathbb{R}_+ \\ 0, & \text{otherwise}, \end{cases} = \begin{cases} v_1(x,t), & (x,t) \in [a,c_j) \times \mathbb{R}_+ \\ v_2(x,t), & (x,t) \in [c_j,b] \times \mathbb{R}_+ \\ 0, & \text{otherwise} \end{cases}$$

$$(\widehat{rv})(x,t) = \begin{cases} r(x)v(x,t), & (x,t) \in [a,b] \times \mathbb{R}_+ \\ 0, & \text{otherwise} \end{cases} = \begin{cases} r_iv_1(x,t), & (x,t) \in [a,c_j) \times \mathbb{R}_+ \\ r_2v_2(x,t), & (x,t) \in [c_j,b] \times \mathbb{R}_+ \\ r_2v_2(x,t), & (x,t) \in [c_j,b] \times \mathbb{R}_+ \\ 0, & \text{otherwise} \end{cases} = \begin{cases} r_iv_1(x,t), & (x,t) \in [a,c_j) \times \mathbb{R}_+ \\ r_2v_2(x,t), & (x,t) \in [c_j,b] \times \mathbb{R}_+ \\ 0, & \text{otherwise} \end{cases}$$

$$\hat{q}(x,t) = q(x,t)\chi(x)H(t), \ \hat{T}(x,t) = T(x,t)\chi(x)H(t), \ \hat{M}(x,t) = M(x,t)\chi(x)H(t), \\ \hat{P}_i(t) = P_i(t)H(t), \ \hat{m}_i(t) = m_i(t)H(t). \end{cases}$$

$$(2.4)$$

The writing of these quantities using the functions $\chi(x)$ and H(t) is abbreviated and formal.

For these functions and their derivatives with respect to the variable $x \in \mathbb{R}$, the points c_i , $i = \overline{1, n}$ are, generally, discontinuity points of the first order. Thus, the point of action of a concentrated force is a point of discontinuity of the first order for shear force \hat{T} and for the derivative $\tilde{\partial}_x \hat{M}$ of the bending moment.

Also, the point of action of a concentrated moment is a point of discontinuity of the first order for the bending moment \hat{M} and ordinary point for the shear force \hat{T} . We note that the jumps of the shear force and of the bending moment at the points $c_i, i = \overline{1, n}$ have the expressions

$$\left[\hat{T}(x,t)\right]_{c} = \hat{T}(c_{i}+0,t) - \hat{T}(c_{i}-0,t) = -\hat{P}_{i}(t), \\ \left[\hat{M}(x,t)\right]_{c} = \hat{M}(c_{i}+0,t) - \hat{M}(c_{i}-0,t) = -\hat{m}_{i}(t).$$
(2.5)

As regards the deflection $\hat{v}(x,t)$, it is continuous with respect to the variable x, for $x \in [a,b]$. Instead, the points $c_1 = a$ and $c_n = b$ are points of discontinuity of the first order for \hat{v} and \hat{rv} , as well as for the derivatives $\tilde{\partial}_x \hat{v}, \tilde{\partial}_x (\hat{rv})$. We note that the joint point $c_j \in (a,b)$ is point of continuity for \hat{v} and point of discontinuity for $\hat{rv}, \tilde{\partial}_x \hat{v}, \tilde{\partial}_x (\hat{rv})$. For jumps of the quantities \hat{rv} and $\tilde{\partial}_x (\hat{rv})$ at the points a, b, c_j we have the expressions

$$\begin{split} \left[\widehat{rv} \right]_{a} &= (\widehat{rv})(a+0,t) - (\widehat{rv})(a-0,t) = r_{1}v_{1}(a+0,t)H(t); \\ \left[\widehat{rv} \right]_{b} &= (\widehat{rv})(b+0,t) - (\widehat{rv})(b-0,t) = -r_{2}v_{2}(b-0,t)H(t); \\ \left[\widetilde{\partial}_{x} \widehat{rv} \right]_{a} &= \widetilde{\partial}_{x} \widehat{rv}(a+0,t) - \widetilde{\partial}_{x} \widehat{rv}(a-0,t) = r_{1}\widetilde{\partial}_{x}v_{1}(a+0,t)H(t); \\ \left[\widetilde{\partial}_{x} \widehat{rv} \right]_{b} &= \widetilde{\partial}_{x} \widehat{rv}(b+0,t) - \widetilde{\partial}_{x} \widehat{rv}(b-0,t) = -r_{2}\widetilde{\partial}_{x}v_{1}(b-0,t)H(t); \\ \left[\widehat{rv} \right]_{c} &= (\widehat{rv})(c_{j}+0,t) - (\widehat{rv})(c_{j}-0,t) = r(c_{j}+0)\widehat{v}(c_{j}+0,t) - r(c_{j}-0)\widehat{v}(c_{j}-0,t) = (r_{2}-r_{1})v(c_{j},t)H(t) = \\ &= [r]_{c} v(c_{j},t)H(t), \text{ this relation takes place due to the deflection continuity }\widehat{v} \text{ in the joint } c_{j}; \\ \left[\widetilde{\partial}_{x} (\widehat{rv}) \right]_{c} &= \widetilde{\partial}_{x} (\widehat{rv})(c_{j}+0,t) - \widetilde{\partial}_{x} (\widehat{rv})(c_{j}-0,t) = (r_{2}\widetilde{\partial}_{x}v(c_{j}+0,t) - r_{1}\widetilde{\partial}_{x}v(c_{j}+0,t)) H(t) = \left[\widetilde{\partial}_{x} (rv) \right]_{c} H(t). \end{split}$$

The quasi-static equations (2.3), with respect to the quantities defined by (2.4) and (2.4'), are rewritten in the form

$$\hat{T}(x,t) + \hat{q}(x,t) = 0, \ \hat{T}(x,t) = \tilde{\partial}_x \hat{M}(x,t), \ \hat{M}(x,t) = -\tilde{\partial}_x^2 (\widehat{rv})(x,t), \ (x,t) \in J \times \mathbb{R}_+.$$
(2.7)

Next, we denote by ∂_x the derivative in the sense of distributions and with $\tilde{\partial}_x$ the derivative in the ordinary sense. The dependence [1], of the two derivatives is given by

PROPOSITION 2.1. If the function f is of class $C^1(\mathbb{R})$, except the points $x_i, i = \overline{1, p}$, where it has discontinuities of the first order with the jumps $[f]_x = f(x_i + 0) - f(x_i - 0)$, then

$$\partial_x f = \tilde{\partial}_x f + \sum_{i=1}^p [f]_x \,\delta(x - x_i),\tag{2.8}$$

where $\delta(x - x_i)$ is the Dirac delta distribution concentrated at the point x_i .

Applying this formula and taking into account (2.5) we obtain

$$\partial_x \hat{T} = \tilde{\partial}_x \hat{T} + \sum_{i=1}^n [\hat{T}]_c \times \delta(x - c_i) = \tilde{\partial}_x \hat{T} - \sum_{i=1}^n \hat{P}_i(t) \times \delta(x - c_i),$$
(2.9)

$$\partial_x \hat{M} = \tilde{\partial}_x \hat{M} + \sum_{i=1}^n [\hat{M}]_c \times \delta(x - c_i) = \tilde{\partial}_x \hat{M} - \sum_{i=1}^n \hat{m}_i(t) \times \delta(x - c_i), \qquad (2.10)$$

where \times is the direct product symbol v.

Analogously, applying the formula (2.8) for the deflection rigidity \hat{rv} we obtain

$$\partial_x(\widehat{rv}) = \tilde{\partial}_x(\widehat{rv}) + [\widehat{rv}]_a \times \delta(x-a) + [\widehat{rv}]_b \times \delta(x-b) + [\widehat{rv}]_c \times \delta(x-c_j) = \tilde{\partial}_x(\widehat{rv}) + \sum_{k=1,j,n} [\widehat{rv}]_c \times \delta(x-c_k), \quad (2.11)$$

$$\partial_x^2(\widehat{rv}) = \widetilde{\partial}_x^2(\widehat{rv}) + \sum_{k=1,j,n} [\widehat{rv}]_c \times \delta'(x - c_k) + \sum_{k=1,j,n} [\widetilde{\partial}_x \widehat{rv}]_c \times \delta(x - c_k),$$
(2.12)

where $c_1 = a$, $c_n = b$, $c_j \in (a,b)$ is the joint point of the bar, and the jumps of the rigidity deflection \hat{rv} and of the derivative $\tilde{\partial}_x \hat{rv}$ at the point a, b and c_j are given by expressions (2.6).

On the base of the formulas (2.9), (2.10), (2.12) and of the equations (2.7) we obtain

$$\partial_x \hat{T} + \hat{q} = -\sum_{i=1}^n \hat{P}_i(t) \times \delta(x - c_i), \ \partial_x \hat{M} - \hat{T} = -\sum_{i=1}^n \hat{m}_i(t) \times \delta(x - c_i),$$

$$\partial_x^2 \hat{rv} + \hat{M} = \sum_{k=1,j,n} [\hat{rv}]_c \times \delta'(x - c_k) + \sum_{k=1,j,n} [\tilde{\partial}_x \hat{rv}]_c \times \delta(x - c_k).$$
(2.13)

These equations represent the complete system of quasi-static equations of the transverse vibrations in the distributions space $\mathcal{D}'(\mathbb{R}^2)$, for elastic bars with zero initial conditions and with one joint.

Eliminating from these equations the quantities \hat{T} and \hat{M} , we obtain

$$\hat{\sigma}_{x}^{4}\widehat{rv}(x,t) = \hat{q} + \sum_{i=1}^{n} \hat{P}_{i}(t) \times \delta(x-c_{i}) + \sum_{i=1}^{n} \hat{m}_{i}(t) \times \delta'(x-c_{i}) + \sum_{k=1,j,n} [\widehat{rv}]_{c} \times \delta'''(x-c_{k}) + \sum_{k=1,j,n} [\tilde{\partial}_{x}\widehat{rv}]_{c} \times \delta''(x-c_{k}).$$
(2.14)

The equation (2.14) is the quasi-static equation in $\mathcal{D}'(\mathbb{R}^2)$, with respect to the deflection rigidity $\widehat{rv}(x,t)$, for the transverse vibrations of the elastic bars with zero initial conditions and with one joint. The

unknown of the equation is the deflection rigidity rv which expresses the change of the mechanical properties of the bar on the two segments $[a, c_i]$ and $[c_i, b], c_i \in (a, b)$.

The jumps of the unknown \hat{rv} and its derivative $\tilde{\partial}_x \hat{rv}$ at bar ends $c_1 = a, c_n = b$ and in the joint c_j are incorporated into the equation. These jumps, like the quantity \hat{q} , can be interpreted as charge density due to the discontinuity points a, b, c_j of the deflection rigidity \hat{rv} and its derivative. Also, the quasi-static equation (2.14) incorporates both the density \hat{q} of the distributed loads and the loads and moments densities, whether

given or constraint, by the terms $\sum_{i=1}^{n} \hat{P}_{i}(t) \times \delta(x-c_{i})$ and $\sum_{i=1}^{n} \hat{m}_{i}(t) \times \delta'(x-c_{i})$.

Consequently, the right side of equation (2.14), namely

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$$\hat{Q}(x,t) = \hat{q}(x,t) + \sum_{i=1}^{n} \hat{P}_{i}(t) \times \delta(x-c_{i}) + \sum_{i=1}^{n} \hat{m}_{i}(t) \times \delta'(x-c_{i}) + \sum_{k=1,j,n} [\hat{rv}]_{c} \times \delta'''(x-c_{k}) + \sum_{k=1,j,n} [\tilde{\partial}_{x}\hat{rv}]_{c} \times \delta''(x-c_{k})$$
(2.15)

is a load density which depends on the following factors: the density of the distributed loads \hat{q} ; the density of the concentrated forces $\sum_{i=1}^{n} \hat{P}_{i}(t) \times \delta(x-c_{i})$; the density of the concentrated moments $\sum_{i=1}^{n} \hat{m}_{i}(t) \times \delta'(x-c_{i})$; the jumps density of the deflection rigidity $\sum_{k=1,j,n} [\hat{rv}]_{c} \times \delta'''(x-c_{k})$ at the points of discontinuity $c_{1} = a, c_{n} = b, c_{j}$; the jumps density of the derivative of the deflection rigidity $\sum_{k=1,j,n} [\tilde{\partial}_{x} \hat{rv}]_{c} \times \delta''(x-c_{k})$ at the points of discontinuity $c_{1} = a, c_{n} = b, c_{j}$.

The justification that the quantity \hat{Q} should be interpreted as a charge density, results from its dimensional equation. Because the force F has the dimensional equation $[F] = MLT^{-2}$, it follows that the deflection rigidity \hat{rv} and the density have the dimensions

$$\left[\hat{rv}\right] = ML^{-1}T^{-2}L^{4}L = ML^{4}T^{-2}, \ \left[\hat{q}\right] = MT^{-2}.$$
 (2.16)

Because the Dirac delta distribution $\delta^{(p)}(x-c)$, p = 0, 1, 2, ... has the dimension $[\delta^{(p)}(x-c)] = L^{-p-1}$, we have the following dimensional equations

$$\begin{bmatrix} \hat{P}_{i}(t) \times \delta(x-c_{i}) \end{bmatrix} = \mathrm{MLT}^{-2} \cdot \mathrm{L}^{-1} = \mathrm{MT}^{-2}, \quad \left[\hat{m}_{i}(t) \times \delta'(x-c_{i}) \right] = \mathrm{ML}^{2}\mathrm{T}^{-2} \cdot \mathrm{L}^{-2} = \mathrm{MT}^{-2}, \\ \begin{bmatrix} \widehat{rv} \end{bmatrix}_{c} \times \delta'''(x-c_{i}) \end{bmatrix} = \mathrm{ML}^{4}\mathrm{T}^{-2} \cdot \mathrm{L}^{-4} = \mathrm{MT}^{-2}, \quad \left[\left[\tilde{\partial}_{x} \widehat{rv} \right]_{c} \times \delta''(x-c_{i}) \right] = \frac{\mathrm{ML}^{4}\mathrm{T}^{-2}}{\mathrm{L}} \cdot \mathrm{L}^{-3} = \mathrm{MT}^{-2}. \quad (2.17)$$

From (2.16) and (2.17) it follows that the quantity \hat{Q} given by (2.15), representing the right side of the equation (2.14), has the dimension of a load density.

As a result, the quasi-static equation (2.14) can be written in compact form in the distributions space $\mathcal{D}'(\mathbb{R}^2)$, thus

$$\partial_x^4 \widehat{rv}(x,t) = \hat{Q}(x,t). \tag{2.18}$$

PROPOSITION 2.2. Let be the function type distribution $\widehat{rv}(x,t) \in \mathcal{D}'(\mathbb{R}^2)$, called the rigidity of the deflection $\hat{v} \in \mathcal{D}'(\mathbb{R}^2)$. Then, the quasi-static equation, in the distributions space $\mathcal{D}'(\mathbb{R}^2)$, of the boundary-value problems regarding the elastic bars with one joint is given by (2.18). The quantity $\hat{Q} \in \mathcal{D}'(\mathbb{R}^2)$ given by (2.15) is the resultant of the distributed forces densities, of the concentrated forces and moments as well

as of the densities jumps of \hat{rv} and $\hat{\partial}_x \hat{rv}$ corresponding to the discontinuity points $c_1 = a, c_n = b$ and to the joint $c_i \in (a,b)$.

Taking into account (2.6) we have $[\widehat{rv}]_c = [r]_c v(c_j, t), [\tilde{\partial}_x \widehat{rv}]_c = [r]_c \tilde{\partial}_x v(c_j, t) H(t)$, because $\tilde{\partial}_x v(c_j + 0, t) = \tilde{\partial}_x v(c_j - 0, t)$.

If c_j is the joint point but is not a material discontinuity point, then according to (2.6) we have

$$[\widehat{rv}]_{c} = [r]_{c} v(c_{j}, t) H(t) = 0, [\widetilde{\partial}_{x} \widehat{rv}]_{c} = r(c_{j}) [\widetilde{\partial}_{x} v]_{c} H(t).$$
(2.19)

Generalization. The quasi-static equation (2.18) written for a single point of discontinuity (geometric and material) can be generalized to any number of points of discontinuity.

Let $c_{\alpha}, c_{\beta}, ..., c_{p}, c_{q} \in (a, b)$ be points of geometric and material discontinuities. In this case the quasistatic equation (2.18) takes the form

$$\partial_x^4 \widehat{rv}(x,t) = \hat{Q}^*(x,t), \qquad (2.20)$$

where the load density $\hat{Q}^*(x,t)$ has the expression

$$\hat{Q}^{*}(x,t) = \hat{q}(x,t) + \sum_{i=1}^{n} \hat{P}_{i}(t) \times \delta(x-c_{i}) + \sum_{i=1}^{n} \hat{m}_{i}(t) \times \delta'(x-c_{i}) + \sum_{k=1,\alpha,\beta,\dots,p,q,n} \left([\widehat{rv}]_{c} \times \delta'''(x-c_{k}) + [\tilde{\partial}_{x}\widehat{rv}]_{c} \times \delta''(x-c_{k}) \right).$$

$$(2.21)$$

It follows that, the new charge density \hat{Q}^* is obtained from the density \hat{Q} , to which are added the densities of the jumps of the quantities \hat{rv} and $\hat{\partial}_x \hat{rv}$, corresponding to the discontinuity points $c_{\alpha}, c_{\beta}, ..., c_p, c_q \in (a, b)$.

As regards the rigidity r = EI of the bar and the rigidity $\hat{rv}(x,t)$ of the deflection we have the expressions

$$r(x) = \begin{cases} r_{\alpha}, \ x \in [a, c_{\alpha}), \\ r_{\beta}, \ x \in [c_{\alpha}, c_{\beta}), \\ \dots, \\ r_{q}, \ x \in [c_{p}, c_{q}), \\ r_{n}, \ x \in [c_{q}, b], \end{cases} \qquad \widehat{rv}(x,t) = \begin{cases} r_{\alpha}v_{\alpha}(x,t), \quad (x,t) \in [a, c_{\alpha}) \times \mathbb{R}_{+}, \\ r_{\beta}v_{\beta}(x,t), \quad (x,t) \in [c_{\alpha}, c_{\beta}) \times \mathbb{R}_{+}, \\ \dots, \\ r_{q}v_{q}(x,t), \quad (x,t) \in [c_{p}, c_{q}) \times \mathbb{R}_{+}, \\ r_{n}v_{n}(x,t), \quad (x,t) \in [c_{q}, b] \times \mathbb{R}_{+}, \\ 0, \qquad \text{otherwise.} \end{cases}$$
(2.22)

The quantities \hat{rv} and $\hat{\partial}_x \hat{rv}$ have discontinuities of the first order at the points $c_1 = a, c_\alpha, c_\beta, ..., c_p, c_q, b = c_n$.

We note that the quasi-static equation (2.20) contains as a particular case the bending equation of the elastic bars with geometric and material discontinuities. For this, it is sufficient for the loads $\hat{q}(x,t), P_i(t)$ and

 $m_i(t)$ to be taken under the form $\hat{q}(x,t) = \hat{q}_0(x)H(t) = \begin{cases} q_0(x), & (x,t) \in [a,b] \times \mathbb{R}_+, \\ 0, & \text{othewise,} \end{cases}$, $P_i(t) = P_i^0 = \text{const.},$

 $m_i(t) = m_i^0 = \text{const.}$ Consequently, the deflection $\hat{v}(x,t)$ and the rigidity rv(x,t) of the deflection \hat{v} have the expressions

$$\hat{v}(x,t) = \hat{v}_0(x)H(t), \ \hat{v}_0(x) = \begin{cases} v_0(x), & x \in [a,b], \\ 0, & \text{otherwise,} \end{cases} \quad \widehat{rv}(x,t) = \widehat{rv_0}(x)H(t) = \begin{cases} r(x)v_0(x), & (x,t) \in [a,b] \times \mathbb{R}_+, \\ 0, & \text{othewise,} \end{cases}$$
(2.23)

where $v_0(x), x \in [a,b]$ is the deflection of the elastic bar produced by the action of the static loads $q_0(x)$, $x \in [a,b], q_0 \in C^0(J), P_i^0, m_i^0, J = \bigcup_{i=1}^{n-1} (c_i, c_{i+1}), c_1 = a, c_n = b$. For the jumps $[\widehat{rv}]_c$ and $[\tilde{\partial}_x \widehat{rv}]_c$ we have the expressions $[\widehat{rv}]_c = [\widehat{rv}_0(x)]_c \times H(t), [\tilde{\partial}_x \widehat{rv}]_c = [\tilde{\partial}_x \widehat{rv}_0(x)]_c \times H(t)$. Taking into account (2.20) and (2.21) we obtain

$$\hat{\partial}_x^4 \widehat{rv_0}(x) \times H(t) = \hat{Q}_0(x) \times H(t), \qquad (2.24)$$

where

$$\hat{Q}_0(x) = \hat{q}_0(x) + \sum_{i=1}^n P_i^0 \delta(x - c_i) + \sum_{i=1}^n m_i^0 \delta'(x - c_i) + \sum_{i=1,\alpha,\beta,\dots,p,q,n} \left(\left[\widehat{rv_0} \right]_c \delta''(x - c_i) + \left[\tilde{\partial}_x \widehat{rv_0} \right]_c \delta''(x - c_i) \right).$$
(2.25)

From (2.24) it follows

$$\partial_x^4 \widehat{rv_0}(x) = \hat{Q}_0(x).$$
 (2.26)

This equation is the generalized equation, in the distributions space $\mathcal{D}'(\mathbb{R})$, of the bending of the elastic bars with geometric and material discontinuities.

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3. CONCLUSIONS

By defining the quantity \hat{rv} , called the rigidity of the deflection \hat{v} , the quasi-static equation written in distributions space $\mathcal{D}'(\mathbb{R}^2)$ has a uniform and general form. It is distinguished by the feature of incorporating all types of loads that can act on the bar, as well as all the material, geometric and external discontinuities of the elastic bar. This concise writing of the quasi-static equation allows us to obtain general solution of the boundary value problem using the notion of fundamental solution of linear operators.

We note that the traditional approach to boundary problems can be applied only to portions of the bar where the loads and the mechanical properties of the bar have no discontinuities. In this way we obtain equations for each segment of the bar, with some joint conditions. This mode does not allow writing a single equation that incorporates the influence of geometrical, material and external discontinuities of the bar.

These show the effectiveness of the method using the distribution theory and interest in writing the quasi-static equation in a compact form in the space $\mathcal{D}'(\mathbb{R}^2)$.

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