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HYSTERETIC TYPE BEHAVIORS OF THE COMPLEX FLUIDS *VIA* NON-DIFFERENTIABILITY

Daniel TIMOFTE [1], Lăcrămioara OCHIUZ [2], Lucian EVA [3], Evelina MORARU [4], Bogdan STANA [4], Maricel AGOP [5,6], Decebal VASINCU [1]

[1] "Gr. T. Popa" University of Medicine and Pharmacy, Surgery Department, "Sf. Spiridon" Hospital, University str. no. 16, 700115, Iaşi, Romania
[2] "Gr. T. Popa" University of Medicine and Pharmacy, Faculty of Pharmacy, Department of Pharmaceutical Technology, University str. no 16, 700115, Iaşi, Romania
[3] "Prof. Dr. Nicolae Oblu" Emergency Clinical Hospital, Ateneului 2, Iaşi - 700309, Romania
[4] "Gr. T. Popa" University of Medicine and Pharmacy, Second Pedriatic Clinic, Hospital "Sf. Maria", University str. no 16, 700115, Iaşi, Romania
[5] University of Science and Technology, Lasers, Atoms and Molecules Physics Laboratory, Villeneuve d'Ascq, 59655, Lille, France
[6] "Gh. Asachi" Technical University, Department of Physics, D. Mangeron str. no. 67, Iaşi 700050, Romania
Corresponding author: Maricel AGOP, E-mail: m.agop@yahoo.com

In a non-standard Scale Relativity approach, the specific momentum, states density and internal energy conservations laws are obtained. Then, the chaoticity, either through turbulence in the fractal hydrodynamics approach, or through stochasticization in the Schrödinger type approach, is generated only by the non-differentiability of the movement trajectories of the complex fluid entities. Eliminating the time between normalized internal stress tensor and normalized internal energy for various given positions, by numerical simulations using the conservation laws mentioned above, hysteretic type behaviours (hysteresis type cycle) occur.

Key words: complex fluid, non-standard Scale Relativity Theory, non-differentiability, hysteretic type behaviours.

1. INTRODUCTION

A great variety of materials is categorized as complex fluids: polymers (elastomers, thermoplastics, composites), colloidal fluids, biological fluids (DNA, proteins, cells, dispersions of biopolymers and cells, human blood), foams, suspensions, emulsions, gels, micelar and liquid-crystal phases, molten materials, etc. Therefore, fluids with non-linear viscous behaviors, as well as viscoelastic materials are complex fluids [1, 2].

Particle dynamics in complex fluids is highly nonlinear. For example, the formation of amorphous solids (glasses, granular or colloids) do not comply to the physical mechanism explaining solids crystallization. So, in amorphous solids, either lowering the temperature or increasing the density, the dynamic process achieves a level where the system cannot totally relax and therefore becomes rigid. This phenomenon is known as *glass transition* (when the temperature lowers) or *jamming transition* (when density increases) [3, 4]. Also, the stress of a viscoelastic fluid, unlike the Newtonian fluid, depends not only on the actually stress applied, but on the one applied during previous deformation of the fluid [5].

Since the theoretical models which describe the complex fluids dynamics are sophisticated [1, 2, 5], in recent papers [6], a new topic was developed using, either the standard Scale Relativity Theory (SRT) [7], or the non-standard Scale Relativity Theory [8–20]. According to previous references, the dynamics of complex fluids entities take place on continuous but non-differentiable curves (fractal curves), so that all physical phenomena involved depend on space-time coordinates and on space-time scale resolution, the complex fluid entities may be reduced to and identified with their own trajectories, the complex fluids should behave as a special "fluid" lacking interactions (fractal fluid) etc.

In the present paper, using the non-Standard Scale Relativity Theory, the hysteretic type behaviours of the complex fluids are analysed.

2. MOTION EQUATION VIA NON-DIFFERENTIABILITY

We can simplify the dynamics of a complex fluid supposing that complex fluid entities move on continuous but non-differentiable curves, i.e. fractal curves (for example, the Peano curve, the Koch curve or the Weierstrass curve [7, 21]).

Once accepted such a hypothesis, the dynamics of the complex fluid are given by the fractal operator $\hat{\mathrm{d}}/\mathrm{d}t$ [13]:

$$\frac{\hat{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\hat{\boldsymbol{V}}\cdot\nabla-\mathrm{i}\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\Delta\,, \tag{1}$$

where

$$\hat{\boldsymbol{V}}=\boldsymbol{V}_D-\mathrm{i}\boldsymbol{V}_F \tag{2}$$

is the complex velocity, $\boldsymbol{V}_D$ is the differentiable and resolution scale independent velocity, $\boldsymbol{V}_F$ is the non-differentiable and resolution scale dependent velocity, $\hat{\boldsymbol{V}}\cdot\nabla$ is the convective term, $\lambda^2/\tau\left(\mathrm{d}t/\tau\right)^{(2/D_F)-1}\Delta$ is the dissipative term, Δ is the Laplacian operator, D_F is the fractal dimension of the movement curve, λ is the space scale, τ is the time scale and λ^2/τ is a specific coefficient associated to the fractal-non-fractal transition. For D_F any definition can be used (the Hausdorff-Besikovici fractal dimension, the Kolmogorov fractal dimension etc. [21]) but once accepted such a definition for D_F, it has to be constant over the entire analysis of the complex fluid dynamics. In a particular case, for motions on Peano curves, $D_F = 2$ [21] of the complex fluid entities, the fractal operator (1) reduces to Nottale's standard operator [7].

Applying the fractal operator (1) to the complex speed (2) and accepting the principle of scale covariance [7] in the form:

$$\frac{\hat{\mathrm{d}}\hat{\boldsymbol{V}}}{\mathrm{d}t}=-\nabla U\,, \tag{3}$$

we obtain the motion equation:

$$\frac{\hat{\mathrm{d}}\hat{\boldsymbol{V}}}{\mathrm{d}t}=\frac{\partial\hat{\boldsymbol{V}}}{\partial t}+\left(\hat{\boldsymbol{V}}\cdot\nabla\right)\hat{\boldsymbol{V}}-\mathrm{i}\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\Delta\hat{\boldsymbol{V}}=-\nabla U\,, \tag{4}$$

where U is an external scalar potential. Equation (4) is a Navier – Stokes type equation. It means that at any point of a fractal path, the local acceleration term, $\partial_t\hat{\boldsymbol{V}}$, the non-linearly (convective) term, $\left(\hat{\boldsymbol{V}}\cdot\nabla\right)\hat{\boldsymbol{V}}$, the dissipative term, $\left(\lambda^2/\tau\right)\left(\mathrm{d}t/\tau\right)^{(2/D_F)-1}\Delta\hat{\boldsymbol{V}}$, and the external free term ∇U make their balance. Therefore, the complex fluid is assimilated to a "rheological" fluid, whose dynamics are described by the complex velocities field, $\hat{\boldsymbol{V}}$ and by the imaginary viscosity type coefficient, $\mathrm{i}\left(\lambda^2/\tau\right)\left(\mathrm{d}t/\tau\right)^{(2/D_F)-1}$. The "rheology" of the fluid can provide hysteretic type properties to the complex fluid.

3. CHAOTICITY THROUGH TURBULENCE AND STOCHATICIZATION *VIA* NON-DIFFERENTIABILITY

For irrotational motions of the complex fluid entities

$$\nabla\times\hat{\boldsymbol{V}}=0,\quad \nabla\times\boldsymbol{V}_D=0,\quad \nabla\times\boldsymbol{V}_F=0 \tag{5}$$

we can choose $\hat{V}$ of the form

$$\hat{\boldsymbol{V}}=-\mathrm{i}\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\nabla\ln\psi\,, \tag{6}$$

where $\phi\equiv\ln\psi$ is the velocity scalar potential. By substituting (6) in (4) and using the method described in [8, 9], it results

$$\frac{\mathrm{d}\hat{\boldsymbol{V}}}{\mathrm{d}t}=-\mathrm{i}\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\nabla\left[\frac{\partial\ln\psi}{\partial t}-\mathrm{i}\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\frac{\nabla\psi}{\psi}+U\right]=0\,. \tag{7}$$

This equation can be integrated, up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ and yields

$$\frac{\lambda^4}{\tau^2}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{4}{D_F}\right)-2}\Delta\psi+\mathrm{i}\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\frac{\partial\psi}{\partial t}-\frac{U}{2}\psi=0\,. \tag{8}$$

Relation (8) is a Schrödinger type equation. For motions on Peano curves [21], D_F = 2 [21] at a Compton scale $\lambda^2/\tau=\hbar/2m_0$ [7], with $\hbar$ the reduced Planck constant and m_0 the rest mass of the complex fluid entities, the relation (8) becomes the standard Schrödinger equation.

If $\psi=\sqrt{\rho}\mathrm{e}^{\mathrm{i}S}$, with $\sqrt{\rho}$ the amplitude and S the phase of ψ, the complex velocity field (6) takes the explicit form:

$$\hat{\boldsymbol{V}}=\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\nabla S-\mathrm{i}\frac{\lambda^2}{2\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\nabla\ln\rho\,. \tag{9}$$

By substituting (9) in (4) and separating the real and the imaginary parts, up to an arbitrary phase factor which may be set at zero by a suitable choice of the phase of ψ, we obtain:

$$\frac{\partial\boldsymbol{V}_D}{\partial t}+\left(\boldsymbol{V}_D\cdot\nabla\right)\boldsymbol{V}_D=-\nabla\left(Q+U\right),\quad \frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\boldsymbol{V}_D\right)=0\,, \tag{10}$$

with Q the specific fractal potential

$$Q=-2\frac{\lambda^4}{\tau^2}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{4}{D_F}\right)-2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}=-\frac{\boldsymbol{V}_F^2}{2}-\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1}\nabla\cdot\boldsymbol{V}_F\,. \tag{11}$$

The first equation (10) represents the specific momentum conservation law, while the second equation (10) represents the states density conservation law. Through the fractal velocity, $\boldsymbol{V}_F$, the specific fractal potential Q is a measure of non-differentiability of the complex fluid entities trajectories, *i.e.* of their chaoticity. The equations (10 and (11) define the fractal hydrodynamics model (FHM). In such a context, the complex fluid is assimilated to a fractal fluid.

Now, certain conclusions are evident: i) For motions on Peano curves at Compton scale [7], the FHM reduces to a quantum hydrodynamic model (QHM). The fractal velocity $\boldsymbol{V}_F$ does not represent actual mechanical motion, but it contributes to the transfer of specific momentum and the concentration of energy.

This may be seen clearly from the absence of $\boldsymbol{V}_F$ from the states density conservation law, and from its role in the variational principle. Any interpretation of Q should take cognizance of the "self" or internal nature of the specific momentum transfer. While the energy is stored in the form of the mass motion and potential energy (as it is classically), some is available elsewhere and only the total is conserved. It is the conservation of energy and specific momentum that ensures reversibility and the existence of eigenstates, but denies a Brownian motion [21] form of interaction with an external medium; ii) For Peano curves motions , at spatial scales higher than the dimension of the boundary layer and at temporal scales higher than the oscillation periods of the pulsating velocities which overlaps the average velocity of the complex fluid motions, the FHM reduces to the standard hydrodynamical model [22]. iii) Since the position vector of the complex fluid entity is assimilated with a stochastic process of Wiener type [7,21], ψ is not only the scalar potential of a complex velocity (through $\phi \equiv \ln\psi$) in the frame of FHM, but also states density (through $|\psi|^2$) in the frame of a Schrödinger type model. It results the equivalence between the formalism of the FHM and the one of Schrödinger type. Moreover, the chaoticity, either through turbulence in the fractal hydrodynamics approach, or through stochasticization in the Schrödinger type approach, is generated only by the non – differentiability of the movement trajectories in a fractal space; iv) In the standard model (Landau's scenario [22]) the Fourier spectrum is always discrete and cannot approximate a continuum spectrum than in case of a large number of frequencies that will generate a unlimited number of spectral components as a result of their beats which appear thanks to the presence of nonlinearities in the complex fluid. Yet, considering standard model, the flow can never be truly chaotic because, in case of multiple periodic functions, correlations tend to be not null, but having an oscillating character. Therefore, Landau's scenario can describe transition towards chaotic behavior only in a complex fluid with an infinite number of degrees of freedom. In our case, when $\delta t/\tau \to 0$ for $D_F \neq 2$ the physical quantities that describe the dynamics of the complex fluid are no longer defined. So, in this approximation, a simulation of a system with an infinite number of degrees of freedom is used. Moreover, dynamic states could be generated, characterized by windows of regular oscillations interrupted by chaotic bursts, the transition between the two states being spontaneous, unpredictable and independent of any of the control parameters variation (turbulence through intermittency).

4. HYSTERETIC TYPE BEHAVIOURS *VIA* NON-DIFFERENTIABILITY

Applying the fractal operator (1) to the internal energy per unit volume, ε, and adopting the principle of scale covariance[7], we obtain the internal energy per unit volume conservation law:

$$\frac{\hat{\mathrm{d}}(\rho\varepsilon)}{\mathrm{d}t} = \frac{\partial(\rho\varepsilon)}{\partial t} + \left(\hat{\boldsymbol{V}} \cdot \nabla\right)(\rho\varepsilon) - \mathrm{i}\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_{F\partial}}\right)-1} \Delta(\rho\varepsilon) = -\nabla U . \tag{12}$$

For the types of movements mentioned above, separating the real part from the imaginary one in equation (16), we shall obtain:

$$\frac{\partial(\rho\varepsilon)}{\partial t} + \nabla \cdot (\rho\varepsilon \boldsymbol{V}_D) = (\rho\varepsilon)\nabla \cdot \boldsymbol{V}_D - \nabla U \quad , \quad \boldsymbol{V}_F \cdot \nabla(\rho\varepsilon) = -\frac{\lambda^2}{\tau}\left(\frac{\mathrm{d}t}{\tau}\right)^{\left(\frac{2}{D_F}\right)-1} \Delta(\rho\varepsilon) . \tag{13}$$

One can notice that, although there is internal energy per unit volume transport at differentiable scale, a similar phenomenon (convection transport at fractal scale – see the second equation (13)) occurs.

For a plane symmetry, in normalized coordinates, the equations (10), (11) and the first equation (13) were numerically integrated with adequate initial and boundary conditions (for details see [19, 20, 23]) via finite differences [24]. By means of numerical solutions in Figs. 1a–f the contour curves of the normalized states density N(a), normalized internal energy per unit volume Θ (b), normalized velocity V_ξ (c), normalized velocity V_η (d) , normalized current density $J = N\left(V_\xi^2 + V_\eta^2\right)^{1/2}$ (e) and diagonal component of the normalized internal stress tensor type $\bar{\sigma} = N\left(V_\xi^2 + V_\eta^2\right)$ (f) on the normalized spatial coordinates (ξ, η) at the normalized times = 0.65 for N_0 = 1 and Θ_0 = 1 are plotted. The following results are obtained: i) The generation of structures in complex fluid by means of solitons packet solutions [25] (see the pronounced contours from Figs. 1a–f; ii) The normalized velocity V_ξ (which is normal to the “complex fluid streamline”) is symmetric with respect to the symmetry axis of the space-time Gaussian, while vertices are induced at the periphery of the structures of the normalized velocity V_η (which is along the “complex fluid streamline”); iii) Potential movement couplings at fractal scale as well as the potential one at differentiable scale are performed through the internal stress tensor type. As a result, the complex fluid entity acquires additional kinetic energy (induced by non-differentiability) that allows jumps from its own “stream line” to another; iv) Eliminating the time between the diagonal component of the normalized internal stress tensor type and normalized internal energy per unit volume for various given positions, by numerical simulations one can obtain hysteresis type effects. For the same ξ, such a tendency is more emphasized for small η (Fig. 2a – hysteresis type cycle), while for bigger η it vanishes (Fig. 2b – absence of hysteresis type cycle).

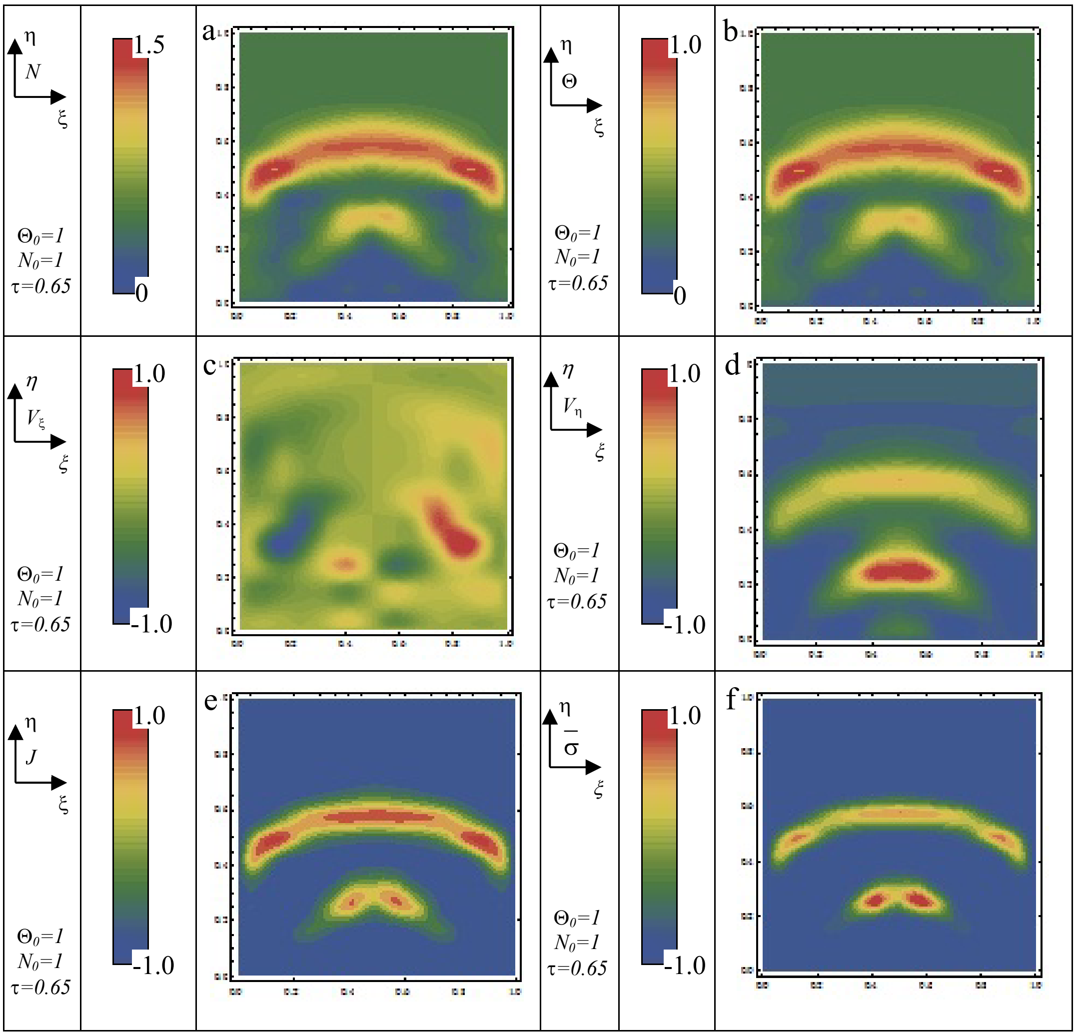

Fig. 1 – Contours curves of the normalized states density N (a), normalized internal energy Θ (b), normalized velocity V_ξ (c), normalized velocity V_η (d), normalized current density J (e) and diagonal component of the normalized internal stress tensor type $\bar{\sigma}$ (f) on the normalized times $\tau = 0.65$ for $N_0 = 1$ and $\Theta_0 = 1$.

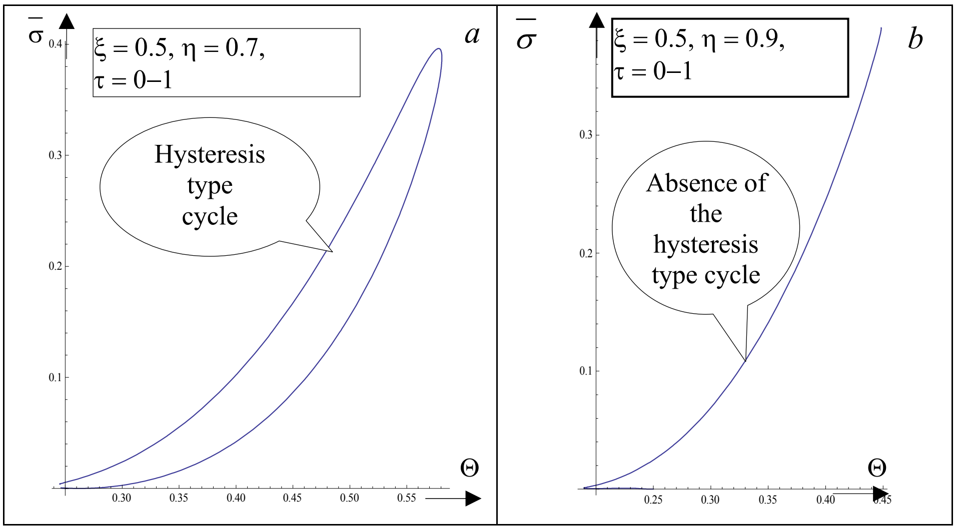

Fig. 2 – The dependence of the diagonal component of the normalized internal stress tensor type $\bar{\sigma}$, on the normalized internal energy, Θ for $\xi = 0.5$, $\eta = 0.7$, $\tau = 0\text{-}1$ (hysteresis cycle (a)) and for $\xi = 0.5$, $\eta = 0.9$, $\tau = 0\text{-}1$ (absence of hysteresis cycle (b)).

5. CONCLUSIONS

The main conclusions of the present paper are as follows:

i) Assuming that the particle movements of a complex fluid occur on continuous but non-differentiable curves, the specific momentum, states density and internal energy conservation laws are obtained.

ii) For irrotational motion the chaoticity, either through turbulence in the fractal hydrodynamics approach, or through stochasticization in the Schrodinger type approach, is generated only by the non-differentiability of the movement's trajectories of the complex fluid entities. In such a context, for Peano curves motions at a Compton scale the FHM reduces to QHM. Moreover, for Peano curves motions at spatial scales higher than the dimension of the boundary layer and at temporal scales higher than the oscillation periods of the pulsating velocities which overlaps the average velocity of the complex fluid motions, the fractal hydrodynamic model reduces to the standard hydrodynamical model;

iii) By numerical simulations using the FHM and internal energy per unit volume conservation law, the generation of structures in complex fluid by means of solitons packet solutions, the symmetry of the velocity field with respect of symmetry axis of a space-time Gaussian, vertices at the structures periphery of the velocity field are obtained;

iv) Potential movement coupling at fractal scale as well as the potential one at differentiable scale is performed through the internal stress tensor type. As a result, the complex fluid entity acquires additional kinetic energy that allows jumps from its own "stream line" to another;

v) By the same numerical simulations, eliminating the time between internal stress tensor and internal energy, for various given positions, one can obtain hysteresis type effects.

REFERENCES

1. THOMAS Y. Hou., *Multi-scale phenomena in complex fluids: modeling, analysis and numerical simulations*, World Scientific Publishing Company, Singapore, 2009.
2. MITCHELL O. D., THOMAS B. G., *Mathematical modeling for complex fluids and flows*, Springer, Berlin 2012.
3. BIROLI G. (2007), *Jamming: A new kind of phase transition?*, Nat. Phys., **3**, *4*, pp. 222–223, 2012.
4. TRAPPE V., PRASAD V., CIPELETTI L., SEGRE P. N., WEITZ D. A., *Jamming phase diagram for attractive particles*, Nature, **411**, *6839*, pp. 772–775, 2001.
5. CHEN S., DOOLEN G. D., *Lattice Boltzmann method for fluid flows*, Annual Review of Fluid Machanics **30**, *1*, pp. 329–364, 1998.
6. NEDEFF V., BEJENARIU C., LAZAR G., AGOP M., *Generalized lift force for complex fluid*, Powder Technology, **235**, pp. 685–695, 2013.
7. NOTALLE L., *Scale relativity and fractal space-time – a new approach to unifying relativity and quantum mechanics*, Imperial College Press, London, 2011.

8. AGOP M., NICA P., NICULESCU O., DUMITRU D. G., *Experimental and theoretical investigations of the negative differential resistance in a discharge plasma*, Journal of the Physical Society of Japan, **81**, 2012; DOI: 10.1143/JPSJ.81.064502.
9. AGOP M., FORNA N., CASIAN-BOTEZ I. et al., *New theoretical approach of the physical processes in nanostructures*, Journal of Computational and Theoretical Nanoscience, **5**, *4*, pp. 483–489, 2008.
10. AGOP M., MURGULET C., *El Naschie's epsilon (infinity) space-time and scale relativity theory in the topological dimension D=4*, Chaos Solitons & Fractals., **32**, *3*, pp. 1231–1240, 2007.
11. AGOP M., NICA P., GIRTU M., *On the vacuum status in Weyl-Dirac theory*, General Relativity and Gravitation, **40**, *1*, pp. 35–55, 2008.
12. AGOP M., PAUN V., HARABAGIU A., *El Naschie's epsilon (infinity) theory and effects of nanoparticle clustering on the heat transport in nanofluids*, Chaos, Solitons and Fractals, **37**, *5*, pp. 1269–1278, 2008.
13. CASIAN-BOTEZ I., AGOP M., NICA P., PAUN V., MUNCELEAU G.V., *Conductive and convective types behaviors at nano-time scales*, Journal of Computational and Theoretical Nanoscience, **7**, pp. 2271–2280, 2010.
14. CIUBOTARIU C., AGOP M., *Absence of a gravitational analog to the Meissner effect*, General Relativity and Gravitation, **28**, *4*, pp. 405–412, 1996.
15. COLOTIN M., POMPILIAN G. O., NICA P., Gurlui S., PAUN V., AGOP M., *Fractal transport phenomena through the scale relativity model*, Acta Physica Polonica A, **116**, *2*, pp. 157–164, 2009.
16. GOTTLIEB I., AGOP M., JARCAU M., *El Naschie's Cantorian space-time and general relativity by means of Barbilian's group. A Cantorian fractal axiomatic model of space-time*, Chaos, Solitons and Fractals, **19**, *4*, pp. 705–730, 2004.
17. GURLUI S., AGOP M., NICA P., ZISKIND M., FOCSA C., *Experimental and theoretical investigations of transitory phenomena in high-fluence laser ablation plasma*, Phys. Rev. E, **78**, 026405, 2008.
18. GURLUI S., AGOP M., STRAT M., BACAITA S., *Some experimental and theoretical results on the anodic patterns in plasma discharge*, Physics of Plasmas, **13**, *6*, 063503, 2006.
19. NICA P., AGOP M., GURLUI S., BEJINARIU C., FOCSA C., *Characterization of aluminum laser produced plasma by target current measurements*, Japanese Journal of Applied Physics, **51**, 2012; DOI: 10.1143/JJAP.51.106102.
20. NICA P., VIZUREANU P., AGOP M., et al., *Experimental and theoretical aspects of aluminum expanding laser plasma*, Japanese Journal of Applied Physics, **48**, *6*, 2009; DOI: 10.1143/JJAP.48.066001.
21. MANDELBROT B. (1983), *The fractal geometry of nature* (Updated and augm. ed.), W. H. Freeman, New York.
22. LANDAU L., LIFSHITZ E. M., *Fluid Mechanics*, 2nd edition, Butterworth-Heinemann, Oxford, 1987.
23. MUNCELEANU G.V., PAUN V.P., CASIAN-BOTEZ I., AGOP M., *The microscopic-macroscopic scale transformation through a chaos scenario in the fractal space-time theory*, International Journal of Bifurcation and Chaos, **21**, pp. 603–618, 2011.
24. ZINKIEWIKZ O.C., TAYLOR R.L., ZHU J.Z., *The finite element method: its basis and fundamentals*, Butterworth-Heinemann, Oxford, 2005.
25. JACKSON E. A., *Perspectives in nonlinear dynamics*, Cambridge University Press, Cambridge, London, 1991.

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