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AN EFFICIENT COLLOCATION TECHNIQUE FOR SOLVING GENERALIZED FOKKER-PLANCK TYPE EQUATIONS WITH VARIABLE COEFFICIENTS

A.H. BHRAWY [1,2], Engy A. AHMED [2], D. BALEANU [3,4,5]

[1] King Abdulaziz University, Faculty of Science, Department of Mathematics, Jeddah, Saudi Arabia
[2] Beni-Suef University, Faculty of Science, Department of Mathematics, Beni-Suef, Egypt
[3] King Abdulaziz University, Faculty of Engineering, Department of Chemical and Materials Engineering, Jeddah, Saudi Arabia
[4] Cankaya University, Faculty of Arts and Sciences, Department of Mathematics and Computer Sciences, Ankara, Turkey
[5] Institute of Space Sciences, Magurele-Bucharest, Romania
E-mail: dumitru@cankaya.edu.tr

This paper proposes an efficient numerical integration process for the generalized Fokker-Planck equation with variable coefficients. For spatial discretization the Jacobi-Gauss-Lobatto collocation (J-GL-C) method is implemented in which the Jacobi-Gauss-Lobatto points are used as collocation nodes for spatial derivatives. This approach has the advantage of obtaining the solution in terms of the Jacobi parameters α and β. Using the above technique, the problem is reduced to the solution of a system of ordinary differential equations in time. This system can be also solved by standard numerical techniques. Our results demonstrate that the proposed method is a powerful algorithm for solving nonlinear partial differential equations.

Key words: time-dependent Fokker-Plank equation, generalized Fokker-Plank equation, real Newell-Whitehead equation, collocation method, implicit Runge-Kutta method.

1. INTRODUCTION

Many physicists and mathematicians have paid much attention to the *Fokker-Planck equation* in recent years due to its importance in mathematical physics. The Fokker-Planck equation has various applications in the fields of logistic population growth, flame propagation, neurophysiology, autocatalytic chemical reaction, branching Brownian motion process, and nuclear reactor theory; see, e.g., [1, 2]. In this direction, Tatari *et al*. [3] investigated the application of the Adomian decomposition method for solving the Fokker-Planck equation. Kim and Tranquilli [4] proposed a numerical solution of the Fokker-Planck equation that is a good approximation to the radiative transport equation when scattering was peaked sharply in the forward direction (it is the case for light propagation in tissues). The modified *path integral method* was proposed by Narayanana and Kumar [5] to solve the Fokker-Planck equation and to study the nature of the stochastic and chaotic response of the nonlinear systems. Recently, the authors of [6] developed the differential transform method to propose a simple scheme for solving the Fokker-Planck equation and some similar equations. Kazem *et al*. [7] proposed and developed two numerical meshless methods based on radial basis functions to approximate the solution of Fokker-Planck equation.

Spectral method [8–13] is a weighted residuals method that provided the highest levels of accuracy attainable so it has become popular in numerical solutions of linear and nonlinear initial-boundary partial differential equations. Among spectral methods, collocation method has become increasingly popular due to its accuracy. *Spectral collocation method* is often used to solve different kinds of problems, e.g., nonlinear variable coefficient differential equations [14–17], fractional differential equations [18, 19], integral and integro-differential equations [20–23], function approximation, and variational problems [24]. Some other very interesting methods for solving differential equations are given in [25–28].

The use of Jacobi polynomials for solving differential equations has gained increasing popularity in recent years [29–30]. Indeed, there are no results on J-GL-C method for solving Fokker-Planck type equations with variable coefficients and linear dispersion term subject to initial-boundary conditions.

Therefore, the objective of this work is to present a numerical algorithm for solving three Fokker-Planck type equations based on Jacobi pseudo-spectral method. The problem is then reduced to the solution of a system of ordinary differential equations in time. This system may be solved by *implicit Runge-Kutta method* (see, [31]). Finally some illustrative examples are implemented to illustrate the efficiency and applicability of the proposed approach.

This paper is organized as follows. A brief review of Jacobi polynomials is given in the following Section. In Section 3, the way of constructing the Gauss-Lobatto collocation technique for nonlinear time-dependent generalized Fokker-Planck equation is described using the Jacobi polynomials, and in Section 4 the proposed method is applied to three test problems. In the last section our conclusions are presented.

2. BASIC PROPERTIES OF JACOBI POLYNOMIALS

The classical Jacobi polynomial of degree k ($P_k^{(\alpha,\beta)}(x)$, $k = 0,1,\cdots$), associated with the two real parameters $\alpha > -1, \beta > -1$, are obeying the following relation

$$P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\alpha,\beta)}(x),\ P_k^{(\alpha,\beta)}(-1) = \frac{(-1)^k \Gamma(k+\beta+1)}{k!\Gamma(\beta+1)},\ P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}. \tag{1}$$

The Jacobi polynomials are eigenfunctions of the well-known singular Sturm-Liouville equation:

$$(1-x^2)\phi''(x) + [\vartheta - \theta + (\theta + \vartheta + 2)x]\phi'(x) + n(n+\theta+\vartheta+1)\phi(x) = 0. \tag{2}$$

Let us define the weighted space and the weighted function as

$$\begin{aligned} &L^2_{w^{(\alpha,\beta)}} = \{v \mid v \text{ is measurable such that } \| u \|_{w^{(\alpha,\beta)}} < \infty\}, \\ &w^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta. \end{aligned} \tag{3}$$

They are equipped with the following inner product and norm,

$$(u,v)_{w^{(\alpha,\beta)}} = \int_{-1}^{1} u(x)v(x)w^{(\alpha,\beta)}(x)\mathrm{d}x,\ \| u \|_{w^{(\alpha,\beta)}} = (u,u)^{\frac{1}{2}}_{w^{(\alpha,\beta)}}. \tag{4}$$

The set of Jacobi polynomials satisfy orthogonality relation:

$$\int_{-1}^{1} P_j^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(x)w^{(\alpha,\beta)}(x)\mathrm{d}x = \begin{cases} h_k = \dfrac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}, & j = k; \\ 0, & j \neq k. \end{cases} \tag{5}$$

Let $S_N(-1,1)$ be the set of polynomials of degree at most N, and due to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\phi \in S_{2N-1}(-1,1)$,

$$\int_{-1}^{1} w^{(\alpha,\beta)}(x)\phi(x)\mathrm{d}x = \sum_{j=0}^{N} \varpi_{N,j}^{(\alpha,\beta)} \phi(x_{N,j}^{(\alpha,\beta)}), \tag{6}$$

where $x_{N,j}^{(\alpha,\beta)}$ ($0 \le j \le N$) and $\varpi_{N,j}^{(\alpha,\beta)}$ ($0 \le j \le N$) are the nodes and the corresponding Christoffel numbers of the Jacobi-Gauss-quadrature formula on the interval $(-1,1)$, respectively. Now, we introduce the following discrete inner product and norm

$$(u,v)_{w^{(\alpha,\beta)}} = \sum_{j=0}^{N} u(x_{N,j}^{(\alpha,\beta)})v(x_{N,j}^{(\alpha,\beta)})\varpi_{N,j}^{(\alpha,\beta)},\ \| u \|_{w^{(\alpha,\beta)}} = (u,u)^{\frac{1}{2}}_{w^{(\alpha,\beta)}}. \tag{7}$$

3. JACOBI SPECTRAL COLLOCATION METHOD

Since the collocation method approximates differential equations, it is very easy to implement it to various problems, including variable coefficient and nonlinear differential equations (see, for instance [32]). In this section, we develop a J-GL-C method to solve numerically the Fokker-Planck equation.

3.1. Fokker-Planck equation with variable coefficients

In what follows, we construct a mathematical algorithm based on J-GL-C method to solve the generalized Fokker-Planck problem for the variable x given by:

$$\frac{\partial u}{\partial t}=\left[-2\frac{\partial}{\partial x}A(x)+4\frac{\partial^2}{\partial x^2}B(x)\right]u,\ \{x,t\}\in[-1,1]\times[0,T], \tag{8}$$

where $A_1(x)$ and $B_1(x)$ are referred to as the drift and diffusion coefficients and depend on the particular application considered. The equation is subject to the boundary conditions

$$u(-1,t)=g_1(t),\qquad u(1,t)=g_2(t),\quad t\in[0,T], \tag{9}$$

and the initial condition

$$u(x,0)=f(x),\quad x\in[-1,1]. \tag{10}$$

Now we will design an efficient algorithm for solving Eqs. (8–10). Let the numerical solution $u(x,t)$ be approximated by the Jacobi polynomials of degree N in the following form:

$$u(x,t)\ =\sum_{j=0}^{N}a_j(t)P_j^{(\alpha,\beta)}(x), \tag{11}$$

thus it follows from Eqs. (5–7), that

$$a_j(t)=\frac{1}{h_j}\sum_{i=0}^{N}P_j^{(\alpha,\beta)}(x_{N,i}^{(\alpha,\beta)})\varpi_{N,i}^{(\alpha,\beta)}u(x_{N,i}^{(\alpha,\beta)},t). \tag{12}$$

In this way, we find

$$u(x,t)\ =\sum_{j=0}^{N}(\frac{1}{h_j}\sum_{i=0}^{N}P_j^{(\alpha,\beta)}(x_{N,i}^{(\alpha,\beta)})\varpi_{N,i}^{(\alpha,\beta)}u(x_{N,i}^{(\alpha,\beta)},t))P_j^{(\alpha,\beta)}(x), \tag{13}$$

or equivalently

$$u(x,t)\ =\sum_{i=0}^{N}(\sum_{j=0}^{N}\frac{1}{h_j}P_j^{(\alpha,\beta)}(x_{N,i}^{(\alpha,\beta)})P_j^{(\alpha,\beta)}(x)\varpi_{N,i}^{(\alpha,\beta)})u(x_{N,i}^{(\alpha,\beta)},t). \tag{14}$$

The spatial partial derivatives with respect to x in Eq. (8) can be easily computed at the J-GL-C points to give

$$\begin{aligned}u_x(x_{N,n}^{(\alpha,\beta)},t)\ &=\sum_{i=0}^{N}(\sum_{j=0}^{N}\frac{1}{h_j}P_j^{(\alpha,\beta)}(x_{N,i}^{(\alpha,\beta)})(P_j^{(\alpha,\beta)}(x_{N,n}^{(\alpha,\beta)}))'\varpi_{N,i}^{(\alpha,\beta)})u(x_{N,i}^{(\alpha,\beta)},t)\\ &=\sum_{i=0}^{N}A_{ni}u(x_{N,i}^{(\alpha,\beta)},t),\ n=0,1,\cdots,N,\end{aligned} \tag{15}$$

and

$$\begin{aligned}u_{xx}(x_{N,n}^{(\alpha,\beta)},t)\ &=\sum_{i=0}^{N}(\sum_{j=0}^{N}\frac{1}{h_j}P_j^{(\alpha,\beta)}(x_{N,i}^{(\alpha,\beta)})(P_j^{(\alpha,\beta)}(x_{N,n}^{(\alpha,\beta)}))''\varpi_{N,i}^{(\alpha,\beta)})u(x_{N,i}^{(\alpha,\beta)},t)\\ &=\sum_{i=0}^{N}D_{ni}u(x_{N,i}^{(\alpha,\beta)},t),\end{aligned} \tag{16}$$

where

$$C_{ni} = \sum_{j=0}^{N} \frac{1}{h_j} P_j^{(\alpha,\beta)}(x_{N,i}^{(\alpha,\beta)})(P_j^{(\alpha,\beta)}(x_{N,n}^{(\alpha,\beta)}))' \varpi_{N,i}^{(\alpha,\beta)}, \tag{17}$$

$$D_{ni} = \sum_{j=0}^{N} \frac{1}{h_j} P_j^{(\alpha,\beta)}(x_{N,i}^{(\alpha,\beta)})(P_j^{(\alpha,\beta)}(x_{N,n}^{(\alpha,\beta)}))'' \varpi_{N,i}^{(\alpha,\beta)}. \tag{18}$$

In the proposed J-GL-C method the residual of (8) is set to zero at $N-1$ of collocation points. Moreover, the conditions (24) will be enforced at -1 and 1. Therefore, adopting (14–18), enables one to write (8–10) in the form:

$$\begin{aligned} \dot{u}_n(t) + \left[2A_2(x_{N,n}^{(\alpha,\beta)}) - 4B_3(x_{N,n}^{(\alpha,\beta)})\right]u_n(t) + \left[2A(x_{N,n}^{(\alpha,\beta)}) - 8B_2(x_{N,n}^{(\alpha,\beta)})\right]\sum_{i=0}^{N} C_{ni}u_i(t) \\ - 4B(x_{N,n}^{(\alpha,\beta)})\sum_{i=0}^{N} D_{ni}u_i(t) = 0, \end{aligned} \tag{19}$$

where

$$u_n(t) = u(x_{N,n}^{(\alpha,\beta)}, t),\ \dot{u}_n(t) = u_t(x_{N,n}^{(\alpha,\beta)}, t),\ n = 1, \cdots, N-1,$$

$$A_2(x) = \frac{\partial A(x)}{\partial x},\ B_2(x) = \frac{\partial B(x)}{\partial x},\ B_3(x) = \frac{\partial^2 B(x)}{\partial x^2}.$$

Equations (19) (using the two-point boundary conditions (24)), generate a system of $(N-1)$ ODEs in time:

$$\begin{aligned} \dot{u}_n(t) + \left[2A_2(x_{N,n}^{(\alpha,\beta)}) - 4B_3(x_{N,n}^{(\alpha,\beta)})\right]u_n(t) + \left[2A(x_{N,n}^{(\alpha,\beta)}) - 8B_2(x_{N,n}^{(\alpha,\beta)})\right]\left(\sum_{i=1}^{N-1} C_{ni}u_i(t) + d_n(t)\right) - \\ - 4B(x_{N,n}^{(\alpha,\beta)})\left(\sum_{i=1}^{N-1} D_{ni}u_i(t) + \hat{d}_n(t)\right) = 0,\ n = 1, \cdots, N-1, \end{aligned} \tag{20}$$

where

$$d_n(t) = (C_{n0}g_1(t) + C_{nN}g_2(t)),\ \text{ and }\ \hat{d}_n(t) = (D_{n0}g_1(t) + D_{nN}g_2(t)).$$

Therefore, the problem (8–10) transforms to the system of ODEs

$$\begin{aligned} \dot{u}_n(t) + \left[2A_2(x_{N,n}^{(\alpha,\beta)}) - 4B_3(x_{N,n}^{(\alpha,\beta)})\right]u_n(t) + \left[2A(x_{N,n}^{(\alpha,\beta)}) - 8B_2(x_{N,n}^{(\alpha,\beta)})\right]\left(\sum_{i=1}^{N-1} C_{ni}u_i(t) + d_n(t)\right) - \\ - 4B(x_{N,n}^{(\alpha,\beta)})\left(\sum_{i=1}^{N-1} D_{ni}u_i(t) + \hat{d}_n(t)\right) = 0,\ n = 1, \cdots, N-1, \\ u_n(0) = f(x_{N,n}^{(\alpha,\beta)}), \end{aligned} \tag{21}$$

which can be written in the following matrix form:

$$\dot{u}(t) = F(t, u(t)),\ u(0) = f, \tag{22}$$

where

$$\begin{aligned} \dot{u}(t) &= [\dot{u}_1(t), \dot{u}_2(t), \cdots, \dot{u}_{N-1}(t)]^T, \\ f &= [f(x_{N,1}), f(x_{N,2}), \cdots, f(x_{N,N-1})]^T, \\ F(t,u(t)) &= [F_1(t,u(t)), F_2(t,u(t)), \cdots, F_{N-1}(t,u(t))]^T, \end{aligned}$$

$$F_n(t,u(t)) = -\left[2A_2(x_{N,n}^{(\alpha,\beta)}) - 4B_3(x_{N,n}^{(\alpha,\beta)})\right]u_n(t) - \left[2A(x_{N,n}^{(\alpha,\beta)}) - 8B_2(x_{N,n}^{(\alpha,\beta)})\right](\sum_{i=1}^{N-1} C_{ni}u_i(t) + d_n(t)) +$$

$$+4B(x_{N,n}^{(\alpha,\beta)})(\sum_{i=1}^{N-1} D_{ni}u_i(t) + \hat{d}_n(t)) = 0,\ n = 1,\cdots,N-1.$$

The system of ODEs (22) in time can be solved using any standard numerical technique.

3.2. BACKWARD KOLMOGOROV EQUATION

In this subsection, we derive a J-GL-C method to solve numerically the backward Kolmogorov model problem given by:

$$\frac{\partial u}{\partial t} = \left[-A(y,t)\frac{\partial}{\partial y} + B(y,t)\frac{\partial^2}{\partial y^2}\right]u,\ \{y,t\} \in [0,1]\times[0,T], \tag{23}$$

subject to the boundary conditions

$$u(-1,t) = g_1(t),\ u(1,t) = g_2(t),\ t \in [0,T], \tag{24}$$

and the initial condition

$$u(y,0) = f(y), y \in [0,1]. \tag{25}$$

The same procedure of subsection 3.1 is used for solving numerically the equations (23–25). The difference between the measured value of numerical solution and its actual value (absolute error), is given by $E(x,t) = |u(x,t) - \tilde{u}(x,t)|$, where $u(x,t)$ and $\tilde{u}(x,t)$ are the exact solution and the numerical solution at the point (x,t), respectively. Moreover, the maximum absolute error is given by $M_E = \text{Max}\{E(x,t) : \forall (x,t) \in [0,1]\times[0,T]\}$. The root mean square (RMS) and N_e errors are given by:

$$\text{RMS} = \sqrt{\sum_{i=0}^{N} \frac{(u(x_{N,i}^{(\alpha,\beta)}),t_i) - \tilde{u}(x_{N,i}^{(\alpha,\beta)}),t_i)^2}{n+1}},\ N_e = \sqrt{\frac{\sum_{i=0}^{N}(u(x_{N,i}^{(\alpha,\beta)}),t_i) - \tilde{u}(x_{N,i}^{(\alpha,\beta)}),t_i)^2}{\sum_{i=0}^{N}(u(x_{N,i}^{(\alpha,\beta)}),t_i)}}. \tag{26}$$

4. NUMERICAL RESULTS

This section reports some numerical results obtained using the algorithm presented in the previous section. We implement the spectral collocation method to solve three types of Fokker-Planck equations with time-dependent coefficients to confirm the good accuracy of the method. Comparison of the results obtained by proposed method and radial basis functions method reveal that the present method is a very efficient one.

Example 1. Consider the Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]u,\ \{x,t\} \in [0,1]\times[0,1], \tag{27}$$

with $A(x) = -1$, $B(x) = 1$, and the exact solution of this problem is $u(x) = x + t$. Table 1 lists the RMS and N_e obtained by our method and the results of radial basis functions method (RBF [7]). Meanwhile the maximum absolute errors using the spectral collocation method for $N = 12$ with various choices of α and β are summarized in Table 2.

Table 1

Comparison of values of RMS and N_e for *J-GL-C* for Example 1

Our method N=12				RBF [7]		
α	β	RMS	N_e	N	RMS	N_e
0	0	2.15×10^{-14}	1.74×10^{-14}	25	1.22×10^{-2}	2.81×10^{-2}
0	0.5	1.18×10^{-14}	9.51×10^{-15}	36	6.52×10^{-4}	1.50×10^{-3}
0.5	0	7.29×10^{-15}	6.05×10^{-15}	49	5.83×10^{-5}	1.34×10^{-4}
0.5	0.5	3.42×10^{-15}	2.79×10^{-15}	64	1.71×10^{-5}	3.93×10^{-5}
0.5	– 0.5	7.29×10^{-15}	6.05×10^{-15}	81	4.00×10^{-6}	9.20×10^{-6}
– 0.5	0.5	2.97×10^{-14}	2.35×10^{-14}	100	1.53×10^{-6}	3.51×10^{-6}

Table 2

Maximum absolute errors for $N = 12$ with various choices of α and β for Example 1

α	β	M_E	α	β	M_E
0	0	6.439×10^{-15}	0.5	0.5	8.43×10^{-15}
0	0.5	3.21×10^{-15}	0.5	0	5.99×10^{-15}
0.5	- 0.5	5.99×10^{-15}	- 0.5	0.5	2.70×10^{-14}

Example 2. Consider the Fokker-Planck equation with variable coefficients:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u, \quad \{x,t\} \in [0,1]\times[0,1] \tag{28}$$

with $A(x) = x$, $B(x) = \dfrac{x^2}{2}$; the exact solution is $u(x) = xe^t$.

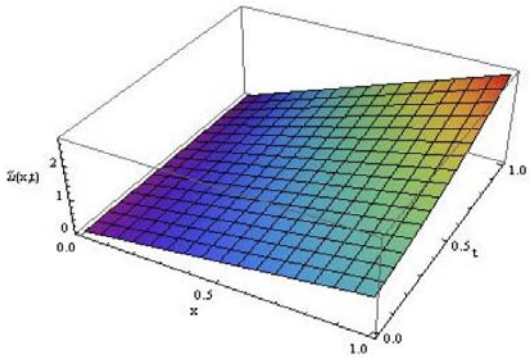

Fig. 1 – The approximate solution of problem (28) for $\alpha = \beta = 0.5$ and $N = 12$.

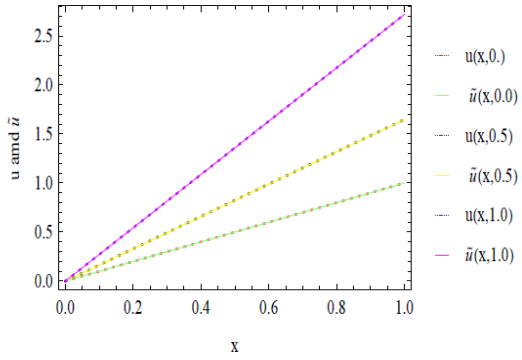

Fig. 2 – Curves of approximate and exact solutions for $t = 0.0, 0.5, 1.0$ of problem (30) for $\alpha = \beta = 0.5$ and $N = 12$.

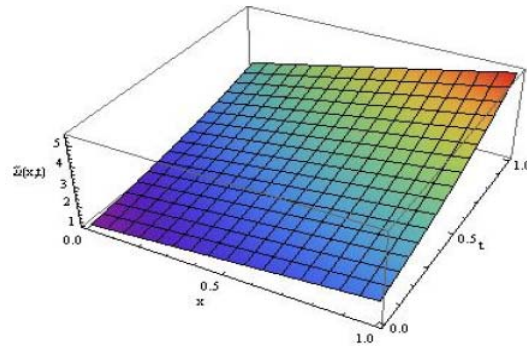

Fig. 3 – The approximate solution of problem (29) for $\alpha = \beta = 0.5$ and $N = 12$.

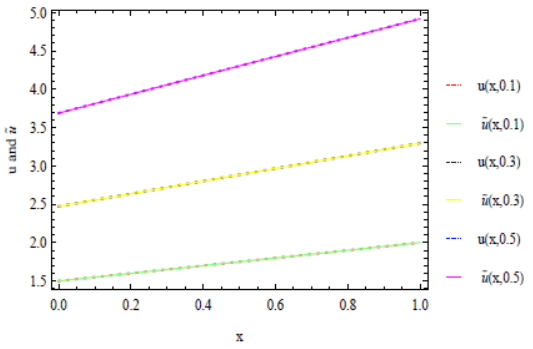

Fig. 4 – Curves of approximate and exact solutions for $t = 0.1, 0.3, 0.5$ of problem (31) for $\alpha = 0, \beta = 0.5$ and $N = 12$.

In Table 3, we list the maximum absolute values of $u(x,t)-u_N(x,t)$ for (28) using the J-GL-C method with four special values of Jacobi parameters $\alpha=\beta=0.5$, $\alpha=\beta=0$, $\alpha=-\beta=0.5$ and $\alpha=0,\beta=0.5$ at $N=12$. A comparison of RMS and N_e obtained by our method and those of radial basis functions methods (RBF [7]) are listed in Table 4.

We plot in Fig. 1, the numerical solution $\tilde{u}(x,t)$, with values of parameters listed in its caption to make it easier to compare with the analytic solution. Moreover, the curves of exact and approximate solution are given in Fig. 2 at various choices of t.

Table 3

Maximum absolute errors for $N=12$ with various choices of α and β for Example 2

α	β	M_E	α	β	M_E
0	0	4.17×10^{-7}	0.5	0.5	4.12×10^{-7}
0	0.5	4.20×10^{-7}	0.5	0	4.15×10^{-7}
0.5	– 0.5	4.18×10^{-7}	– 0.5	0.5	4.51×10^{-7}

Table 4

Comparison of values of *RMS* and N_e for *J-GL-C* for Example 2

Our method $N=12$				RBF [7]		
α	β	RMS	N_e	N	RMS	N_e
0.5	0	1.81×10^{-7}	1.25×10^{-7}	49	1.13×10^{-3}	8.68×10^{-4}
0.5	0.5	2.03×10^{-7}	1.39×10^{-7}	64	1.10×10^{-4}	8.44×10^{-5}
0.5	– 0.5	1.67×10^{-7}	1.16×10^{-7}	81	2.53×10^{-5}	1.94×10^{-5}
– 0.5	0.5	2.46×10^{-7}	1.62×10^{-7}	100	1.83×10^{-5}	1.42×10^{-5}

Example 3. Consider the general form of backward Kolomogorov equation:

$$\frac{\partial u}{\partial t}=\left[-A(x,t)\frac{\partial}{\partial x}+B(x,t)\frac{\partial^2}{\partial x^2}\right]u,\ \{x,t\}\in[0,1]\times[0,1] \tag{29}$$

with drift and diffusion coefficients given respectively by: $A(x,t)=-(x+1)$, $B(x)=x^2\mathrm{e}^t$; the exact solution of this problem is $u(x)=(x+1)\mathrm{e}^t$. The boundary and initial conditions have been derived from the exact solution.

Table 5

Comparison of values of RMS and N_e for *J-GL-C* for Example 3

Our method $N=12$				RBF [7]		
α	β	RMS	N_e	N	RMS	N_e
0.5	0	3.35×10^{-7}	1.03×10^{-7}	49	2.04×10^{-3}	7.23×10^{-4}
0.5	0.5	3.08×10^{-7}	9.48×10^{-8}	64	1.54×10^{-3}	5.46×10^{-4}
0.5	– 0.5	3.75×10^{-7}	1.17×10^{-7}	81	6.18×10^{-5}	2.18×10^{-5}
– 0.5	0.5	4.39×10^{-7}	1.31×10^{-7}	100	6.11×10^{-6}	2.19×10^{-6}

Table 5 presents a comparison of RMS and N_e acquired by our method and those of radial basis functions methods (RBF [7]). We plot the approximate solution $\tilde{u}(x,t)$ in Fig. 3. As shown in Fig. 4, the numerical and the exact solutions fit for different values of *t*. Therefore, this example indicates that the spectral Jacobi Gauss-Lobatto collocation method is compared favorably with the analytical solution.

5. CONCLUSIONS

In this paper, we proposed an efficient numerical scheme based on J-GL-C spectral method for solving nonlinear time-dependent generalized Fokker-Planck equation with time-dependent coefficients and Dirichlet boundary conditions. Numerical examples were given to demonstrate the validity and applicability of the method. Because of the accuracy, applicability, and validity of the proposed method, this technique is competitive with other numerical methods.

REFERENCES

1. H. RISKEN, *The Fokker-Plank equation: method of solution and applications*, Berlin, Springer Verlag, 1989.
2. M. LAKESTANI, M. DEHGHAN, *Numerical solutions of Fokker-Plank equation using the cubic B-spline scaling functions*, Numer. Meth. Partial Diff. Equat., **25**, pp. 418–429, 2009.
3. M. TATARI, M. DEGHAN, M. RAZZAGHI, *Application of the Adomian decomposition method for the Fokker-Planck equation*, Mathematical and Computer Modelling, **45**, pp. 639–650, 2007.
4. A.D. KIM, P. TRANQUILLI, *Numerical solution of the Fokker-Planck equation with variable coefficients*, Journal of Quantitative Spectroscopy & Radiative Transfer, **109**, pp. 727–740, 2008.
5. S. NARAYANAN, P. KUMAR, *Numerical solutions of Fokker-Planck equation of nonlinear systems subjected to random and harmonic excitations*, Probabilistic Engineering Mechanics, **27**, pp. 35–46, 2012.
6. S. HESAM, A.R. NAZEMI, A. HAGHBIN, *Analytical solution for the Fokker-Planck equation by differential transform method*, Scientia Iranica B, **4**, pp. 1140–1145, 2012.
7. S. KAZEM, J.A. RAD, K. PARAND, *Radial basis functions methods for solving Fokker-Planck equation*, Engineering Analysis with Boundary Elements, **36**, pp. 181–189, 2012.
8. R. S. BEIDOKHTI, A. MALEK, *Solving initial-boundary value problems for systems of partial differential equations using neural networks and optimization techniques*, Journal of the Frank lin Institute, **346**, pp. 898–913, 2009.
9. D. BALEANU, A.H. BHRAWY, T. M. TAHA, *Two Efficient Generalized Laguerre Spectral Algorithms for Fractional Initial Value Problems*, Abstract and Applied Analysis,Volume 2013, Article ID 546502, 10, 2013.
10. E.H. DOHA, A. H. BHRAWY, *An efficient direct solver for multidimensional elliptic Robin boundary value problems using a Legendre spectral-Galerkin method*, Computers & Mathematics with Applications, **64**, pp. 558–571, 2012.
11. E. TOHIDI, A.H. BHRAWY, K. ERFANI, *A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation*, Applied Mathematical Modelling, **37**, pp. 4283–4294, 2013.
12. G. EBADI, M.Y. RAHIMI-ARDABILI, S. SHAHMORAD, *Numerical solution of the nonlinear Volterra integro-differential equations by the Tau method*, Applied Mathematics and Computation, **188,** pp. 1580–1586, 2007.
13. D. CONTE, B. PATERNOSTER, *Multistep collocation methods for Volterra Integral Equations*, Applied Numerical Mathematics, **59,** pp. 1721–1736, 2009.
14. E. BAUTU, E. PELICAN, *Numerical solution for Fredohom first kind integral equations occurring in synthesis of electro-magnetic fields*, Rom. J. Phys., **52**, pp. 245–256, 2007.
15. A.H. BHRAWY, *A Jacobi-Gauss-Lobatto collocation method for solving generalized Fitzhugh-Nagumo equation with time-dependent coefficients*, Applied Mathematics and Computation, **222**, pp. 255–264, 2013.
16. E.H. DOHA, A.H. BHRAWY, R.M. HAFEZ, M.A. ABDELKAWY, *A Chebyshev-Gauss-Radau scheme for nonlinear hyperbolic system of first order*, Applied Mathematics and Information Science, **8**, pp. 1–10, 2014.
17. Z. JACKIEWICZ, B. ZUBIK-KOWAL, *Spectral collocation and waveform relaxation methods for nonlinear delay partial differential equations*, Applied Numerical Mathematics, **56,** pp. 433–443, 2006.
18. E.H. DOHA, A.H. BHRAWY, S.S. EZZ-ELDIEN, *Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations*, Applied Mathematical Modelling, **35**, pp. 5662–5672, 2011.
19. E.H. DOHA, A.H. BHRAWY, S.S. EZZ-ELDIEN, *A new Jacobi operational matrix: An application for solving fractional differential equations*, Applied Mathematical Modelling, **36**, pp. 4931–4943, 2012.
20. J. LI, X. WANG, S. XIAO, T. WANG, *A rapid solution of a kind of 1D Fredholm oscillatory integral equation*, Journal of Computational and Applied Mathematics, **236**, pp. 2696–2705, 2012.
21. A.H. BHRAWY, E. TOHIDI, F. SOLEYMANI, *A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals*, Applied Mathematics and Computation, **219**, pp. 482–497, 2012.
22. J.WU, Y. WANG, W. LI, W. SUN, *Toeplitz-type approximations to the Hadamard integral operator and their applications to electromagnetic cavity problems*, Applied Numerical Mathematics, **58**, pp. 101–121, 2008.
23. E.V. CHISTYAKOVA, *Regularizing properties of difference schemes for singular integral-differential equations*, Applied Numerical Mathematics, **62**, pp. 1302–1311, 2012.
24. C. CANUTO, M.Y. HUSSAINI, A. QUARTERONI, T.A. ZANG, *Spectral Methods: Fundamentals in Single Domains*, Springer-Verlag, New York, 2006.
25. D. KUMAR, J. SINGH, SUSHILA, *Application of homotopy analysis transform method to fractional biological population model*, Rom. Rep. Phys., **65**, pp. 63–75, 2013.
26. D. ROSTAMY, M. ALIPOUR, H. JAFARI, D. BALEANU, *Solving multi-term orders fractional differential equations by operational matrices of BPs with convergence analysis*, Rom. Rep. Phys., **65**, pp. 334–349, 2013.

27. F.A. ALIEV, N.I. VELIEVA, Y.S. GASIMOV, N.A. SAFAROVA, L.F. AGAMALIEVA, *High algorithms to the solution of the optimal output feedback problem for the linear systems*, Proc. Romanian Acad. A, **13**, pp. 207–214, 2012.
28. J.A. TENREIRO MACHADO, P. STEFANESCU, O. TINTAREANU, D. BALEANU, *Fractional calculus analysis of the cosmic microwave background*, Rom. Rep. Phys., **65**, pp. 316–323, 2013.
29. E.H. DOHA, A.H. BHRAWY, W.M. ABD-ELHAMEED, *Jacobi spectral Galerkin method for elliptic Neumann problems*, Numer. Algorithms, **50**, pp. 67–91, 2009.
30. E.H. DOHA, A.H. BHRAWY, *Jacobi spectral Galerkin method for the integrated forms of fourth-order elliptic differential equations*, Numer. Meth. Partial Diff. Equat., **25**, pp. 712–739, 2009.
31. H. ARABSHAHI, *A study of dynamic characterizations of GaAs/AlGaAs self-assembled quantum dot lasers*, Rom. Rep. Phys., **63**, pp. 1061–1069, 2011.
32. A.H. BHRAWY, D. BALEANU, *A Spectral Legendre- Gauss-Lobatto collocation method for space-fractional advection diffusion equations with variable coefficients*, Rep. Math. Phys., **72**, pp. 219–233, 2013.

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