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A SHORT PROOF FOR THE CHARACTERIZATION BY ORDER AND DEGREE PATTERN OF $PGL(2,q)$ AND $L_2(q)$

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The degree pattern of a finite group G is denoted by $D(G)$. In [14] and [19] the characterization of $L_2(q)$ and $PGL(2,q)$ by their orders and their degree patterns are proved. In this paper we give a very short proof for the main results of these papers.

Key words: projective special linear group, projective general linear group, degree pattern, prime graph.

1. INTRODUCTION

Let N and P denote the set of natural numbers and the set of prime numbers, respectively. If $n \in N$, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of orders of the elements of G is denoted by $\pi_e(G)$. Obviously, $\pi_e(G)$ is closed and partially ordered by divisibility, hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements. The prime graph of G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$. Denote by $t(G)$ the numbers of connected components of $\Gamma(G)$ and by $\pi_i = \pi_i(G)$, where $i = 1, 2, \ldots, t(G)$, the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then always we assume that $2 \in \pi_1$ and $\pi_2, \ldots, \pi_{t(G)}$ are called the odd component(s) of $\Gamma(G)$.

Let $\pi(G) = \{p_1, p_2, \ldots, p_m\}$ and $p_1 < p_2 < \cdots < p_m$. The degree pattern of G is denoted by $D(G)$ and defined as follows: $D(G) = (\deg(p_1), \deg(p_2), \ldots, \deg(p_m))$, where $\deg(p_i)$ is the degree of vertex p_i in $\Gamma(G)$. A group G is called OD-characterizable if G is uniquely determined by $|G|$ and $D(G)$.

It is proved that sporadic simple groups and their automorphism groups except $Aut(J_2)$ and $Aut(McL)$, the alternating groups A_p, A_{p+1}, A_{p+2} and the symmetric groups S_p and S_{p+1}, where $p \in P$ are OD-characterizable [7]. In [14], it is proved that all finite simple groups with exactly four prime divisors are OD-characterizable, except A_{10}. Also in [16, 17] finite groups with the same order and degree pattern as an almost simple group related to $L_2(49)$ or $U_3(5)$ are determined. Recently in [18] and [13] it is proved that every special linear group $L_2(q)$ and every projective general linear groups $PGL(2,q)$ are OD-characterizable. In fact, in this paper we give a very simple proof for these results. More results can be found in [1, 5, 6, 8, 9, 10, 15, 19, 20]. All further unexplained notations are standard and can be found in [2]. If $p \in P$ and $k, n \in N$, then $p^k \parallel n$ means that $p^k \mid n$ and $p^{k+1} \nmid n$.

2. PRELIMINARY RESULTS

The next lemma summarizes the structural properties of a Frobenius group and a 2-Frobenius group [2, 3]:

LEMMA 2.1. a) *Let* G *be a Frobenius group with Frobenius kernel* K *and Frobenius complement* H. *Then* $t(G)=2$, $\pi(K)$ *and* $\pi(H)$ *are the components of* $\Gamma(G)$.

b) *Let* G *be a 2-Frobenius group, i.e.,* G *has a normal series* $1\le H\le K\le G$, *such that* K *and* G/H *are Frobenius groups with kernels* H *and* K/H, *respectively. If* G *has even order, then*

(i) $t(G)=2$, $\pi_1=\pi(G/K)\cup\pi(H)$ *and* $\pi_2=\pi(K/H)$;

(ii) G/K *and* K/H *are cyclic*, $|G/K|$ *divides* $|Aut(K/H)|$ *and* $(|G/K|,|K/H|)=1$;

(iii) H *is a nilpotent group and* G *is a solvable group*.

By using [12, Theorem A] we have the following result:

LEMMA 2.2. *Let* G *be a finite group with* $t(G)\ge 2$. *Then one of the following holds*:

a) G *is a Frobenius or* 2*-Frobenius group*;

b) *there exists a nonabelian simple group* S *such that* $S\le G/N\le Aut(S)$ *for some nilpotent normal* π_1*-subgroup* N *of* G *and* G/S *is a* π_1*-group*.

3. MAIN RESULTS

Throughout this section let $p\in P$, $n\in N$ and $q=p^n$.

THEOREM 3.1. *Let* p *be an odd prime and* $\varepsilon=1$ *or* 2. *If* G *is a finite group such that* $|G|=q(q^2-1)/\varepsilon$ *and* $\deg(p)=0$ *in* $\Gamma(G)$, *then* $L_2(q)\le G\le Aut(L_2(q))$.

Proof. We can easily see that if $|G|=12$ or 24 and $\deg(3)=0$ in $\Gamma(G)$, then $G\cong A_4$ or S_4, respectively. Also by using GAP we obtain that if $q=5$, then $G\cong A_5$ or S_5, as required. Therefore let $q>5$. First we show that G is not a Frobenius or 2-Frobenius group.

Step 1. If G is a Frobenius group with kernel K and complement C, then by Lemma 2.1, $\pi(K)$ and $\pi(C)$ are the connected components of $\Gamma(G)$. Since $\deg(p)=0$ in $\Gamma(G)$, $\pi(K)=\{p\}$ or $\pi(C)=\{p\}$. If $\pi(K)=\{p\}$, then $|K|=q$ and $|C|=(q^2-1)/\varepsilon$. We know that $|K|\equiv 1(\mathrm{mod}\,|C|)$, which is impossible. If $\pi(C)=\{p\}$, then $|C|=q$ and $|K|=(q^2-1)/\varepsilon$. Hence $(q^2-1)/\varepsilon\equiv 1\ (\mathrm{mod}\ q)$, which is a contradiction, since $q>5$. Therefore, G is not a Frobenius group. Now let G be a 2-Frobenius group with normal series $1\le H\le K\le G$, such that G/H and K are Frobenius groups with kernels K/H and H, respectively. By using Lemma 2.1, $|K/H|=q$ and $|H||G/K|=(q^2-1)/\varepsilon$, since $\deg(p)=0$ in $\Gamma(G)$ and p is an odd prime number. Also by Lemma 2.1, we have $|G/K|\,\big|\,(p-1)$, which is a divisor of $q-1$. Therefore, $q-1=m|G/K|$, for some $m\ge 1$ and so $|H|=(q+1)m/\varepsilon$. We know that $|H|\equiv 1(\mathrm{mod}\,|K/H|)$. So $(q+1)m/\varepsilon\equiv 1(\mathrm{mod}\,q)$. Then $m\equiv\varepsilon(\mathrm{mod}\,q)$ and so $m=\varepsilon$, since $1\le m\le q$. Hence $|H|=q+1$ and so $|G/K|=(q-1)/\varepsilon$. Also $|G/K|\,\big|\,(p-1)$ and $(p-1)\,\big|\,(q-1)$, which implies that $q=p$. Therefore, $|H|=p+1$ and $|K|=p(p+1)$. Since K is a solvable group, if t is an odd prime divisor of $p+1$, then K has a $\{p,t\}$-Hall subgroup, say T. Let $s\in N$ and $t^s\,\|\,(p+1)$. Then $|T|=pt^s$ and if n_t is the numbers of Sylow t-subgroups of T, then $n_t=1$ or p. If $n_t=p$, then

$1+tr=p$, for some $r>0$, which is a contradiction, since $t\,|\,(p+1)$ and t is odd. We note that $t^i \equiv 1 \pmod p$, where $1 \le i \le s$, since $p+1$ is even and so $p>t^s$. Therefore $n_p=1$, where n_p is the numbers of Sylow p-subgroups of T. Hence by using Sylow Theorem it follows that T is a nilpotent subgroup of K and so $p \sim t$ in $\Gamma(G)$, which is a contradiction, since $\deg(p)=0$ in $\Gamma(G)$. Therefore, H is a $\{2\}$-group, i.e., there exists a natural number α such that $|H|=p+1=2^\alpha$ ($\alpha \ge 3$, since by assumption $p=q>5$). Let P be a Sylow p-subgroup of G. Since $\Phi(H) \lhd G$, if $\Phi(H)=\{1\}$, then $|\Phi(H)| \equiv 1 \pmod p$.

Since $|\Phi(H)|<p+1$, $\Phi(H) \cap C_G(P)=\{1\}$, which is a contradiction, since $\deg(p)=0$ in $\Gamma(G)$. Hence $\Phi(H)=\{1\}$ and so H is an elementary abelian 2-group. Let $F=GF(2^\alpha)$ and so H is the additive group of F. Also $|P|=p=2^\alpha-1$ and so P is the multiplicative group of F. Now G/K acts by conjugation on H and similarly G/K acts by conjugation on P and this action is faithful. Then G/K keeps the structure of the field F and so G/K is isomorphic to a subgroup of the automorphism group of F. Hence $|G/K|=2^\alpha-2 \le |Aut(F)|=\alpha$, which is impossible, since $\alpha \ge 3$. Therefore, G is not a 2-Frobenius group.

Step 2. By Lemma 2.2, there exists a nonabelian simple group S such that $S \le G/N \le Aut(S)$ where N is a nilpotent subgroup of G. Also by Lemma 2.2, since G/S is a π_1-group and $\deg(p)=0$ in $\Gamma(G)$, we conclude that $\{p\}$ is an odd component of $\Gamma(S)$ and $|S|=qm$, where $m\,|\,(q^2-1)$. All of the nonabelian simple groups with at least two connected components are given in [11, Tables 1a, 2b and 2c]. Now we must consider each possibility separately. For convenience we omit the details of the proof and only state a few of them. We remark that in these tables, $p' \in P \setminus \{2\}$, q' is a prime power and $n' \in N$.

Case 1. Let $S=A_{n'}$, where $6<n'=p'$, $p'+1$ or $p'+2$; n' or $n'-2$ is not prime. By using [11, Table 1a], we have $p'=q$, since the odd component of $\Gamma(A_{n'})$ is $\{p'\}$. As we mentioned above we have $(p'-1)!\,|\,(p'^2-1)$, which is a contradiction, since in this case $p' \ge 7$.

Case 2. Let $S=A_{n'}$, where $6<n'=p', p'-2$ are primes. By using [11, Table 2b], we have $p'=q$ or $p'-2=q$, since the odd component of $\Gamma(A_{n'})$ are $\{p'\}$ and $\{p'-2\}$. Then we must have $\alpha(p'-1)(p'-3)!$ divides p'^2-1 or $(p'-2)^2-1$, where $\alpha=p'$ if $p'-2=q$ and $\alpha=p'-2$ if $p'=q$, which is a contradiction.

Case 3. Let $S=A_{p'-1}(q')$, where $(p',q') \ne (3,2),(3,4)$. We know that $m\,|\,(q^2-1)$, where $|S|=qm$. By using [11, Table 1a], we have

$$q=(q'^{p'}-1)/(q'-1)(p',q'-1).$$

We can easily see that $q^2-1<q'^{2p'}$, which is a contradiction, since $q'^{p'(p'-1)/2}\,|\,(q^2-1)$ and $q'^{p'(p'-1)/2} \ge q'^{2p'}$.

Case 4. Let $S=G_2(q')$ be a Chevalley group, where $q' \equiv 0 \pmod 3$. By using [11, Table 2b], we have $q'^2-q'+1=q$ or $q'^2+q'+1=q$, since the odd components of $\Gamma(S)$ are $\pi(q'^2-q'+1)$ or $\pi(q'^2+q'+1)$. Let $q'^2+\beta q'+1=q$, where $\beta=1$ or -1. Then $q'^6\,|\,(q^2-1)=((q'^2+\beta q'+1)^2-1)$. We can easily see that $(q'^2 \pm q'+1)^2-1<q'^6$, which is a contradiction.

Similarly, we can prove that S is not isomorphic to all other simple groups in Tables in [11], except $A_1(q')$.

Case 5. Let $S = A_1(q')$. If q' is even and $q' > 2$, then by [11, Table 2b], the odd components of $\Gamma(S)$ are $\pi(q'-1)$ or $\pi(q'+1)$. If $\pi(q'-1) = \{p\}$, then $q'-1 = q$ and so $(q'+1) \mid ((q'-1)^2 - 1)$, which is impossible. If $\pi(q'+1) = \{p\}$, then $q'+1 = q$ and so $(q'-1) \mid ((q'+1)^2 - 1)$. So we have $q' = 4$ and $q = 5$, which is impossible, since $q > 5$. Hence q' is not even.

Therefore $3 < q' \equiv \varepsilon \pmod 4$, where $\varepsilon = 1$ or -1. By [11, Table 2b], the odd components of $\Gamma(S)$ are $\pi(q')$ and $\pi((q'+\varepsilon)/2)$. If $\pi((q'+\varepsilon)/2) = \{p\}$, then $(q'+\varepsilon)/2 = q$ and so $q' \mid (((q'+\varepsilon)/2)^2 - 1)$, which is a contradiction, since $q' = 3$. So we conclude that $\pi(q') = \{p\}$ and $q' = q$. Therefore we have $S = A_1(q) = L_2(q)$.

This argument shows that $L_2(q) \le G/N \le Aut(L_2(q))$ and so $|N| = 1$ or 2. If $|N| = 2$, then we have $N \le Z(G)$, which is a contradiction since $\deg(p) = 0$ in $\Gamma(G)$. Therefore $|N| = 1$ and $L_2(q) \le G \le Aut(L_2(q))$. □

COROLLARY 3.2. *The finite group* $L_2(q)$ *is* OD-*characterizable*.

Proof. Let G be a finite group such that $|G| = |L_2(q)|$ and $D(G) = D(L_2(q))$. We know that $|L_2(q)| = q(q^2-1)/d$, where $d = (2, q-1)$. By using [11, Table 2b], we have $\deg(p) = 0$ in $\Gamma(L_2(q))$. In [5, Theorem 1.4] it is proved that if q is even, then $G = L_2(q)$. Therefore, let q be odd. So $|G| = q(q-1)/2$ and $\deg(p) = 0$ in $\Gamma(G)$. By using Theorem 3.1, we have $L_2(q) \le G \le Aut(L_2(q))$. On the other hand $|G| = |L_2(q)|$ and so $G = L_2(q)$. □

THEOREM 3.3. *The finite group* $PGL(2,q)$ *is* OD-*characterizable*.

Proof. Let G be a finite group such that $|G| = |PGL(2,q)|$ and $D(G) = D(PGL(2,q))$. If q is even, then $PGL(2,q) = L_2(q)$ and by Corollary 3.2, we have $G = PGL(2,q)$. Therefore, let q be odd. We know that $|PGL(2,q)| = q(q^2-1)$ and $\mu(PGL(2,q)) = \{q-1, p, q+1\}$. Hence $\deg(p) = 0$ and $\deg(2) = |\pi(G)| - 2$ in $\Gamma(G)$.

By using Theorem 3.1, we have $L_2(q) \le G \le Aut(L_2(q))$. Thus G is an extension of $L_2(q)$ by an involution, since $|G| = 2|L_2(q)|$. We know that $|Out(L_2(p^n))| = 2n$. In fact every element of $Out(L_2(p^n))$ is a product of a field automorphism and a diagonal automorphism. Let $\varphi \in G/L_2(q)$. If φ is a field automorphism of order 2, then φ centralizes $L_2(p)$ and so $2 \sim p$ in $\Gamma(G)$, which is a contradiction. If φ is a field-diagonal automorphism of order 2, then $\Gamma(L_2(q)) = \Gamma(G)$ (see [3]), which is impossible, since in $\Gamma(L_2(q))$ we have $\deg(2) < |\pi(L_2(q))| - 2$. Therefore φ is a diagonal automorphism of $L_2(q)$ and so $G = PGL(2,q)$.

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