A PERTURBATIVE ANALYSIS OF NONLINEAR CUBIC-QUINTIC DUFFING OSCILLATORS

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Duffing oscillators comprise one of the canonical examples of Hamilton systems. The presence of a quintic term makes the cubic-quintic Duffing oscillator more complex and interesting to study. In this paper, the homotopy analysis method (HAM) is used to obtain the analytical solution for the nonlinear cubic-quintic Duffing oscillators. The HAM helps to obtain the frequency $\omega$ in the form of approximation series of a convergence control parameter $h$. The valid region of $h$ is determined by plotting the $\omega-h$ curve and afterwards we compared the obtained results with the exact solutions.

Key words: homotopy analysis method, homotopy-Padé technique, oscillator.

1. INTRODUCTION

Duffing equation is used to model the conservative double-well oscillators, which can occur, for example, in magneto-elastic mechanical systems [1]. These systems can be presented in the following form

$$\begin{cases}
\frac{d^2 u}{dt^2} + \alpha u(t) + \beta u^3(t) + \gamma u^5(t) = 0, \\
u(0) = B, \quad u'(0) = 0,
\end{cases}
$$

(1)

where $B$ is the amplitude of the oscillator. Under the initial conditions mentioned above the nonlinear cubic-quintic Duffing oscillator has the exact frequency [2]:

$$\omega = \frac{\pi}{2} \sqrt{\frac{\alpha + \frac{3}{2} B^2 + \frac{B^4}{4}}{}}.$$

(2)

$$M = \frac{3\beta B^2 + 2\gamma B^4}{6\alpha + 3\beta B^2 + 2\gamma B^4},$$

$$N = \frac{2\gamma B^4}{6\alpha + 3\beta B^2 + 2\gamma B^4}.$$

Eq. (1) describes an oscillator with an unknown frequency $\omega$. Under the transformations

$$\tau = \omega t, \quad u(t) = BU(\tau),$$

(3)

the original Eq. (1) becomes
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\[
\begin{align*}
\omega^3 \frac{d^2U}{d\tau^2} + \alpha U(\tau) + B^2\beta U^3(\tau) + B^4\gamma U^5(\tau) &= 0, \\
U(0) &= 1, \quad U'(0) = 0.
\end{align*}
\]

(4)

In the present paper, we use the homotopy analysis method (HAM) to obtain the periodic solutions of the nonlinear cubic-quintic Duffing oscillators. It is to be noted that Liao [3] employed the basic ideas of the homotopy to overcome the restrictions of traditional techniques [4–18] namely the HAM. Notice that the HAM contains an auxiliary parameter \( h \) which provides a convenient way to control the convergence region and the rate of approximation series. Liao investigated the influence of \( h \) on the convergence of solution series by means of \( h \)-curves [3].

2. APPLICATION OF HOMOTOPY ANALYSIS METHOD

The periodic solution of \( U(\tau) \) with the frequency \( \omega \) can be written as

\[
U(\tau) = \sum_{m=0}^{\infty} c_m \cos(m\tau),
\]

(5)

where \( c_m \) are coefficients. It is convenient to choose

\[
U_0(\tau) = \cos \tau,
\]

(6)

as the initial guess of \( U(\tau) \). Let \( \omega_0 \) denotes the initial guess of \( \omega \), then we choose the auxiliary linear operator

\[
L[\varphi(\tau;q)] = \omega_0^2 \left[ \frac{d^2 \varphi(\tau;q)}{d\tau^2} + \varphi(\tau;q) \right],
\]

(7)

with the property

\[
L \left[ C \cos \tau + C \sin \tau \right] = 0,
\]

(8)

where \( C_1 \) and \( C_2 \) are coefficients. We define a nonlinear operator

\[
N[\varphi(\tau;q),\Omega(q)] = \Omega^2(q) \frac{d^2 \varphi(\tau;q)}{d\tau^2} + \alpha \varphi(\tau;q) + B^2\beta \varphi^3(\tau;q) + B^4\gamma \varphi^5(\tau;q).
\]

(9)

Let \( h \) denotes a nonzero auxiliary parameter and \( H(\tau) \) a nonzero auxiliary function. Then, we construct the zero-order deformation equation

\[
(1 - q)L[\varphi(\tau;q)] = qhH(\tau)N[\varphi(\tau;q),\Omega(q)], \quad q \in [0,1],
\]

(10)

such that

\[
\varphi(0,q) = 1, \quad \frac{\partial \varphi(\tau;q)}{\partial \tau} \bigg|_{\tau=0} = 0.
\]

(11)

When \( q \in [0,1] \), the solution \( \varphi(\tau;q) \) varies from \( U_0(\tau) \) to \( U(\tau) \) so does \( \Omega(q) \) from \( \omega_0 \) to \( \omega \). The Taylor’s series with respect to \( q \) can be constructed for \( \varphi(\tau;q) \) and \( \Omega(q) \) and if these two series are convergent at \( q = 1 \), we have:

\[
U(\tau) = U_0(\tau) + \sum_{j=0}^{\infty} U_j(\tau), \quad \omega = \omega_0 + \sum_{j=0}^{\infty} \omega_j,
\]

(12)

where

\[
U_m(\tau) = \frac{\partial^m \varphi(\tau;q)}{m! \partial q^m} \bigg|_{q=0}.
\]

(13)

and

\[
\omega_m = \frac{\partial^m \Omega(q)}{m! \partial q^m} \bigg|_{q=0}.
\]

(14)
Differentiating the zero-order deformation Eq. (10) and Eq. (11) \( m \) times with respect to embedding parameter \( q \) and then dividing them by \( m! \) and finally setting \( q = 0 \), we have the so-called \( m \)th order deformation equation

\[
L[U_m'(\tau) - \chi_m U_{m-1}(\tau)] = hR_m(U_0, \omega_0, \ldots, U_{m-1}, \omega_{m-1}),
\]

\[U_m(0) = 0, \quad U_m'(0) = 0, \quad m \geq 1,
\]

where

\[
R_m(U_0, \omega_0, \ldots, U_{m-1}, \omega_{m-1}) = \frac{\partial^{m-1} N[\varphi(\tau; q), \Omega(q)]}{(m-1)! \partial q^{m-1}},
\]

Note that \( U_m, \omega_m \) are all unknown, but we have only Eq. (15) for \( U_m \), thus an additional algebraic equation is required for determining \( \omega_m \). It is found that the right-hand side of the \( m \)th order deformation Eq. (15) is expressed by

\[
R_m(U_0, \omega_0, \ldots, U_{m-1}, \omega_{m-1}) = \sum_{k=0}^{\psi(m)} c_{m,k}(\omega_{m-1})\cos((2k + 1)\tau),
\]

where \( c_{m,k} \) is a coefficient and \( \psi(m) \) is a positive integer dependent on order \( m \).

If \( R_m(U_0, \omega_0, \ldots, U_{m-1}, \omega_{m-1}) \) contains the term \( \cos \tau \) then the solution of Eq. (15) involves the so-called secular term \( \tau \cos \tau \) that this disobey the rules of solutions expression, thus coefficient \( c_{m,0} \) must be enforced to be zero. This provides with the additional algebraic equation for determining \( \omega_{m-1} \)

\[
c_{m,0}(\omega_{m-1}) = 0.
\]

Consequently, we obtain

\[
U_m(\tau) = \chi_m U_{m-1}(\tau) + \frac{h}{\omega_0^2} \sum_{j=1}^{\psi(m)} \frac{c_{m,j}(\omega_{m-1})}{1 - (2j + 1)^2} \cos[(2j + 1)\tau] + C_1 \cos \tau + C_2 \sin \tau,
\]

where \( C_1, C_2 \) are two coefficients and to be determined by conditions \( U_m(0) = 0 \) and \( U_m'(0) = 0 \).

Thus the \( N \) th order approximation can be given by

\[
U(\tau) = U_0(\tau) + \sum_{j=0}^{N} U_j(\tau),
\]

and

\[
\omega = \omega_0 + \sum_{j=0}^{N} \phi_j.
\]

### 3. NUMERICAL RESULTS AND DISCUSSION

For given \( \alpha = \beta = \gamma = 1 \) the amplitude \( \omega_{m-1} \) can be determined by the analytic approach mentioned above. For \( m = 1, 2 \) we have:

\[
\begin{align*}
\omega_0 &= \frac{1}{\sqrt{10B^4 + 12B^2 + 16}}, \\
\omega_1 &= \frac{1}{96\sqrt{10B^4 + 12B^2 + 16}(5B^4 + 6B^2 + 8)} \left(B^2 h(48 + 100B^2 + 65B^6 + 96B^4)\right).
\end{align*}
\]
Note. The obtained results contain the auxiliary parameter \( h \). It is found that convergence regions of the approximation series are dependent upon \( h \) [3]. For example, consider cases \( B = 0.1, 0.15 \). We plotted the \( \omega - h \) curve to determine the so-called valid region of \( h \), as shown in Fig. 1. Obviously, the valid regions of \( h \) for \( B = 0.1, 0.15 \) are \(-4 < h < 4 \) and \(-3.5 < h < 3.5 \), respectively, for instance for \( B = 0.1 \) we have the result \( \omega = 1.003733 \) as shown in Table 1.

Also, we consider cases \( B = 1, 1.1 \). We plotted the \( \omega - h \) curve to determine the so-called valid region of \( h \) as shown in Fig. 2. It is shown that the valid regions of \( h \) for \( B = 1, 1.1 \) are \(-1 < h < 0.9 \) and \(-0.9 < h < 0.8 \), respectively. Furthermore for \( B = 1 \), we have result \( \omega = 1.538669 \) as shown in Table 1. In Table 2, we compared the 14th order approximations of HAM with the exact solutions.

### Table 1

<table>
<thead>
<tr>
<th>( M )</th>
<th>( B = 0.1(\omega = 1.003770) )</th>
<th>( B = 1(\omega = 1.523590) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1.003753</td>
<td>1.538663</td>
</tr>
<tr>
<td>8</td>
<td>1.003728</td>
<td>1.538669</td>
</tr>
<tr>
<td>9</td>
<td>1.003733</td>
<td>1.538669</td>
</tr>
<tr>
<td>10</td>
<td>1.003732</td>
<td>1.538695</td>
</tr>
<tr>
<td>11</td>
<td>1.003733</td>
<td>1.538669</td>
</tr>
<tr>
<td>12</td>
<td>1.003733</td>
<td>1.538669</td>
</tr>
<tr>
<td>13</td>
<td>1.003733</td>
<td>1.538669</td>
</tr>
<tr>
<td>14</td>
<td>1.003733</td>
<td>1.538669</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>( B )</th>
<th>HAM</th>
<th>Exact</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>1.035516</td>
<td>1.035540</td>
<td>2.32\times10^{-3}</td>
</tr>
<tr>
<td>0.5</td>
<td>1.107092</td>
<td>1.106540</td>
<td>4.99\times10^{-4}</td>
</tr>
<tr>
<td>3</td>
<td>7.636706</td>
<td>7.268630</td>
<td>5.06\times10^{-2}</td>
</tr>
<tr>
<td>5</td>
<td>20.256660</td>
<td>19.181500</td>
<td>5.60\times10^{-2}</td>
</tr>
<tr>
<td>8</td>
<td>51.078098</td>
<td>48.294600</td>
<td>5.75\times10^{-2}</td>
</tr>
<tr>
<td>10</td>
<td>79.536112</td>
<td>75.177400</td>
<td>5.79\times10^{-2}</td>
</tr>
<tr>
<td>20</td>
<td>316.703160</td>
<td>299.223000</td>
<td>5.84\times10^{-2}</td>
</tr>
<tr>
<td>50</td>
<td>1976.897968</td>
<td>1867.570000</td>
<td>5.85\times10^{-2}</td>
</tr>
</tbody>
</table>

Fig. 1 – The \( \omega - h \) curve for \( B = 0.1, 0.15 \).

Fig. 2 – The \( \omega - h \) curve for \( B = 1, 1.1 \).
3.1. Square residual error

We obtained the constant \( h \) using the least square method. In theory, at the \( M \) th-order of approximation, we can define the exact square residual error

\[
\Delta_M(h) = \int_D \left( N \left( \sum_{k=0}^{M} U_k(\tau) \omega_k \right) \right)^2 d\tau. \tag{24}
\]

Clearly, the more rapidly \( \Delta_M(h) \) decreases to zero, the faster the approximation series converges.

Remark 1. The curve of \( \Delta_{10}(h) \) versus \( h \) at \( B = 0.1 \) is shown in Fig. 3, which indicates that the optimal values of \( h \) is about -1.5.

Remark 2. The curve of \( \Delta_{10}(h) \) versus \( h \) at \( B = 1 \) is shown in Fig. 4, which indicates that the optimal values of \( h \) is about -0.78.

4. HOMOTOPY-PADÉ TECHNIQUE

The Padé technique is widely applied to enlarge the convergence region and convergence rate of given series. The so-called homotopy-Padé technique was suggested by means of combining the Padé technique with HAM.

For a given series

\[
S_k(q) = \sum_{j=0}^{k} c_j q^j, \tag{25}
\]

the corresponding \([n,m]\) Padé approximate is expressed by

\[
S_{n,m}(q) = \frac{C_{n,m}(q)}{D_{n,m}(q)} = \frac{\sum_{j=0}^{n} c_j q^j}{1 + \sum_{j=1}^{m} d_j q^j}, \tag{26}
\]

where \( c_j, d_j \) can be determined by solving the linear system:
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\[
\begin{align*}
\omega_j + \sum_{j=0}^{i-1} \omega_{ij} d_{i-j} &= c_j, \quad i = 0, 1, \ldots, n, \\
\omega_j + \sum_{j=i-m}^{i-1} \omega_{ij} d_{i-j} &= 0, \quad i = n + 1, \ldots, n + m.
\end{align*}
\]

(27)

Setting \( q = 1 \) provides the \([n, m]\) homotopy-Padé approximation

\[
\omega^{(n,m)} = S_{n+m} (1) = \sum_{j=0}^{n+m} \omega_j = \frac{C_{n,m} (1)}{D_{n,m} (1)} = \frac{\sum_{j=0}^{n} c_j}{1 + \sum_{j=1}^{n} d_j},
\]

(28)

which accelerate the convergence rate of solution series of HAM. We have applied the homotopy-Padé technique to accelerate the convergence rate of \( M \)th-order approximations of HAM. In Table 3 we compared the approximations of homotopy-Padé technique with exact solutions.

<table>
<thead>
<tr>
<th>( B = 0.1 )</th>
<th>( B = 0.3 )</th>
<th>( B = 1 )</th>
<th>( B = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega^{(0,1)} / \omega_0 )</td>
<td>1.0037</td>
<td>1.00482</td>
<td>1.01757</td>
</tr>
<tr>
<td>( \omega^{(2,1)} / \omega_x )</td>
<td>1.00015</td>
<td>1.00326</td>
<td>1.01317</td>
</tr>
<tr>
<td>( \omega^{(1,1)} / \omega_x )</td>
<td>1.00011</td>
<td>1.00232</td>
<td>1.01219</td>
</tr>
<tr>
<td>( \omega^{(4,1)} / \omega_x )</td>
<td>1.00013</td>
<td>1.00236</td>
<td>1.01087</td>
</tr>
<tr>
<td>( \omega^{(3,1)} / \omega_x )</td>
<td>1.00010</td>
<td>1.00219</td>
<td>1.01058</td>
</tr>
<tr>
<td>( \omega^{(2,2)} / \omega_0 )</td>
<td>1.00010</td>
<td>1.00168</td>
<td>1.00876</td>
</tr>
<tr>
<td>( \omega^{(1,1)} / \omega_0 )</td>
<td>1.00010</td>
<td>1.00143</td>
<td>1.00781</td>
</tr>
</tbody>
</table>

5. CONCLUSIONS

In this paper, the HAM is presented to calculate the frequency and the solution of the nonlinear cubic-quintic Duffing oscillators. According to obtained results, the HAM and homotopy-Padé technique could give efficient frequency approximations for the nonlinear cubic-quintic Duffing oscillators. It is worth mentioning that nonlinear cubic-quintic oscillator models arise in many areas of nonlinear science, e.g., in the study of optical solitons [19]-[20]; thus in nonlinear optics the cubic-quintic Ginzburg-Landau partial differential equation is a generic nonlinear dynamical model describing optical soliton propagation in laser cavities, see, for example, Ref. [21].

REFERENCES


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