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SOLUTIONS OF THE TELEGRAPH EQUATIONS USING A FRACTIONAL CALCULUS APPROACH

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In this paper, the fractional differential equation for the transmission line without losses in terms of the fractional time derivatives of the Caputo type is considered. In order to keep the physical meaning of the governing parameters, new parameters σ and α were introduced. These parameters characterize the existence of the fractional components in the system. A relation between these parameters is also reported. Fractional differential equations are examined with both temporal and spatial fractional derivatives. We show a few illustrative examples when the wave periodicity is broken in either temporal or spatial variables. Finally, we present the output of numerical simulations that were performed with both temporal and spatial fractional derivatives.

Key words: transmission line, fractional differential equations, Mittag-Leffler function.

1. INTRODUCTION

The Fractional Calculus (FC) is the generalization of the ordinary integrals and derivatives of integer orders to arbitrary ones. Mathematical and physical considerations in favor of the use of models based on derivatives of non-integer order are given in [1–21] and the references therein. The authors of Ref. [22] present a numerical method for solving a class of fractional differential equations based on Bernstein polynomials basis; these matrices are utilized to reduce the multi-term orders fractional differential equation to a system of algebraic equations. In Ref. [23] it was established the double Laplace formulas for the partial fractional integrals and derivatives in the sense of Caputo. The telegraph equations are a pair of linear partial differential equations which describe the evolution of voltage and current on an electrical transmission line with distance and time. The telegraph equations are due to Oliver Heaviside who in the 1880s developed the transmission line model. This model demonstrates that the electromagnetic waves can be reflected on the wire, and that appear wave patterns along the transmission line. Recently, the fractional telegraph equations were considered by many authors [24–30]. Unlike the work of the authors mentioned above, in which the passing from an ordinary derivative to a fractional one is direct, here we first analyze the ordinary derivative operator and try to bring it to the fractional form in a consistent manner [31].

This paper is organized as follows: In the second section we present the basic definitions of fractional calculus. The third section presents the fractional transmission line and the methodology proposed. Fractional differential equations are examined separately, with temporal and spatial derivative, respectively. Finally, we show the complete solution and numerical simulations by taking simultaneously both derivatives (time-space derivatives). In the fourth section we depict our conclusions.

2. BASIC DEFINITIONS

The definitions of the fractional order derivative are not unique and there exist several definitions, including the Riemann-Liouville, the Grünwald-Letnikov, and the Caputo representation for Fractional Derivative (CFD). In the Caputo case, the derivative of a constant is zero and the initial conditions for the fractional differential equations have a known physical interpretation; this is important from the physical and engineering point of view. For a function *f(t)*, the CFD is defined as [3]

$$ {}_{o}^{C}D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\gamma-n+1}} \mathrm{d}\tau. \tag{1}$$

In this case, $0 < \gamma \leq 1$, is the order of the fractional derivative, n=1,2,…, N and n–1< $\gamma \leq n$.

The Laplace transform to CFD is given by [3]

$$ L\left[{}_{o}^{C}D_t^{\gamma} f(t)\right] = S^{\gamma} F(s) - \sum_{k=0}^{m-1} S^{\gamma-k-1} f^{(k)}(0). \tag{2}$$

The Mittag-Leffler function has caused extensive interest among physicists due to its vast potential of applications describing realistic physical systems with memory and delay. The Mittag-Leffler function is defined by

$$ E_a(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(am+1)}, \quad (a > 0), \tag{3}$$

where $\Gamma(.)$ is the gamma function. If a=1, from (3), we obtain e^t, the exponential function as a special case of the Mittag-Leffler function [3].

3. THE FRACTIONAL TRANSMISSION LINE

The idea is replace the time derivative operator $\frac{\mathrm{d}}{\mathrm{d}t}$ by a new fractional operator $\frac{\mathrm{d}^{\gamma}}{\mathrm{d}t^{\gamma}}$ (γ represents the order of the derivative). The proposed alternative is introducing an additional parameter, which must have dimension of seconds to be consistent with the dimension of the ordinary derivative operator. Thus, we replace the ordinary time derivative operator by the fractional one as:

$$ \frac{\mathrm{d}}{\mathrm{d}t} \rightarrow \frac{1}{(\sigma_t)^{1-\gamma}} \frac{\mathrm{d}^{\gamma}}{\mathrm{d}t^{\gamma}}, \qquad 0 < \gamma \leq 1, \tag{4}$$

where $0 < \gamma \leq 1$, and, σ_t is a new parameter representing the fractional time components in the system (components that show an intermediate behavior between a conservative and a dissipative system); it has the dimension of time and can be called “cosmic time”; this is a non-local time [32]. In the case γ=1 the expression (4) becomes an ordinary time derivative operator

$$ \left[\frac{1}{(\sigma_t)^{1-\gamma}} \frac{\mathrm{d}^{\gamma}}{\mathrm{d}t^{\gamma}}\right]_{\gamma=1} = \frac{\mathrm{d}}{\mathrm{d}t}, \qquad 0 < \gamma \leq 1. \tag{5}$$

In the case of the spatial derivative operator $\frac{\mathrm{d}}{\mathrm{d}x}$, we can replace the ordinary derivative by the fractional spatial derivative as follows:

$$ \frac{\mathrm{d}}{\mathrm{d}x} \rightarrow \frac{1}{(\alpha_x)^{1-\beta}} \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}, \qquad 0 < \beta \leq 1, \tag{6}$$

in this case, the parameter α_x has the dimension of length and it represents the spatial fractional component. When β=1 the expression (6) becomes a classical space derivative operator. In the following, we will apply this idea to investigate the case of the transmission line.

Fractional time transmission line equation. Using (4) the transmission line equations [24] can be written in terms of the fractional time derivatives as:

$$\frac{\partial^2}{\partial x^2}V(x,t)-\frac{LC}{(\sigma_t)^{2(1-\gamma)}}\frac{\partial^{2\gamma}}{\partial t^{2\gamma}}V(x,t)=0, \qquad 0<\gamma\leq 1, \tag{7}$$

$$\frac{\partial^2}{\partial x^2}I(x,t)-\frac{LC}{(\sigma_t)^{2(1-\gamma)}}\frac{\partial^{2\gamma}}{\partial t^{2\gamma}}I(x,t)=0, \qquad 0<\gamma\leq 1. \tag{8}$$

A particular solution of (7) is

$$V(x,t)=V_0\mathrm{e}^{-\mathrm{i}kx}u(t), \tag{9}$$

where k is the wave vector in the direction x and V_0 is the initial voltage. Substituting (9) in (7) we obtain

$$\frac{\mathrm{d}^{2\gamma}u(t)}{\mathrm{d}t^{2\gamma}}+\omega^2 u(t)=0, \tag{10}$$

where

$$\omega_\gamma^2=\frac{k^2}{LC}\sigma_t^{2(1-\gamma)}. \tag{11}$$

The solution of the equation (10) reads as

$$u(t)=E_{2\gamma}\left\{-\omega^2 t^{2\gamma}\right\}. \tag{12}$$

Substituting the expression (12) in (9) we obtain a particular solution of the equation as

$$V(x,t)=V_0\mathrm{e}^{-\mathrm{i}kx}E_{2\gamma}\left\{-\omega_\gamma^2 t^{2\gamma}\right\}. \tag{13}$$

First case. When $\gamma = 1$, the solution of (7) follows from (13) and it is written by

$$V(x,t)=\mathrm{Re}\,\tilde{V}_0\mathrm{e}^{-\mathrm{i}(\omega t-kx)}. \tag{14}$$

The solution (14) represents a monochromatic wave respect to t and x (Fig. 1).

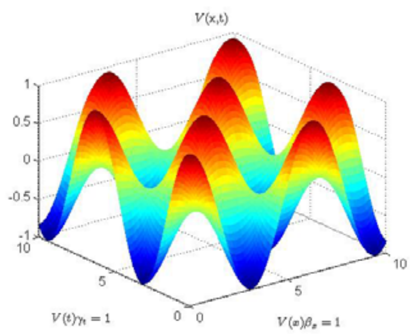

Fig. 1 – Periodic wave with respect to t and x.

Second case. When $\gamma=\frac{1}{2}$ the equation (7) is written as

$$\frac{\partial^2}{\partial x^2}V(x,t)-\frac{LC}{\sigma_t}\frac{\partial}{\partial t}V(x,t)=0, \qquad 0<\gamma\le 1. \tag{15}$$

The solution can be found in the form (9), then we obtain the following equation for *u(t)*

$$\frac{\mathrm{d}u}{\mathrm{d}t}+\omega^2 u(t)=0, \tag{16}$$

where, in this case $\omega^2_{1/2}=\sigma_t$ follows from (11). The solution of the equation (16) is obtained in terms of the Mittag-Leffler function. Then, when $\gamma=\frac{1}{2}$, we have: $u(t)=E_1\left\{-\omega^2 t\right\}=\mathrm{e}^{-\omega^2 t}$.

The particular solution has the form

$$V(x,t)=V_0\mathrm{e}^{-\omega^2 t}\mathrm{e}^{-ikx}. \tag{17}$$

We notice that, for this case the solution is periodic only regarding *x,* and it is not periodic regarding *t*. The solution represents a plane wave with exponential time-decaying amplitude. This is a direct consequence of the fractional time derivative [33]. On the other hand, we have that the velocity is given by $V=\frac{1}{\sqrt{LC}}$, and the angular frequency $\omega=k\upsilon$, then $\omega=\frac{k}{\sqrt{LC}}$, if we take: $\gamma=\sigma_t\omega=\frac{\sigma_t k}{\sqrt{LC}}$.

It is important to see that there exists a direct relation between the parameter σ_t and the period T_0 given by the order γ of the differential equation

$$\gamma=\sigma_t\omega=\frac{\sigma_t}{T_0}, \quad 0<\sigma_t\le T_0. \tag{18}$$

Taking into account the relation (20), the solution (15) can be written as

$$V(x,\tilde{t})=V_0\mathrm{e}^{-ikx}E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}\tilde{t}^{2\gamma}\right\}, \tag{19}$$

where, $\tilde{t}=\frac{t}{T_0}$, is a dimensionless parameter, and $T_0=\frac{\sqrt{LC}}{k}$ is the wave's period. The wave's periodicity is broken in the region: $0<\sigma_t<\frac{\sqrt{LC}}{k}<T_0$.

Fig. 2 shows the loss of periodicity of the wave with respect to *t*.

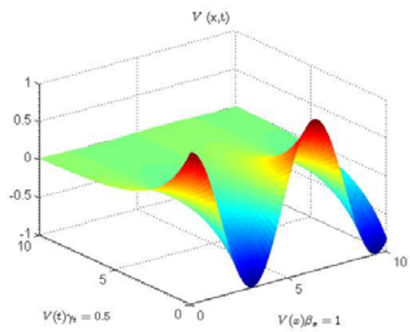

Fig. 2 – Periodic wave with respect to *x*, the periodicity of the wave is broken in *t*.

Fractional space transmission line equation. Now will consider the equation (7) assuming that the spatial derivative is fractional (6) and the time derivative is integer. Then, we have the spatial fractional equation

$$\frac{1}{\alpha_x^{2(1-\beta)}}\frac{\partial^{2\beta}}{\partial x^{2\beta}}V(x,t)-LC\frac{\partial^2}{\partial t^2}V(x,t)=0, \qquad 0<\beta\le 1, \tag{20}$$

where the order of the spatial fractional differential equation is represented by $0<\beta\le 1$, and α_x has dimension of length. A particular solution to the equation (20) may be as follows:

$$V(x,t)=V_0\mathrm{e}^{\mathrm{i}\omega t}u(x), \tag{21}$$

and substituting (21) in (20), we obtain

$$\frac{\mathrm{d}^{2\beta}u(x)}{\mathrm{d}t^{2\beta}}+\tilde{k}^2u(x)=0, \tag{22}$$

where $\tilde{k}^2=\omega^2LC\alpha_x^{2(1-\beta)}=k^2\alpha_x^{2(1-\beta)}$ is the wave vector in the medium in presence of fractional components and k is the wave vector without its presence. The wave vectors are equal, $\tilde{k}=k$, only in the case, $\beta = 1$, when do not exist fractional components. Using (13) the solution to the equation (21) is given in terms of the Mittag-Leffler function

$$u(x)=E_{2\beta}\left\{-\tilde{k}^2x^{2\beta}\right\}=\sum_{n=0}^{\infty}\frac{\left(-\tilde{k}^2x^{2\beta}\right)^n}{\beta\left(2n\beta+1\right)}. \tag{23}$$

Therefore, by placing (23) in (21) we obtain the solution

$$\tilde{V}(x,t)=\tilde{V}_0\mathrm{e}^{\mathrm{i}\omega t}E_{2\beta}\left\{-\tilde{k}^2x^{2\beta}\right\}. \tag{24}$$

First case. For the fractional space case, when $\beta = 1$, the solution follows from

$$\tilde{V}(x,t)=\mathrm{Re}\,\tilde{V}_0\mathrm{e}^{\mathrm{i}(\omega t-kx)}, \tag{25}$$

with $\tilde{k}=k=\omega\sqrt{LC}$, where k is the component of the wave vector in the x direction and is related with the wavelength by $k=\dfrac{1}{\lambda}$.

The solution (25) represents a monochromatic wave with respect to t and x, (Fig. 3).

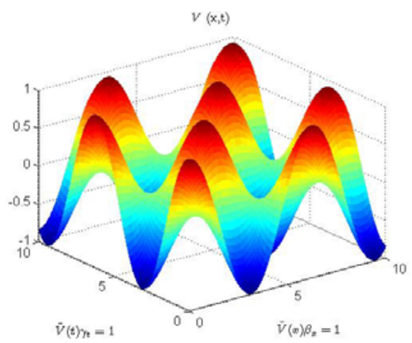

Fig. 3 – Periodic wave with respect to t and x.

Second case. We have $\beta=\dfrac{1}{2}$, $\tilde{k}^2=k^2\alpha_x=\omega LC\alpha_x$, and $\left[\tilde{k}^2\right]=\dfrac{1}{l}$ has dimensions of the inverse of the length. From (23) we conclude that $u(x)=E_1\left\{-\tilde{k}^2x\right\}=\mathrm{e}^{-\tilde{k}^2x}$.

The solution (24) is written as

$$\tilde{V}(x,t)=\tilde{V}_0\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\tilde{k}^2x}. \tag{26}$$

This wave is periodic only with respect to t, but not with respect to x, so in this case the periodicity is lost. In this case, a direct relation between α_x and the wavelength λ given by β is described by: $\beta = k\alpha_x = \frac{\alpha_x}{\lambda}$, for $0 < \alpha_x \le \lambda$. We can use this relation in order to write the equation (24) as follows

$$\tilde{V}(x,t) = \tilde{V}_0 \mathrm{e}^{\mathrm{i}\omega t} E_{2\beta}\left\{-\beta^{2(1-\beta)}\tilde{x}^{2\beta}\right\}. \tag{27}$$

Here $\tilde{x} = \frac{x}{\lambda}$ is a dimensionless parameter. Fig. 4 shows the loss of periodicity of the wave with respect to x.

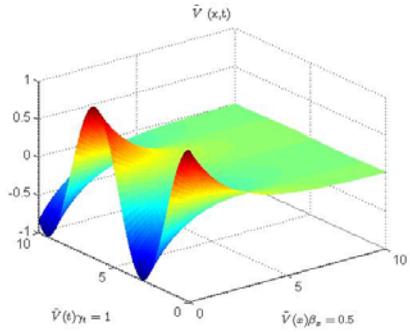

Fig. 4 – Periodic wave with respect to t, the periodicity of the wave is broken in x.

Fractional time-space transmission line equation. Now we consider the equations (9) and (10) assuming that in time and space the derivative are fractional, (Eqs. (4) and (6)). Then, we obtain the time-space fractional equation

$$\frac{1}{(\alpha_x)^{2(1-\beta)}}\frac{\partial^{2\beta}}{\partial x^{2\beta}}V(x,t) - \frac{LC}{(\sigma_t)^{2(1-\gamma)}}\frac{\partial^{2\gamma}}{\partial t^{2\gamma}}V(x,t) = 0, \quad 0<\beta\le 1, \quad 0<\gamma\le 1, \tag{28}$$

$$\frac{1}{(\alpha_x)^{2(1-\beta)}}\frac{\partial^{2\beta}}{\partial x^{2\beta}}I(x,t) - \frac{LC}{(\sigma_t)^{2(1-\gamma)}}\frac{\partial^{2\gamma}}{\partial t^{2\gamma}}I(x,t) = 0, \quad 0<\beta\le 1, \quad 0<\gamma\le 1, \tag{29}$$

where the order of the time-space fractional differential equation is represented by $0 < \beta \le 1$ and $0 < \gamma \le 1$, α_x has dimension of length and σ_t of time. The full solution of the equation (28) is

$$\overline{V}(\overline{x},\overline{t}) = A \cdot E_{2\beta}\left\{-\beta^{2(1-\beta)}\overline{x}^{2\beta}\right\} \cdot E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}\overline{t}^{2\gamma}\right\} = 0, \quad 0<\beta\le 1, \quad 0<\gamma\le 1, \tag{30}$$

where $\overline{x} = kx$, $\overline{t} = \omega t$ are dimensionless parameters and A is a constant. Figs. 5, 6, 7, and 8 show numerical simulations where the fractional time derivative and the spatial fractional derivative are taken at the same time for different particular cases of γ and β.

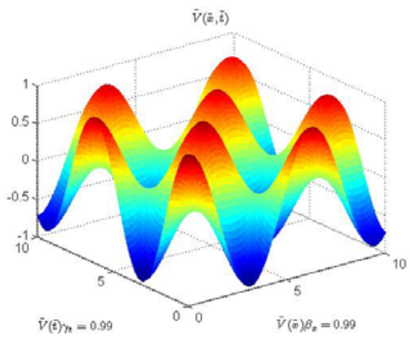

Fig. 5 – Fractional wave with respect to t and x. Here $\gamma = 0.99$ and $\beta = 0.99$.

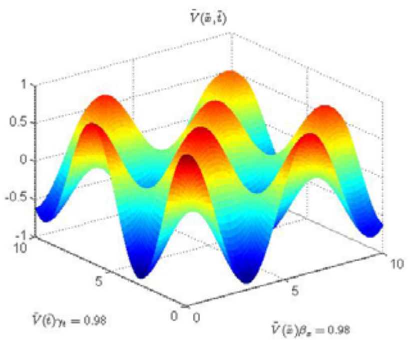

Fig. 6 – Fractional wave with respect to t and x.Here $\gamma = 0.98$ and $\beta = 0.98$.

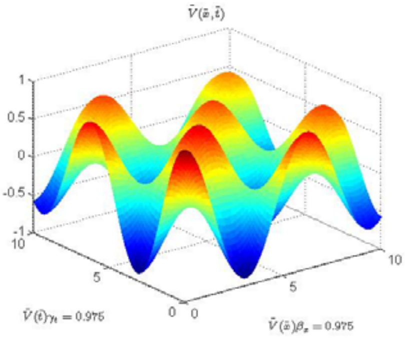

Fig. 7 – Fractional wave with respect to t and x. Here $\gamma = 0.975$ and $\beta = 0.975$.

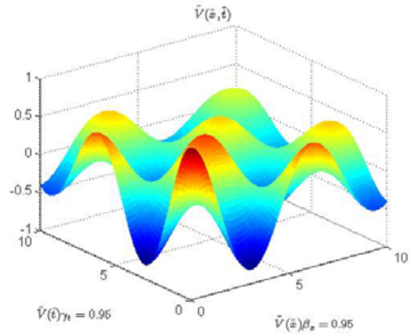

Fig. 8 – Fractional wave with respect to t and x. Here $\gamma = 0.95$ and $\beta = 0.95$.

4. CONCLUSIONS

In this paper we have considered the fractional differential equation for the transmission line without losses in terms of the Caputo fractional derivative (CFD). We have used the idea suggested in [31] to construct the corresponding fractional differential equations. Two new parameters σ_t and α_x were introduced, these parameters representing the components that show an intermediate behavior between conservative and dissipative systems. The general solutions of the CFD, depending only on the parameters γ and β are given in the form of the multivariate Mittag-Leffler functions that preserve the physical units of the studied system. We also show that the periodicity with respect to time t and to space x is broken and the wave behaves like a wave with time-decaying amplitude or spatial-decaying amplitude for the temporal and spatial case, respectively. Also, the equations (19) and (27) have a universal character and they are related with the conditions $\gamma = \sigma_t\omega = \dfrac{\sigma_t}{T_0}$ for $0 < \sigma_t \le T_0$ and $\beta = k\alpha_x = \dfrac{\alpha_x}{\lambda}$ for $0 < \alpha_x \le \lambda$. In the case where both time-space derivatives are simultaneously considered, the equation (30) shows the complete solution by applying the separation of variables method. Besides, it was shown that when the parameters γ and β are less than 1, the wave losses its periodicity with respect to time t and space x. We believe that with this approach it will be possible to have a better understanding of the transient effects in electrical systems and transmission lines.

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