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ANALYTICAL MODELING OF FUNCTIONALLY GRADED PLATES UNDER GENERAL TRANSVERSE LOADS

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In this paper, analytical model of functionally graded plates in the case of small deflections is investigated. The material properties of functionally graded plates (Young's modulus and Poisson's ratio) are assumed to vary continuously through the thickness of the plate, according to a power law distribution of the volume fraction of the constituents. Two types of boundary conditions for plate are presented. The fundamental equations are obtained by the aid of Von-Karman theory considering small transverse deflections. Then, the solution is obtained by minimization of the total potential energy. A comprehensible flowchart of mathematical code is presented to show how the stresses and deflections can be obtained at any point of the plate. The comparison made between results of the current study and other literature shows a good agreement.

Key words: small deflection, non-linear behavior, FGM, power law.

1. INTRODUCTION

Since 1980s functionally graded materials (FGM) have gained significant attention as one of the advanced inhomogeneous composite materials in many engineering applications. FGMs were initially designed as thermal barrier materials for aerospace structural applications and fusion reactors. Currently, FGMs are generally used as structural components in extremely high temperature environments.

With the increased use of these materials, extensive research works have been done on FGMs. A wide range of results on linear behavior of functionally graded plates with different material function models are available in the literature [1–5]. In these studies, various plate theories such as classical or higher order shear deformation theories have been used. However, nonlinear investigations of FGM plates under mechanical loading are limited in number. GhannadPour and Alinia [6] studied large deflection behavior of functionally graded plates under pressure loads. In their studies, the Young's modulus of FGM plates was assumed to vary continuously through the thickness of the plate, according to the simple power law distributions. They also studied large deflection behavior of functionally plates under pressure loads [7]. Yang and Shen [8] investigated the large deflection and post-buckling responses of FGM rectangular plates under transverse and in-plane loads by means of a semi-analytical approach. They assumed material properties to be temperature-dependent, and be graded in thickness direction according to a simple power law distribution in terms of the volume fraction of the constituents. Mechanical behavior of FGM plates under transverse load was studied by Shyang-Ho Chi and Yen-Ling Chung [9]. Woo and Meguid [10] reported an analytical solution for large deflection of thin FGM plates and shallow shells. The material properties were assumed to be temperature independent, however the thermal load due to the one-dimensional steady heat conduction in the plate thickness direction was considered. Alinia and GhannadPour [11] investigated nonlinear analysis of FGM square plates. The plates were subjected to pressure loading and their geometric nonlinearity was introduced in the strain–displacement equations based on Von-Karman assumptions. Alinia and Ghannadpour [11] employed the classical plate theory whereas Khabbaz et al. [12] used higher-order theory. Recently, considering the physical neural surface, Zhang and Zhou [13] and Prakash et al. [14] presented the analytical and finite element analysis of FGM plates, respectively. Singha et al. [15] investigated the nonlinear behaviors of FGM plates under transverse distributed load using a high precision plate bending finite element model. Material properties of the plate were assumed to be graded in the thickness direction according to a simple power- law distribution in terms of volume fractions of the constituents.

In the literature surveyed above, the Young's modulus of FGM plate varies continuously throughout the thickness direction according to the volume fraction of constituents defined by power-law, sigmoid, or exponential function. However, the Poisson's ratios of the FGM plates are assumed to be constant. In this study, small deflection behavior of square plate made of functionally graded material is studied using analytical approach. The material properties of the functionally graded plates such as Young's modulus and Poisson's ratio are assumed to vary continuously through the thickness of the plate, according to the simple power law distribution. The plates are assumed to be both simply supported and clamped along all edges and the Classical Plate Theory (CPT) is applied throughout this work. The solution is obtained by minimization of the total potential energy of the plate. Eventually, the numerical results are presented to discuss the effects of FGM on deflection and stress fields of the plate.

2. FUNDAMENTAL EQUATIONS FOR SMALL DEFLECTION

According to the small deflection assumption, line elements perpendicular to the middle surface of the plate before deformation remain normal and un-stretched after deformation [16]. Consequently the transverse strain components ε_{zz}, γ_{xz} and γ_{yz} are negligibly small. Thus, the displacements at a general point in the *x, y* and *z* directions are

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\frac{\partial w(x,y)}{\partial x}, \\ v(x,y,z) &= v_0(x,y) - z\frac{\partial w(x,y)}{\partial y}, \\ w(x,y,z) &= w_0(x,y), \end{aligned} \tag{1}$$

where $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ are components of displacement at the general point, while $u_0(x,y)$, $v_0(x,y)$ and $w_0(x,y)$ are similar components at the middle surfaces ($z=0$). Using Eq. (1) for in-plane linear strains gives the following expressions for strains at the general point [17]:

$$\varepsilon = \varepsilon_0 + z\delta, \tag{2}$$

where

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \quad \varepsilon_0 = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} \end{Bmatrix}, \quad \delta = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}. \tag{3}$$

Neglecting the transverse shear deformations, leads to significant errors when applied to moderately thick plate with thickness larger than 0.1 of span [16]. However, Shames and Dym [18] indicated that for a plate with the thickness less than 0.1 of its span, the classical plate theory is expected to give reliable results. In this paper, the thickness of the medium-thick FGM plate is assumed to be 0.05 of its span; therefore the transverse shear deformations can be neglected.

We also assume that the plate is in a state of plane stress, and then the stress–strain relationship at a general point for the plate becomes

$$\begin{Bmatrix} \sigma_x \\ \sigma_x \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}, \tag{4}$$

where

$$\begin{aligned} Q_{11} &= Q_{22} = \frac{E(z)}{1-\nu^2}, \\ Q_{12} &= Q_{21} = \frac{\upsilon E(z)}{1-\nu^2}, \\ Q_{66} &= G(z) = \frac{E(z)}{2(1-\nu^2)}. \end{aligned} \tag{5}$$

The axial forces and the bending moments are obtained in the matrix forms as follows [9]:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A].\{e_0\} + [B].\{d\} \tag{6}$$

and

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [B].\{e_0\} + [C].\{d\}. \tag{7}$$

The plate stiffness coefficients (A, B and D) are defined as

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-t/2}^{t/2} Q_{ij}\left(1, z, z^2\right) \mathrm{d}z \qquad i,j = 1,2,6. \tag{8}$$

Since both E and υ vary continuously through the thickness of the plate, according to a power law distribution, the integrations in Eq. (8) will be complicated. Therefore, one of the two following ways can be chosen to represents the formula.

- These equations can be used in integral form, and whenever is needed, are calculated by replacing values for p.
- These equations can be calculated by replacing special values for p.

For example, the components of matrix A for p=1 are computed as:

$$A_{11} = A_{22} = \frac{1}{2u_{12}^2} t \left(\log\left[\frac{u_2-1}{u_1-1}\right] \times \left(E_2(u_1-1) - E_1(u_2-1)\right) + log\left[\frac{u_2+1}{u_1+1}\right] \times \left(E_1(u_2+1) - E_2(u_1+1)\right)\right), \tag{9a}$$

$$\begin{aligned} A_{12} = A_{21} = \frac{1}{2u_{12}^2} t \Bigg(& E_{12} \log\left[\frac{(u_2-1)(u_1+1)}{(u_1-1)(u_2+1)}\right] - 2\left(E_2 u_2 + E_1 u_1\right) + 2\left(E_2 u_1 + E_1 u_1\right) + \\ & + \log\left[\frac{(u_2+1)(u_2-1)}{(u_1+1)(u_1-1)}\right] \times \left(E_2 u_1 - E_1 u_2\right)\Bigg). \end{aligned} \tag{9b}$$

The strain energy per unit volume is $1/2\,\bar{\sigma}^T\bar{\varepsilon}$. Thus, the strain energy of the plate using this equation and Eqs. (3) and (8), can be computed by integrating through the thickness with respect to *z*, which can put into the form:

$$A_{66} = \frac{1}{2u_{12}^2}\,t\left(E_{12}\log\left[\frac{u_2+1}{u_1+1}\right] + \left(E_2u_2 + E_1u_1\right) - \left(E_2u_1 + E_1u_2\right) + \log\left[\frac{u_1+1}{u_2+1}\right]\times\left(E_2u_1 - E_1u_2\right)\right) \quad (9c)$$

$$U = \frac{1}{2}\iint\left\{e^{\mathrm{T}}\left[A\right]e - 2e^{\mathrm{T}}\left[B\right]d + d^{\mathrm{T}}\left[D\right]d\right\}\mathrm{d}x\mathrm{d}y\,. \quad (10)$$

As the external force exists in the current problem, the total potential energy is equal to the summation of strain energy and potential energy of pressure, i.e.

$$V = U + V_p\,. \quad (11)$$

In the above equation, V_p is the potential energy of uniform pressure and is written as:

$$V_p = -\iint q(x,y)\,w\,\mathrm{d}\,x\,\mathrm{d}\,y\,, \quad (12)$$

where, $q(x,y)$ is the general transverse load.

3. MATHEMATICAL CODE

In this section a flowchart of mathematical code is presented. This code can compute the linear and nonlinear deflection and axial stresses of any point of the uniform and triangular transverse loaded FG plate in two mentioned boundary conditions (Fig. 1).

4. RESULTS AND DISCUSSIONS

4.1. Model specification

The analysis of FG plate is conducted for a type of ceramic and metal combination. The set of materials considered is alumina and aluminum. Young's modulus and Poisson's ratio were selected as being 70 GPa and 0.33 for aluminum, and 380 GPa and 0.3 for alumina, respectively. In all cases, the lower surface of the plate is assumed to be metal (aluminum) rich and the upper surface is assumed to be pure ceramic (alumina). The projected length of the plate (a) is 0.2 m and the thickness of plate (t) is considered to be 0.01 m. Two types of load as listed in Table 1 are considered. Fig. 2 clearly illustrates these two types of load on FG plate. As depicted, the magnitude of uniform load is q_0 and the maximum magnitude of the triangular load is $\bar{q}$.

Table 1

Types of load

Uniform transverse load:	$q = q_0$
Triangular transverse load:	$q = \begin{cases} \frac{2\bar{q}}{a}x & 0 < x < \frac{a}{2} \\ -\frac{2\bar{q}}{a}x + 2\bar{q} & \frac{a}{2} < x < a \end{cases}$

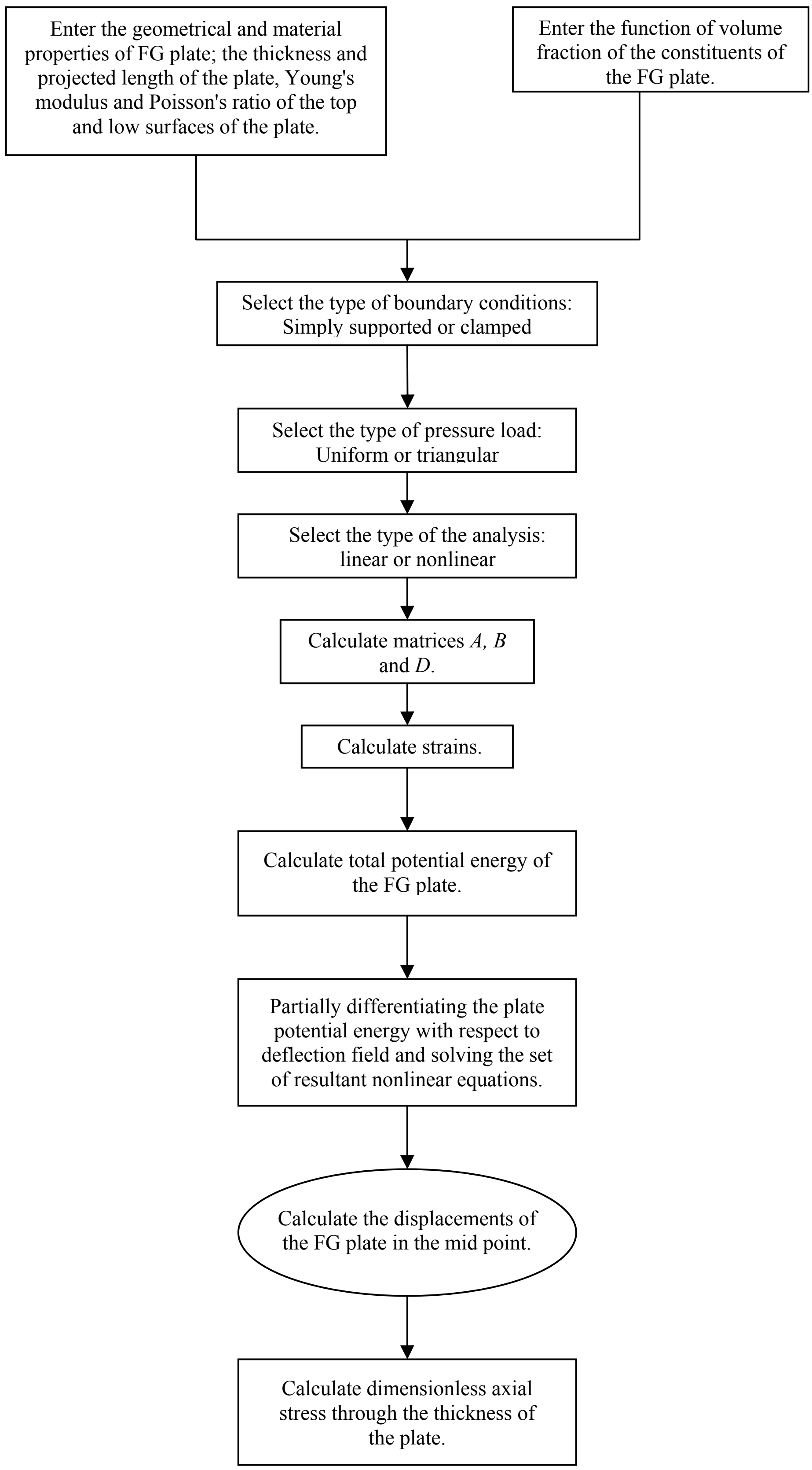

Fig. 1 – Flowchart of the mathematical procedure.

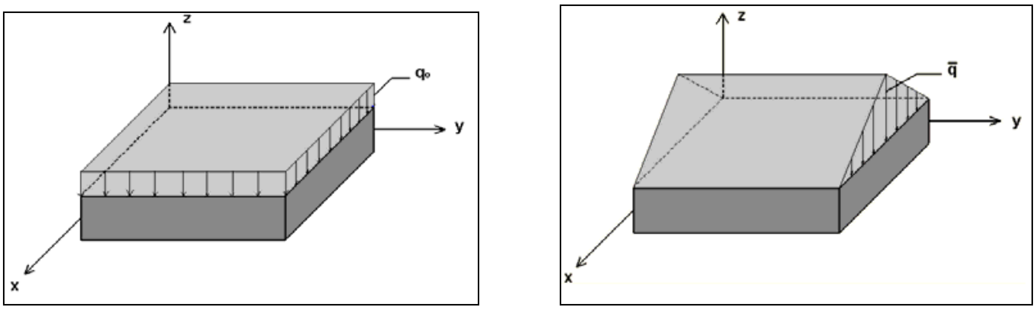

Fig. 2 – FG plate with uniform and triangular transverse load.

4.2. Small deflection

In order to get the meaningful results, the parameters are presented in dimensionless form as listed in Table 2.

Table 2

Dimensionless parameters

	dimensionless parameters
Center deflection	$W = \frac{w}{t}$
Uniform load parameter	$Q = \frac{q\,a^4}{E_2\,t^4}$
Triangular load parameter	$\bar{Q} = \frac{\bar{q}\,a^4}{E_2\,t^4}$
Axial stress	$\sigma = \frac{\sigma_x\,a^2}{E_2\,t^2}$
Thickness coordinate	$Z = \frac{z}{t}$

Fig. 3 shows the volume fraction of the ceramic phase through the dimensionless thickness. Fig. 4 indicates the central small deflection of simply supported FG plate versus uniform and triangular pressure load, respectively. As the overall uniform load is larger than the overall triangular load, then its corresponding central deflection curves behave the same way. In addition, as the linear method is chosen, the load-deflection curve is linear.

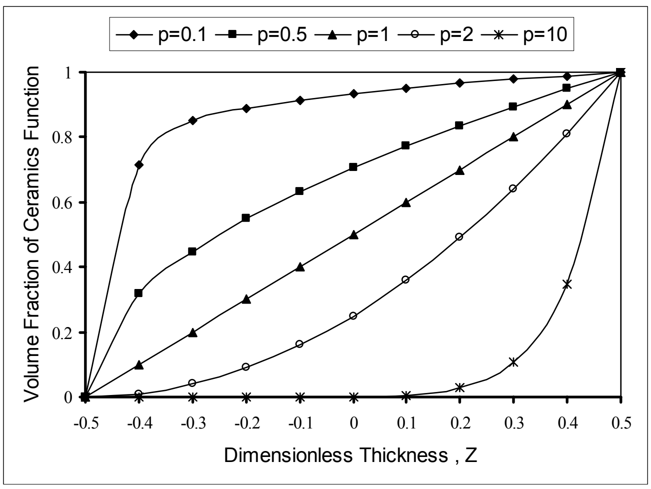

Fig. 3 – Variation of the volume fraction $(Z + 0.5)^p$ through the dimensionless thickness Z.

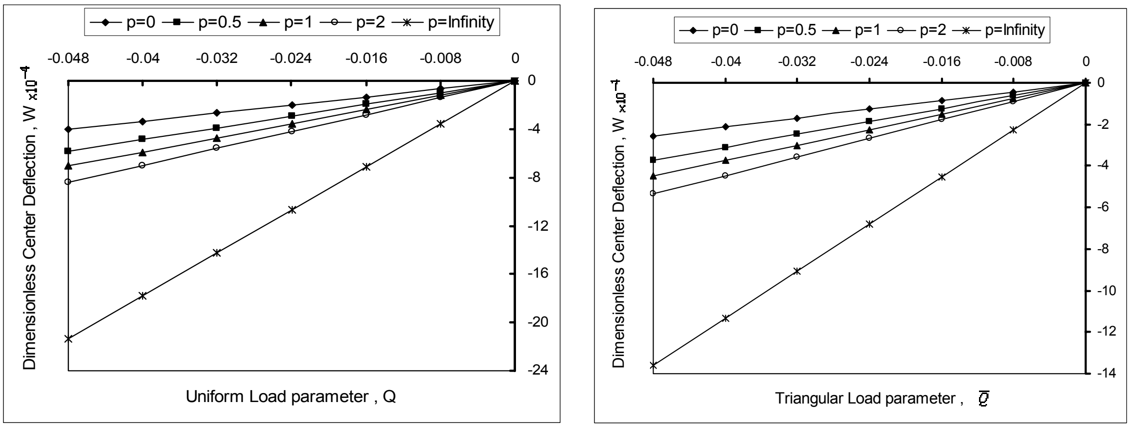

Fig. 4 – Dimensionless central small deflection versus uniform and triangular transverse load for simply supported FG plate.

Fig. 5 illustrates dimensionless axial stress through the thickness Z at the center of the simply supported FG plate under uniform and triangular load parameter, respectively. As illustrated in these two figures, when the material of the structure is isotropic $(p = 0, p = \infty)$, the normal stress is linearly varied at thickness, and also dose not depend on Young's modulus and Poisson's ratio. In other word, the normal stress of isotropic plate relies on geometry, boundaries, and the load conditions. However, at other values of p, the variety of mechanical properties of FGM at thickness, leads to reliance of normal stress on Young's modulus and Poisson's ratio and then the normal stress curves will be nonlinear.

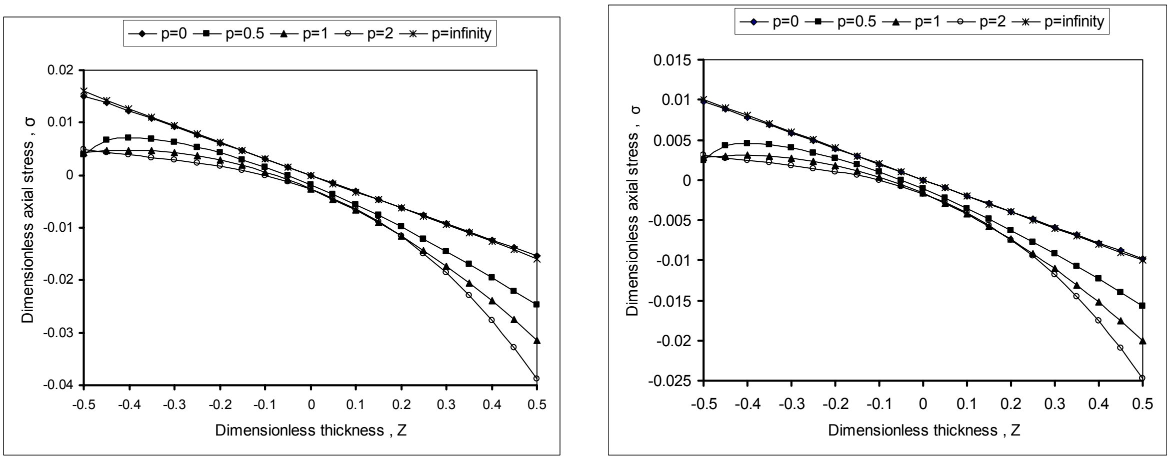

Fig. 5 – Dimensionless axial stress σ through the thickness Z at the center of the simply supported FG plate under uniform and triangular load parameter Q = 0.048.

5. CONCLUSIONS

Small deflection analysis of functionally graded plates was studied. It was assumed that both Young's modulus and Poisson's ratio are varying continuously through the thickness of the plate according to a power law distribution of the volume fraction of the constituents. Two types of boundary conditions were applied. The fundamental equations for the plates of FGM were obtained using the Von Karman theory for small and large transverse deflection and the solution was obtained by minimization of the total potential energy. Also a flowchart of mathematical code was presented for this work, So that the user can obtain the deflection and axial stresses in any point of the FG plate. There was good agreement between the results of Von-Karman and mathematical code. Also, the effects of variable parameter, p, on deflection field were discussed.

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