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QUANTUM BREATHERS IN THE β-FERMI-PASTA-ULAM MODEL

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Based on the time-dependent Hartree approximation and the semidiscrete multiple-scale method, quantum breathers in the β-Fermi-Pasta-Ulam model are investigated analytically. Due to the use of the Hartree approximation, we can deal with the case of quantum breathers with large number of quanta. It is found that, at the Brillouin zone boundary, the equation of motion for the single-phonon wave function has the stationary localized solution whose eigenfrequency is above the top of the harmonic wave band. Using these stationary localized single-phonon wave functions, we have constructed localized n -phonon Hartree states. The result shows that such n -phonon Hartree states can be considered as quantum breathers states. Furthermore, we obtain the energy level formula of quantum breathers, which suggests the energy of such quantum breathers is quantized.

Key words: discrete breathers, quantum breathers, Fermi-Pasta-Ulam model.

1. INTRODUCTION

In quantum regime, nearly degenerate many quanta bound states are considered as quantum breathers [1]. Although these quantum excitations are extended states in a translationally invariant system, they are characterized by exponentially localized weight functions in full analogy to their classical counterparts, i.e., discrete breathers [2]. In recent years，some works have been devoted to the study of quantum breathers in nonlinear lattice models, such as Klein-Gordon (KG) model and Fermi-Pasta-Ulam (FPU) model [3–9]. However, these works are focused on quantum breathers with two-quanta, which is the simplest case. Compared with biphonons, it is more worthwhile to study quantum breathers with more than two-quanta or a large number of quanta. Unfortunately, little attention has been paid to the study of these cases.

In this paper, we shall study n -quanta quantum breathers in the β-FPU model. First, we quantize β-FPU model by introducing local phonon creation and annihilation operators, and retain only number conserving terms. Next, we choose a general n -phonon state as system state vector and use the time-dependent Hartree approximation. Then, with the help of the time-dependent variational method, we obtain an equation of motion for the single-phonon wave function. For the purpose of solving the equation of motion, we will employ the semidiscrete multiple-scale method. Details are presented in the following text.

2. THE β-FPU MODEL AND ITS QUANTIZATION

The Hamiltonian operator for the β-FPU model is given by

$$H=\sum_{j=1}^{f}\left[\frac{p_j^2}{2m}+\frac{K_2}{2}\left(x_{j+1}-x_j\right)^2+\frac{K_4}{4}\left(x_{j+1}-x_j\right)^4\right], \qquad (1)$$

where p_j and x_j are momentum and displacement operators satisfying the commutation relations $[x_j, p_{j'}]=\mathrm{i}\hbar\delta_{jj'}$, f is the number of sites, m is the mass of the particles, and $K_2(>0)$ and $K_4(>0)$ are the harmonic and anharmonic force constants, respectively.

For a periodic lattice system, by assuming $f \to \infty$, one can use the following auxiliary relations [7]:

$$\sum_{j=1}^{f}\left(x_{j+1}-x_{j}\right)^{2}=\frac{1}{2}\sum_{j=1}^{f}\left(x_{j+1}-x_{j}\right)^{2}+\frac{1}{2}\sum_{j=1}^{f}\left(x_{j}-x_{j-1}\right)^{2}=2\sum_{j=1}^{f}x_{j}^{2}-\sum_{j=1}^{f}x_{j}\left(x_{j+1}+x_{j-1}\right), \tag{2}$$

and

$$\begin{aligned}\sum_{j=1}^{f}\left(x_{j+1}-x_{j}\right)^{4}&=\frac{1}{2}\sum_{j=1}^{f}\left(x_{j+1}-x_{j}\right)^{4}+\frac{1}{2}\sum_{j=1}^{f}\left(x_{j}-x_{j-1}\right)^{4}=\\&=2\sum_{j=1}^{f}x_{j}^{4}-4\sum_{j=1}^{f}x_{j}^{3}\left(x_{j+1}+x_{j-1}\right)+3\sum_{j=1}^{f}x_{j}^{2}(x_{j+1}^{2}+x_{j-1}^{2}).\end{aligned} \tag{3}$$

Thus, the Hamiltonian (1) can be recast as

$$\begin{aligned}H=&\sum_{j=1}^{f}(\frac{p_{j}^{2}}{2m}+K_{2}x_{j}^{2})-\frac{K_{2}}{2}\sum_{j=1}^{f}x_{j}\left(x_{j+1}+x_{j-1}\right)+\\&+\frac{K_{4}}{2}\sum_{j=1}^{f}x_{j}^{4}-K_{4}\sum_{j=1}^{f}x_{j}^{3}\left(x_{j+1}+x_{j-1}\right)+\frac{3}{4}K_{4}\sum_{j=1}^{f}x_{j}^{2}(x_{j+1}^{2}+x_{j-1}^{2}).\end{aligned} \tag{4}$$

In order to quantize the Hamiltonian (4), we may introduce a local phonon (or vibron) creation and annihilation operators, i.e., $a_j^{\dagger}$ and a_j, respectively. Then, the displacement and momentum operators can be expanded as

$$x_{j}=\sqrt{\frac{\hbar}{2m\omega_{0}}}\left(a_{j}^{\dagger}+a_{j}\right), \tag{5}$$

$$p_{j}=i\sqrt{\frac{\hbar m\omega_{0}}{2}}\left(a_{j}^{\dagger}-a_{j}\right), \tag{6}$$

where $\omega_0=\sqrt{\frac{2K_2}{m}}$ is the local phonon mode frequency. For the sake of convenience，a dimensionless parameter $\gamma=3\hbar K_4/4mK_2\omega_0$ is introduced to represent anharmonic strength.

Inserting equations (5) and (6) into equation (4), and disregarding number nonconserving terms, one can get the following quantized number conserving FPU Hamiltonian:

$$\begin{aligned}H=&E_{0}+\hbar\omega_{0}\left(1+2\gamma\right)\sum_{j=1}^{f}a_{j}^{\dagger}a_{j}-\left(\frac{1}{4}+\gamma\right)\hbar\omega_{0}\sum_{j=1}^{f}a_{j}^{\dagger}\left(a_{j+1}+a_{j-1}\right)+\\&+\frac{\gamma\hbar\omega_{0}}{2}\sum_{j=1}^{f}\left[a_{j}^{\dagger2}a_{j}^{2}+\frac{a_{j}^{\dagger2}}{2}\left(a_{j+1}^{2}+a_{j-1}^{2}\right)\right]+\\&+\frac{\gamma\hbar\omega_{0}}{2}\sum_{s=\pm1}\sum_{j=1}^{f}\left[a_{j}^{\dagger}a_{j}a_{j+s}^{\dagger}a_{j+s}-\left(a_{j}^{\dagger2}a_{j}a_{j+s}+hc\right)\right],\end{aligned} \tag{7}$$

where $E_0=(1+\gamma)f\frac{\hbar\omega_0}{2}$ is zero point energy. Let us discuss the physical meaning of the terms appearing in some of the above equations. Since the meaning of the linear terms is clear, we comment mainly on the nonlinear terms. The on-site and inter-site repulsion ($\gamma\hbar\omega_0/2>0$) is defined through the terms $a_j^{\dagger2}a_j^2$

and $a_j^\dagger a_{j\pm1}^\dagger a_{j\pm1} a_j$, and the terms $a_j^{\dagger 2} a_{j\pm1}^2$ represent simultaneous tunneling of two phonons (or vibrons) between neighboring sites, i.e., the simultaneous creation of two phonons at site j with a simultaneous annihilation of two phonons at sites $j\pm1$. The terms $a_j^{\dagger 2} a_j a_{j\pm1}$ represent the simultaneous creation of two phonons on the same site, one of which is annihilated at the same site, while the other one tunnels to the neighboring site.

3. THE TIME-DEPENDENT HARTREE APPROXIMATION AND EQUATION OF MOTION

In our work, the Schrödinger picture is chosen to analyze the dynamic behavior of the system. Thus, the system state vector $|\Psi(t)\rangle$ is time dependent, while operators are time independent. In the Schrödinger picture, time evolution of the state vector $|\Psi(t)\rangle$ is governed by the Schrödinger equation:

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\Psi(t)\rangle = H |\Psi(t)\rangle . \tag{8}$$

Theoretically, one may expand any quantum state of the system in the Fock space. Here, we consider a general n-phonon system state vector [10], which can be expanded in the Fock space as

$$|\Psi_n(t)\rangle = \frac{1}{\sqrt{n!}} \sum_{j_1=1}^{f} \sum_{j_2=1}^{f} \cdots \sum_{j_n=1}^{f} \theta_n(j_1, j_2, \cdots, j_n, t) a_{j_1}^\dagger a_{j_2}^\dagger \cdots a_{j_n}^\dagger |0\rangle , \tag{9}$$

where $|0\rangle = |0\rangle_1 |0\rangle_2 \cdots |0\rangle_f$ is the vacuum state, and $\theta_n(j_1, j_2, \cdots, j_n, t)$ is the n-phonon wave function, which is normalized as

$$\sum_{j_1=1}^{f} \sum_{j_2=1}^{f} \cdots \sum_{j_n=1}^{f} |\theta_n(j_1, j_2, \cdots, j_n, t)|^2 = 1. \tag{10}$$

Inserting equations (7) and (9) into equation (8), and considering local phonon creation and annihilation operators satisfying the boson commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$, then one obtains

$$\begin{aligned}
&(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - n\tilde{\omega})\theta_n(j_1, j_2, \cdots, j_n, t) + \left(\frac{1}{4} + \gamma\right)\omega_0 \sum_{k=1}^{n} [\theta_n(j_1, j_2, \cdots, j_{k-1}, j_k + 1, j_{k+1}, \cdots, j_n, t) + \\
&+\theta_n(j_1, j_2, \cdots, j_{k-1}, j_k - 1, j_{k+1}, \cdots, j_n, t)] - \frac{\gamma\omega_0}{2} \sum_{k=1}^{n} \sum_{l \neq k}^{n} [\delta_{j_l, j_k} \theta_n(j_1, j_2, \cdots, j_l, \cdots, j_k, \cdots, j_n, t) + \\
&+\frac{1}{2}\delta_{j_l, j_k} \theta_n(j_1, j_2, \cdots, j_{l-1}, j_l + 1, j_{l+1}, \cdots, j_{k-1}, j_k + 1, j_{k+1}, \cdots, j_n, t) + \\
&+\frac{1}{2}\delta_{j_l, j_k} \theta_n(j_1, j_2, \cdots, j_{l-1}, j_l - 1, j_{l+1}, \cdots, j_{k-1}, j_k - 1, j_{k+1}, \cdots, j_n, t)] - \\
&-\frac{\gamma\omega_0}{2} \sum_{s=\pm1} \sum_{k=1}^{n} \sum_{l \neq k}^{n} [\delta_{j_l, j_{k-s}} \theta_n(j_1, j_2, \cdots, j_l, \cdots, j_k, \cdots, j_n, t) - \delta_{j_l, j_k} \theta_n(j_1, j_2, \cdots, j_l, \cdots, j_{k-1}, j_k + s, j_{k+1}, \cdots, j_n, t) - \\
&-\delta_{j_l - s, j_k} \theta_n(j_1, j_2, \cdots, j_{l-1}, j_l - s, j_{l+1}, \cdots, j_k, \cdots, j_n, t)] = 0,
\end{aligned} \tag{11}$$

where $\tilde{\omega} = E_0 / n\hbar + \omega_0(1 + 2\gamma)$. We note that equation (11) is the Schrödinger equation for a system of n phonons in a lattice with f sites. Moreover, the interaction between pairs of phonons is a Kronecker delta-function. Although it is very difficult to solve equation (11) directly, we can adopt the time-dependent Hartree approximation to simplify the matter. This approximation is well known in quantum theory, and has been applied to studies of nonlinear excitation in quantum lattice systems [11–14].

The basic idea of the time-dependent Hartree approximation is the assumption that each phonon feels the same mean field potential caused by the interaction with other phonons, and the many-body wave function can be approximately described by using single-phonon wave functions. This approximation is valid when the number of phonons n is large. In the Hartree approximation, the n-phonon wave function $\theta_n(j_1, j_2, \cdots, j_n, t)$ is assumed to have the following form [15]

$$\theta_n^{(H)}(j_1, j_2, \cdots, j_n, t) = \prod_{k=1}^{n} \Phi_{n,j_k}(t)\,, \tag{12}$$

where $\Phi_{n,j_k}(t)$ is the single-phonon wave function. Considering these single-phonon wave functions are independent of k, we can write them simply as $\Phi_{n,j}(t)$, where $j = 1, 2, \cdots, f$.

Making use of equation (12), the n-phonon state vector (9) becomes

$$\left|\Psi_n(t)\right\rangle^{(H)} = \frac{1}{\sqrt{n!}}\left(\sum_{j=1}^{f}\Phi_{n,j}(t)a_j^{\dagger}\right)^n \left|0\right\rangle \tag{13}$$

and then the normalization condition is

$$\sum_{j=1}^{f}\left|\Phi_{n,j}(t)\right|^2 = 1\,. \tag{14}$$

With the help of the time-dependent variational method [16], one obtains an equation of motion for $\Phi_{n,j}(t)$, which yields

$$\begin{aligned} &\mathrm{i}\frac{\mathrm{d}\Phi_j}{\mathrm{d}t} - \tilde{\omega}\Phi_j + \left(\frac{1}{4}+\gamma\right)\omega_0(\Phi_{j+1}+\Phi_{j-1}) - \frac{\gamma\omega_0}{2}(n-1)\Phi_j^*(2\Phi_j^2+\Phi_{j+1}^2+\Phi_{j-1}^2) - \\ &-\gamma\omega_0(n-1)[\Phi_j(\left|\Phi_{j+1}\right|^2+\left|\Phi_{j-1}\right|^2) - \left|\Phi_j\right|^2(\Phi_{j+1}+\Phi_{j-1}) - \\ &-\frac{1}{2}(\left|\Phi_{j+1}\right|^2\Phi_{j+1} + \left|\Phi_{j-1}\right|^2\Phi_{j-1} + \Phi_{j+1}^*\Phi_j^2 + \Phi_{j-1}^*\Phi_j^2)] = 0. \end{aligned} \tag{15}$$

It should be pointed out that, in equation (15), we omit the subscript n and substitute $\Phi_j(t)$ for $\Phi_{n,j}(t)$.

4. THE STATIONARY LOCALIZED SOLUTION AND QUANTUM BREATHERS

We note that equation (15) is in fact a discrete nonlinear equation. It is very difficult to solve this equation exactly. Here, we adopt the multiple-scale method combined with semidiscrete approximation (i.e., the semidiscrete multiple-scale method) [17] to look for an asymptotic solution of equation (15). In this treatment, we first make a scale transformation

$$\Phi_j = \varepsilon\psi_j\,, \tag{16}$$

where ε is a small parameter. Substituting equation (16) into equation (15), then one gets

$$\begin{aligned} &\mathrm{i}\frac{\mathrm{d}\psi_j}{\mathrm{d}t} - \tilde{\omega}\psi_j + \left(\frac{1}{4}+\gamma\right)\omega_0(\psi_{j+1}+\psi_{j-1}) - \varepsilon^2\frac{\gamma\omega_0}{2}(n-1)\psi_j^*(2\psi_j^2+\psi_{j+1}^2+\psi_{j-1}^2) - \\ &-\varepsilon^2\gamma\omega_0(n-1)[\psi_j(\left|\psi_{j+1}\right|^2+\left|\psi_{j-1}\right|^2) - \left|\psi_j\right|^2(\psi_{j+1}+\psi_{j-1}) - \\ &-\frac{1}{2}(\left|\psi_{j+1}\right|^2\psi_{j+1} + \left|\psi_{j-1}\right|^2\psi_{j-1} + \psi_{j+1}^*\psi_j^2 + \psi_{j-1}^*\psi_j^2)] = 0. \end{aligned} \tag{17}$$

Next, we introduce the multiple-scale variables t_1, t_2, and x_1, where

$$t_1 = \varepsilon t, t_2 = \varepsilon^2 t, x_1 = \varepsilon ja = \varepsilon x\,. \tag{18}$$

Thus, the wave function ψ_j in equation (17) is regarded as $\psi_j(t, t_1, t_2, x_1)$ and the derivative operators $\mathrm{d}/\mathrm{d}t$ and $\partial/\partial x$ are expanded as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} \quad ; \quad \frac{\partial}{\partial x} = \varepsilon \frac{\partial}{\partial x_1}. \tag{19}$$

Now, we look for modulated wave solutions of the type

$$\psi_j = \phi_j(t_1, t_2, x_1)\mathrm{e}^{\mathrm{i}\theta_j}, \tag{20}$$

where $\theta_j = kja - \omega t$ represents the phase of the carrier wave. In the semidiscrete approximation, the envelope function ψ is considered as a continuum variable while the phase θ is a discrete variable. Therefore, we need to use the continuum approximation for ϕ_j:

$$\phi_j(t_1, t_2, x_1) \to \phi(t_1, t_2, x_1) \quad ; \quad \phi_{j\pm 1} = \phi \pm a\frac{\partial \phi}{\partial x} + \frac{a^2}{2}\frac{\partial^2 \phi}{\partial x^2} + \dots. \tag{21}$$

Inserting equation (20) into equation (17) and using equations (19) and (21), then equation (17) can be rewritten as

$$\begin{aligned}
&[\omega - \tilde{\omega} + 2\left(\frac{1}{4}+\gamma\right)\omega_0 \cos(ka)]\phi \exp(\mathrm{i}\theta_j) + \varepsilon\mathrm{i}[\frac{\partial \phi}{\partial t_1} + 2\left(\frac{1}{4}+\gamma\right)\omega_0 \sin(ka) a \frac{\partial \phi}{\partial x_1}]\phi \exp(\mathrm{i}\theta_j) + \\
&+ \varepsilon^2\{\mathrm{i}\frac{\partial \phi}{\partial t_2} + \left(\frac{1}{4}+\gamma\right)\omega_0 a^2 \cos(ka)\frac{\partial^2 \phi}{\partial x_1^2} - 2\gamma\omega_0 (n-1)[1-\cos(ka)]^2 |\phi|^2 \phi\}\phi \exp(\mathrm{i}\theta_j) + o(\varepsilon^2) = 0.
\end{aligned} \tag{22}$$

Keeping terms to order ε^2 and equating the coefficients at each order of ε to be zero, then equation (22) yields the following equations:

$$\omega = \tilde{\omega} - 2\left(\frac{1}{4}+\gamma\right)\omega_0 \cos(ka), \tag{23}$$

$$\frac{\partial \phi}{\partial t_1} = -2\left(\frac{1}{4}+\gamma\right) a\omega_0 \sin(ka)\frac{\partial \phi}{\partial x_1}, \tag{24}$$

$$\mathrm{i}\frac{\partial \phi}{\partial t_2} + \left(\frac{1}{4}+\gamma\right)\omega_0 a^2 \cos(ka)\frac{\partial^2 \phi}{\partial x_1^2} - 2\gamma\omega_0 (n-1)[1-\cos(ka)]^2 |\phi|^2 \phi = 0. \tag{25}$$

Noting that equation (23) is the *dispersion relation*. Using it, one can obtain the *group velocity*

$$V_g = 2\left(\frac{1}{4}+\gamma\right) a\omega_0 \sin(ka). \tag{26}$$

From equation (24), we deduce

$$\phi(t_1, t_2, x_1) = \phi(t_2, z_1), \tag{27}$$

where $z_1 = x_1 - V_g t_1$ is introduced as the new scale. Thus, equation (25) can be rewritten as

$$i\frac{\partial \phi}{\partial t_2} + P\frac{\partial^2 \phi}{\partial z_1^2} + Q|\phi|^2 \phi = 0, \tag{28}$$

with

$$P = \left(\frac{1}{4} + \gamma\right)\omega_0 a^2 \cos(ka)\,, \tag{29}$$

$$Q = -2\gamma\omega_0(n-1)[1-\cos(ka)]^2\,. \tag{30}$$

Making the transformations $\phi = u/\varepsilon$ and $z = z_1/\varepsilon$, and noting that $t_2 = \varepsilon^2 t$, we can change equation (28) into the following form

$$\mathrm{i}\frac{\partial u}{\partial t} + P\frac{\partial^2 u}{\partial z^2} + Q|u|^2 u = 0\,. \tag{31}$$

Equation (31) is just the nonlinear Schrödinger (NLS) equation, which can be solved by the inverse-scattering transform [18]. In the case of $Q/P > 0$, the one-soliton solution (bright soliton) of the NLS equation (31) is given by

$$u(z,t) = 2\beta \operatorname{sech}\left[2\beta\sqrt{\frac{Q}{2P}}\left(z - z_0\right)\right]\exp\left[\mathrm{i}(2\beta^2 Qt + \theta_0)\right]. \tag{32}$$

This envelope soliton solution is characterized by three real parameters: β, z_0, and θ_0, which determine the height (as well as the width), initial position, and initial phase of the soliton, respectively.

Thus, it is not difficult to find that the single-phonon wave function on the *j*-th site has the following form

$$\Phi_{n,j}(t) = 2\beta \operatorname{sech}\left[2\beta\sqrt{\frac{Q}{2P}}\left(j - j_0\right)a - 2\beta\sqrt{\frac{Q}{2P}}V_g t\right] \times \exp[\mathrm{i}(kja - \omega t + 2\beta^2 Qt + \theta_0)]\,, \tag{33}$$

where j_0 is arbitrary, which represents the central position of envelope soliton at $t = 0$. Noting that, at $k = k_0 = \pm\pi/a$, we have $V_g = 0$, $\omega = \omega_{\max} = \tilde{\omega} + (\frac{1}{2} + 2\gamma)\omega_0$, and $Q = -8\gamma\omega_0(n-1)$. Then equation (33) becomes

$$\Phi_{n,j}(t) = 2\beta \operatorname{sech}\left[2\beta\sqrt{C_0}\left(j - j_0\right)a\right] \times \exp\{\mathrm{i}(k_0 ja + \theta_0) - \mathrm{i}[\omega_m + 16\gamma\omega_0(n-1)\beta^2]t\}\,, \tag{34}$$

where $C_0 = (Q/2P)_{k=k_0}$. Equation (34) gives the stationary localized solution. The eigenfrequency of localized solution is $\Omega = \omega_m + 16\gamma\omega_0(n-1)\beta^2$; we see that it is above the top of the harmonic wave band.

It should be noted that, different from the classical case, β cannot be arbitrary since $\Phi_{n,j}(t)$ has to satisfy the normalization condition (14). Noting that the envelope part of $\Phi_{n,j}(t)$ is considered as continuum variable in the semidiscrete approximation, thus one can convert discrete sum into continue integral, namely,

$$\sum_{j=1}^{f}\left|\Phi_{n,j}(t)\right|^2 = \frac{2\beta}{a\sqrt{C}}\int_{-\infty}^{+\infty}\operatorname{sec h}^2(\eta)\mathrm{d}\eta = \frac{4\beta}{a\sqrt{C_0}} = 1\,, \tag{35}$$

where $\eta = 2\beta\sqrt{C_0}(x - x_0)$. Then, one gets the following quantization condition

$$\beta = \frac{a\sqrt{C_0}}{4} = \sqrt{\frac{\gamma(n-1)}{(1+4\gamma)}}. \tag{36}$$

Substituting equation (34) into equation (13) and making use of equation (36), we can construct the Hartree product eigenstates:

$$\left|\Psi_n(t)\right\rangle^{(H)} = \exp(-\mathrm{i}n\Omega t + \mathrm{i}n\theta_0) \times \frac{1}{\sqrt{n!}}\left(\frac{a\sqrt{C_0}}{2}\right)^n \left(\sum_{j=1}^{f} \mathrm{sech}\left[\frac{C_0 a^2}{2}\left(j - j_0\right)\right]\exp(\mathrm{i}k_0 ja)a_j^{\dagger}\right)^n \left|0\right\rangle. \tag{37}$$

It is obvious that the approximate Hartree n-phonon states (37) are stationary states, and are localized quantum states. Moreover, using equation (37), we can get the mean number of phonons on the j-th site, which is written in the form

$$\left\langle n_j(t)\right\rangle^{(H)} = {}^{(H)}\left\langle \Psi_n(t)\right| a_j^{\dagger}a_j \left|\Psi_n(t)\right\rangle^{(H)} = \frac{a^2 C_0 n}{4}\mathrm{sech}^2\left[\frac{C_0 a^2}{2}\left(j - j_0\right)\right], \tag{38}$$

which suggests that most of phonons are localized in sites of the vicinity of the central position $j = j_0$. Theoretically, when *quantum breather states* superpose, the result is a spatially localized excitation with very lone time to tunnel from one lattice site to another, which is fully analogous to their classical counterparts [19]. Hence, we confirm that these localized Hartree n-phonon states can be viewed as quantum breather states. For such quantum breather states, the corresponding Hartree energy is then given by

$$E_n = n\hbar\Omega = E_0 + \left(\frac{3}{2} + 4\gamma\right)n\hbar\omega_0 + \frac{16\gamma^2}{\left(1+4\gamma\right)}n(n-1)^2\hbar\omega_0. \tag{39}$$

Obviously, equation (39) gives the energy level formula of the system for quantum breather states (37). This means that the energy of such quantum breathers is quantized.

5. CONCLUSIONS

In this work, we have studied that n-quanta quantum breathers in the β-FPU model. We found that, at the Brillouin zone boundary, the equation of motion for the single-phonon wave function has the stationary localized solution whose eigenfrequency is above the top of the harmonic wave band. Making use of these stationary localized single-phonon wave functions, we have constructed localized n-phonon Hartree states. The obtained result indicates that such n-phonon Hartree states can be considered as quantum breathers states. As the quantum counterparts of discrete breathers, quantum breathers are also spatially localized excitations. In addition, we obtained the energy level formula of the system for quantum breather states, which suggests the energy of quantum breathers is quantized. It is worth noting that quantum breathers that we have found have obvious quantum properties, which are different from two-phonon bound states or biphonons [5–7]. We believe that our work may be useful for understanding localization phenomenon in some quantum systems.

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