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FRACTIONAL OPTICAL SOLITARY WAVE SOLUTIONS OF THE HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION

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Based on a specific fractional transformation, we find new types of fractional optical solitary wave solutions of the higher-order nonlinear Schrödinger equation with the group velocity dispersion, self-phase modulation, third order dispersion, self-steepening and self-frequency shift, which describes the propagation of femtosecond light pulses in nonlinear optical fibers. For different values of a real parameter R, these new solutions can describe concavely W-shaped ($R > 2$), convexly W-shaped ($-1 < R < 0$), bright ($0 < R < 1$), and dark ($1 < R \leq 2$) optical solitary wave solutions. In particular, for the concavely W-shaped solution ($R > 0$), the larger the parameter R becomes, the larger the maximum value of the center part of the intensity is, and when $R \to \infty$ this maximum value approaches a constant. While for the convexly W-shaped solution ($-1 < R < 0$), the smaller the absolute value of the parameter R becomes, the larger the maximum value of the center part of the intensity is.

Key words: higher-order nonlinear Schrödinger equation; fractional transformation; optical solitary wave solutions; periodic wave solutions.

1. INTRODUCTION

The nonlinear Schrödinger (NLS) equation is used to model the propagation of picosecond optical pulses and is of fundamental importance in optical fiber communications [1–6] (the NLS equation is known as the Gross-Pitaevskii equation in the study of Bose-Einstein condensates [7, 8]). It is well known that the NLS equation is a relatively simple soliton equation and is completely integrable by using the inverse scattering transform, the Painlevé analysis, the bilinear transformation, etc. [1–5].

However, a lot of experimental and theoretical works have shown that the NLS equation is no longer valid for the study of femtosecond optical pulses and of ultrashort (few-cycle) optical solitons [3, 4]. In order to model the propagation of much shorter (femtosecond) optical pulses, third order and nonlinear dispersion terms have been added to the NLS equation. As a result, the higher-order nonlinear Schrödinger (HONLS) equation was put forward in the form [9–12]

$$E_z = i(\alpha_1 E_{tt} + \alpha_2 |E|^2 E) + \alpha_3 E_{ttt} + \alpha_4 (|E|^2 E)_t + \alpha_5 E(|E|^2)_t, \tag{1}$$

where $E \equiv E(z,t)$ stands for the slowly varying envelope of the electric field, the subscripts t and z are the temporal and spatial partial derivatives, respectively, α_i (i = 1, 2,..., 5) are the real parameters related to the group velocity dispersion (GVD), self-phase modulation (SPM), third order dispersion (TOD), self-steepening (SS), and self-frequency (SF) shift via stimulated Raman scattering, respectively. Equation (1) includes many types of complex nonlinear wave equations for the different parameters α_i $(i = 1, 2, 3, 4, 5)$ in the area of nonlinear optics:

i) When $\alpha_i = 0$ (i = 3, 4, 5), Eq. (1) becomes the cubic NLS equation with GVD and SPM, i.e., $E_z = i(\alpha_1 E_{tt} + \alpha_2 |E|^2 E)$, which admits many integrability properties [1–6].

ii) When $\alpha_i = 0$ (i = 2,3,4), Eq. (1) becomes the derivative NLS equation including GVD and SS, i.e., $E_z = i\alpha_1 E_{tt} + \alpha_5 E(|E|^2)_t$, which describes the propagation of circularly polarized nonlinear van Alfvén waves in plasmas [13,14]; it is integrable via the inverse scattering transform and admits multi-soliton solutions [15].

iii) When $\alpha_i = 0$ (i = 3, 5), Eq. (1) becomes the extended derivative NLS equation including GVD, SPM and SS, i.e., $E_z = i(\alpha_1 E_{tt} + \alpha_2 |E|^2 E) + \alpha_4 (|E|^2 E)_t$ [16, 17], whose *N*-soliton solutions have been obtained, see Ref. [16].

iv) When $\alpha_4 + \alpha_5 = 0$, Eq. (1) becomes the Hirota-type HONLS equation with GVD, SPM, TOD, SS and SF, i.e., $E_z = i(\alpha_1 E_{tt} + \alpha_2 |E|^2 E) + \alpha_3 E_{ttt} + \alpha_4 |E|^2 E_t$, whose *N*-soliton solutions were given in Ref. [18].

v) When $\alpha_5 = 0$, Eq. (1) becomes the generalized NLS equation with GVD, SPM, TOD, and SS, i.e., $E_z = i(\alpha_1 E_{tt} + \alpha_2 |E|^2 E) + \alpha_3 E_{ttt} + \alpha_4 (|E|^2 E)_t$; solitary wave solutions and *N*-soliton solutions of this equation were also deduced [19, 20].

vi) When $\alpha_1 = \alpha_2 = 0$, Eq. (1) reduces to the complex modified KdV equation with TOD, SS, and SF, i.e., $E_z = \alpha_3 E_{ttt} + \alpha_4 (|E|^2 E)_t + \alpha_5 E(|E|^2)_t$ [9, 21].

For Eq. (1), some types of exact solutions have been also investigated using different methods [9, 10, 22–26]. Here we will study new types of optical solitary wave solutions of (1), which might adequately describe some specific physical phenomena for the chosen values of the parameters.

The structure of the paper is as follows. In Section 2, we consider two distinct fractional transformations involving (a) hyperbolic functions and (b) Jacobi elliptic functions, in order to obtain new optical solitary wave solutions of Eq. (1). Finally, Section 3 concludes the paper.

2. FRACTIONAL TRANSFORMATION AND EXACT SOLUTIONS

We consider the ansatze

$$E(z,t) = A(z,t)\exp[\mathrm{i}\varphi(z,t)],\ \ \varphi(z,t) = kz - \Omega t, \tag{2}$$

where $A(z,t)$ is a complex function, and k, Ω are both real constants. Then the substitution of this expression into Eq. (1) yields the complex nonlinear wave equation:

$$A_z + b_1 A_t - \mathrm{i}b_2 A_{tt} - \alpha_3 A_{ttt} - \mathrm{i}b_3 |A|^2 A + b_4 |A|^2 A_t + b_5 A^2 A_t^* + \mathrm{i}b_6 A = 0, \tag{3}$$

where A^* is the complex conjugate of A, and

$$b_1 = -2\alpha_1\Omega + 3\alpha_3\Omega^2,\ b_2 = \alpha_1 - 3\alpha_3\Omega\ \ ,\ b_2 = \alpha_2 - \alpha_4\Omega,$$
$$b_4 = 2\alpha_4 + \alpha_5,\ b_5 = \alpha_4 + \alpha_5,\ b_6 = k + \alpha_1\Omega^2 - \alpha_3\Omega^3.$$

Li *et al.* [10] used the following polynomial transformation of hyperbolic functions

$$A(z,t) = \mathrm{i}\beta + \lambda\tanh[\eta(t - \chi z)] + \mathrm{i}\rho\,\mathrm{sech}[\eta(t - \chi z)], \tag{4}$$

to study solutions of Eq. (3) such that they gave some exact solutions of Eq. (1) for certain values of a set of parameters defining the solitary wave solutions.

2.1. Solitary waves due to the use of hyperbolic functions

In what follows, we would like to find some new types of optical solitary wave solutions of Eq. (3) via the fractional transformation [27]:

$$A(z,t)=\frac{\mathrm{i}\beta+\lambda\tanh[\eta(t-\chi z)]+\mathrm{i}\rho\ \ \mathrm{sech}[\eta(t-\chi z)]}{1+R\ \ \mathrm{sech}[\eta(t-\chi z)]}, \tag{5}$$

where $\beta,\ \lambda,\ \rho,\ \eta,\ \chi,\ R$ are real-valued constants. Thus the corresponding amplitude is written as

$$|A(z,t)|=\frac{\{(\rho^2-\lambda^2)\mathrm{sech}^2[\eta(t-\chi z)]+2\beta\rho\ \ \mathrm{sech}[\eta(t-\chi z)]+\lambda^2+\beta^2\}^{1/2}}{|1+R\ \ \mathrm{sech}[\eta(t-\chi z)]|}. \tag{6}$$

In particular if $R=0$, then the fractional transformation (5) reduces to the known form (4). Here we do not consider this case. We will consider the case $R\neq 0$, which leads to new types of optical solitary wave solutions of HONLS equation (1) via the ansatze (5).

With the aid of symbolic computation, we substitute Eq. (5) into Eq. (3) and separate the real and imaginary parts, and further balance the coefficients of these linearly independent terms $\sinh^i[\eta(t-\chi z)]\cosh^j[\eta(t-\chi z)]$ $(i=0,1;\ j=0,1,2,...)$ to yield a set of algebraic equations with respect to unknowns. Here we omit the derivation of the set of these rather cumbersome algebraic equations. In the following, when $R\neq 0$, we present new optical solitary wave solutions of Eq. (1) using the fractional transformation (5):

Case 1. $3\alpha_2\alpha_3=\alpha_1\alpha_4$. We obtain the following optical solitary wave solution of HONLS equation (1)

$$E(z,t)=\frac{\mathrm{i}\beta-\mathrm{i}\beta(R-2/R)\ \ \mathrm{sech}[\eta(t-\chi z)]}{1+R\ \ \mathrm{sech}[\eta(t-\chi z)]}\exp[\mathrm{i}(kz-\Omega t)], \tag{7}$$

and its intensity is defined by

$$|E(z,t)|^2=\frac{\{\beta-\beta(R-2/R)\ \ \mathrm{sech}[\eta(t-\chi z)]\}^2}{\{1+R\ \ \mathrm{sech}[\eta(t-\chi z)]\}^2}, \tag{8}$$

where the parameters are given by

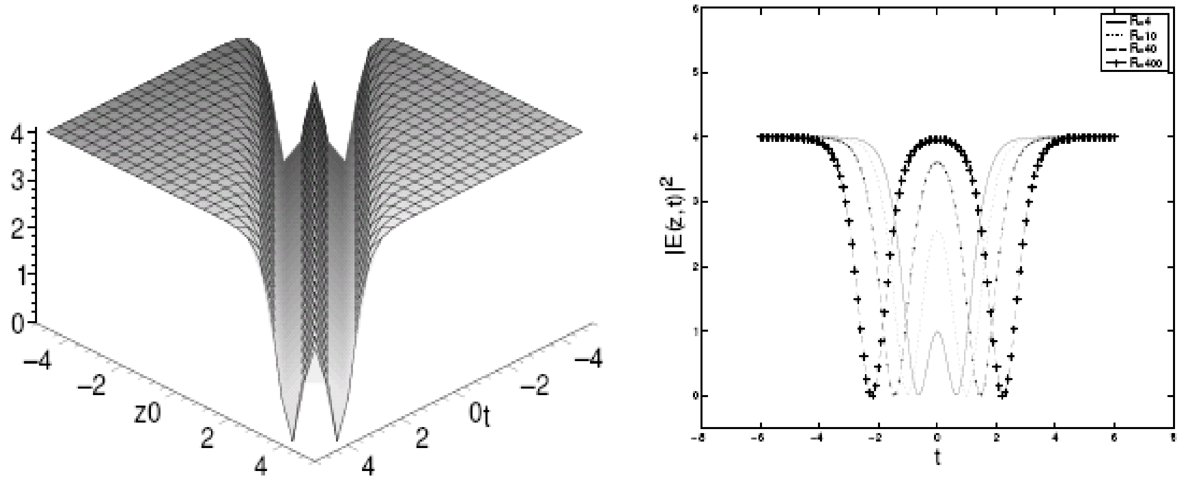
$$\begin{gathered}\Omega=\frac{\alpha_2}{\alpha_4},\ k=-\frac{2\alpha_1\alpha_2^2}{3\alpha_4^2},\ \eta^2=\frac{2\beta^2(R^2-1)(3\alpha_4+2\alpha_5)}{\alpha_3R^2},\\ \chi=-2\alpha_1\Omega+3\alpha_3\Omega^2-\frac{\beta^2(R^2+2)(3\alpha_4+2\alpha_5)}{3R^2}.\end{gathered} \tag{9}$$

From Eq. (9) with $\eta\neq 0$, we know that $R\neq\pm 1$. When $R<-1$, the solution (7) has a singular surface with $\eta(t-\chi z)=\mathrm{arcsech}(R-1)$. Therefore if $R>-1$ and $R\neq 1$, then solution (7) is a new regular optical solitary wave solution of HONLS equation (1). For different values of the parameter R, Eq. (7) admits abundant structures, which include the W-shaped optical solitary wave, bright optical solitary wave, and dark optical solitary wave:

i) We take parameters as $\beta=2,\rho=3,\chi=1$. When $R>2$, we have the W-shaped optical solitary wave solution of (1), and the maximum value of the center part of the intensity $|E(z,t)|^2$ is typically less than the maximum values of the two additional intensity "shoulders" (Figs. 1a, 1b). This kind of W-shaped solitary wave is called a *concavely* W-shaped solitary wave. Moreove, Fig. 1b shows that the maximum value of the center part of the intensity approaches the constant value $\beta^2=4$, when $R\to\infty$.

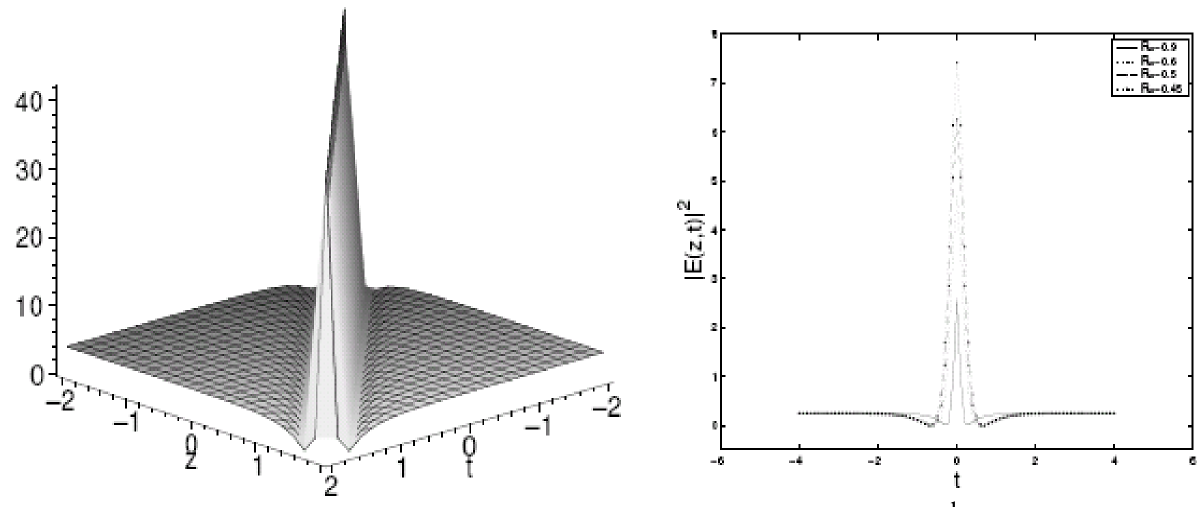
ii) We take parameters as $\beta=0.5,\rho=3,\chi=1$. If $-1<R<0$, then we also get a W-shaped optical solitary wave, which differs from the previous W-shaped solitary wave because in this case the maximum value of the center part of the intensity $|E(z,t)|^2$ is larger than the maximum values of the two additional "shoulders" (Figs. 2a, 2b). We call this kind of solitary wave as a *convexly* W-shaped solitary wave. Moreover, Fig. 1b shows that the larger R becomes, the larger the maximum value of the soliton's center part is.

iii) We take parameters as $\beta = 2, \rho = 3, \chi = 1$. When $0 < R < 1$, we have from Eq. (8) the bright solitary wave solution, which is shown in Fig. 3a. When $0 < R \le 2$, we have the dark solitary wave solution, which is shown in Fig. 3b.



a b

Fig. 1 – The intensity of W-shaped optical solitary wave solution (7) with $R > 2$: a) the three-dimensional intensity plot for $R = 4$; b) the intensity plot at $z = 0$ for $R = 4$, 10, 40, and 400.



a b

Fig. 2 – The intensity of W-shaped optical solitary wave solution (7) for $-1 < R < 0$: a) The three-dimensional intensity plot for $R = -0.9$; b) the intensity plot at $z = 0$ for $R = -0.45$, -0.5, -0.6, and -0.9.

Case 2. $3\alpha_2\alpha_3 = \alpha_1\alpha_4$, $\alpha_4 + \alpha_5 = 0$. We obtain the dark optical solitary wave solution of Eq. (1)

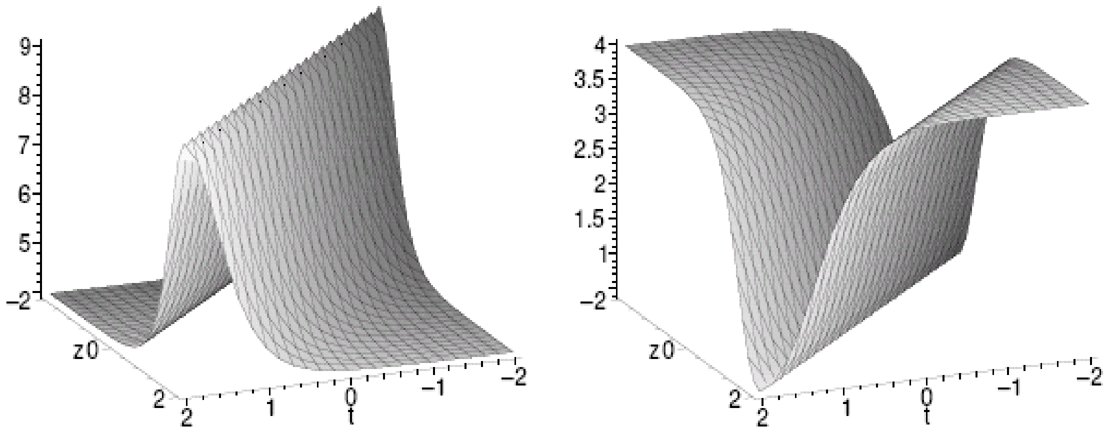
$$E(z,t) = \left\{ \mathrm{i}\beta + \frac{\lambda \tanh[\eta(t - \chi z)]}{1 + \mathrm{sech}[\eta(t - \chi z)]} \right\} \exp[\mathrm{i}(kz - \Omega t)], \tag{10}$$

and its intensity is defined by

$$| E(z,t) |^2 = \beta^2 + \frac{\lambda^2 \tanh^2[\eta(t - \chi z)]}{\{1 + \mathrm{sech}[\eta(t - \chi z)]\}^2}, \tag{11}$$

where the parameters are determined by

$$\Omega=\frac{\alpha_2}{\alpha_4},\ k=-\frac{2\alpha_1\alpha_2^2}{3\alpha_4^2},\ \eta^2=-\frac{2\alpha_4\lambda^2}{3\alpha_3},\ \chi=-\alpha_4(1/3\lambda^2+\beta^2)-2\alpha_2\Omega+3\alpha_3\Omega^2, \tag{12}$$



a b

Fig. 3 – a) The intensity of bright optical solitary wave solution (7) for R = 0.8; b) the intensity of dark optical solitary wave solution (7) for R = 1.5.

It follows from Eq. (12) that we need the condition $\alpha_3\alpha_4<0$. From the expression of intensity (11), we see that the solution (10) is a dark optical solitary wave solution of Eq. (1), which is different from the usual dark solitary wave solutions.

Case 3. $\alpha_3=0,\ \alpha_4+\alpha_5=0.$ We obtain the optical solitary wave solution of Eq. (1)

$$E(z,t)=\frac{\sqrt{2}\lambda\tanh[\eta(t-\chi z)]-\mathrm{i}\lambda\ \mathrm{sech}[\eta(t-\chi z)]}{\sqrt{2}+\mathrm{sech}[\eta(t-\chi z)]}\exp[\mathrm{i}(kz-\Omega t)], \tag{13}$$

and its intensity is defined by

$$|E(z,t)|^2=\frac{\lambda^2\{2-\mathrm{sech}^2[\eta(t-\chi z)]}{\{\sqrt{2}+\mathrm{sech}[\eta(t-\chi z)]\}^2}, \tag{14}$$

where the parameters are determined by

$$k=-\frac{\alpha_1(\alpha_2-\alpha_4\Omega)^2}{\alpha_4},\ \eta=\frac{\alpha_2}{\alpha_4}-\Omega,\ \chi=-2\alpha_2\Omega,\ \lambda^2=\frac{\alpha_1(\alpha_4\Omega-\alpha_2)}{\alpha_4}, \tag{15}$$

Case 4. $\alpha_1=\alpha_3=3\alpha_4+2\alpha_5=0.$ We obtain the optical solitary wave solution of Eq. (1)

$$E(z,t)=\frac{\mathrm{i}\beta+\lambda\tanh[\eta(t-\chi z)]+\mathrm{i}\rho\ \mathrm{sech}[\eta(t-\chi z)]}{1+R\ \mathrm{sech}[\eta(t-\chi z)]}\exp[\mathrm{i}(kz-\Omega t)], \tag{16}$$

and its intensity is defined by

$$|E(z,t)|^2=\frac{(\rho^2-\lambda^2)\ \mathrm{sech}^2[\eta(t-\chi z)]+2\beta\rho\ \mathrm{sech}^2[\eta(t-\chi z)]+\lambda^2+\beta^2}{\{1+R\ \mathrm{sech}[\eta(t-\chi z)]\}^2}, \tag{17}$$

where the parameters are determined by

$$\lambda=\left(\frac{2\rho\beta^2-2R\beta^3-\rho^3+\rho R^2\beta^2}{2R\beta-\rho R^2-\rho}\right)^2,\ k=\frac{\rho(\beta^2-\rho^2)(\alpha_2-\alpha_4\Omega)}{2R\beta-\rho R^2-\rho},$$
$$\eta=\frac{(R\beta^2-\rho R^2\beta-\rho\beta+R\rho^2)(\alpha_2-\alpha_4\Omega)}{\lambda(2R\beta-\rho R^2-\rho)},\ \chi=0, \tag{18}$$

From Eq. (18), we need the condition $(2\rho\beta^2-2R\beta^3-\rho^3+\rho R^2\beta^2)(2R\beta-\rho R^2-\rho)>0$.

In particular, when $R=0$, the obtained solution (16) reduces to the known solution (27) in Ref. [9]. For the case $R=>-1$ and $R\neq 0$, the solution (16) admits abundant structures, which contain the W-shaped optical solitary wave, bright optical solitary wave, and dark optical solitary wave, see the soliton intensity (17). However, here we do not analyze them in detail.

2.2. Periodic waves due to the use of Jacobi elliptic functions

In addition, according to the idea put forward in Refs. [28-30], we assume that Eq. (3) has the Jacobi elliptic function solution

$$A(z,t)=\frac{\mathrm{i}\beta+\lambda\ \ \mathrm{sn}[\eta(t-\chi z),m]+\mathrm{i}\rho\ \ \mathrm{cn}[\eta(t-\chi z),m]}{1+R\ \ \mathrm{cn}[\eta(t-\chi z),m]}, \tag{19}$$

When $m\to 1$, the solution (19) reduces to the considered solution (5). For the general case $0<m<1$, we can obtain new doubly periodic optical solitary wave solutions of Eq. (1).

For example, when $3\alpha_2\alpha_3=\alpha_1\alpha_4$, we have the doubly periodic wave solution of Eq. (1):

$$E(z,t)=\frac{\mathrm{i}\beta+\mathrm{i}\rho\ \ \mathrm{cn}[\eta(t-\chi z),m]}{1+R\ \ \mathrm{cn}[\eta(t-\chi z),m]}\exp[\mathrm{i}(kz-\Omega t)], \tag{20}$$

and its intensity is given by

$$|E(z,t)|^2=\frac{\{\beta+\rho\ \ \mathrm{cn}[\eta(t-\chi z),m]\}^2}{\{1+R\ \ \mathrm{cn}[\eta(t-\chi z),m]\}^2},$$

where the parameters of the solitary wave solution are given by

$$\Omega=\frac{\alpha_2}{\alpha_4},\ k=-\frac{2\alpha_1\alpha_2^2}{3\alpha_4^2},\ \rho=\frac{\beta[R^2(2m^2-1)-2m^2]}{R[2R^2(m^2-1)-2m^2+1]},$$
$$\eta^2=-\frac{2\rho^2(R^2-1)[R^2(1-m^2)+m^2](3\alpha_4+2\alpha_5)}{3\alpha_3[R^2(2m^2-1)-2m^2]^2},$$
$$\chi=\alpha_1+\frac{\alpha_3\rho^2[2R^2(2m^2-1)(m^2-1)-R^2(8m^4-8m^2-1)+2m^2(2m^2-1)]}{2(R^2-1)[R^2(1-m^2)+m^2]}.$$

Similar to Case 1, the solution (20) contains different types of optical solitary wave solutions for different values of the parameter R and the modulus m. Moreover, for other cases, we can also obtain new doubly periodic wave solutions of (1) via the new ansatz (19).

3. CONCLUSIONS

In summary, we have found some new types of optical solitary wave solutions of the HONLS equation with GVD, SPM, TOD, SS, and SF terms. The obtained solutions differ from the known optical solitary wave solutions reported in the literature. Moreover, for different values of the free parameter R, the obtained solutions are the concavely W-shaped ($R>2$), the convexly W-shaped ($-1<R<0$), the bright ($0<R<1$), and the dark ($1<R\leq 2$) optical solitary wave solutions. In particular, for the concavely W-

shaped solution ($R > 0$), the larger the parameter R becomes, the larger the maximum value of the center part of the intensity is, and this center intensity approaches a constant value, when $R \to \infty$ (see Fig. 1b). While for the convexly W-shaped solution ($-1 < R < 0$), the smaller the absolute value of the parameter R becomes, the larger the maximum value of the center part of the intensity is (Fig. 2b). These new solitary wave solutions might be useful to describe other nonlinear optics phenomena. Moreover, the approach developed in this paper can also be extended to other nonlinear wave models, in order to generate new solitary wave solutions describing certain physical phenomena.

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