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KINKS AND DOMAIN WALLS OF THE ZAKHAROV EQUATION IN PLASMAS

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This paper obtains the topological 1-soliton solution to the Zakharov equation, which appears in plasma physics. The ansatz method is applied to extract the soliton solution. The power law nonlinearity is taken into account. The study is also extended to (1+2) dimensions. A couple of numerical simulations are given to illustrate the analytical results.

Key words: Zakharov equations, topological solution, ansatz method.

1. INTRODUCTION

The study of nonlinear evolution equations (NLEEs) is very important in several areas of Applied Mathematics and Theoretical Physics [1–20]. In fact this study is going on for several decades and is still burning bright. There are several avenues to venture in order to conduct research in this field. Integrability of these NLEEs is one of the most important features that need to be addressed. A closed form analytical soliton solution or other waves is always beneficial in this area of research. This paper is going to address the integrability aspect of the Zakharov equation (ZE) that appears in Plasma Physics. In order to keep the equation on a generalized setting, the ZE is considered with power law nonlinearity. While there are several integration tools that are available to retrieve solutions to the NLEEs, this paper is going to stay focused on the *ansatz method* that is a relatively simple approach to extract soliton solution to any given NLEE. The target will be to obtain the topological 1-soliton solution to the ZE. These are also known as *shock waves* or *kinks*. In multi-dimensional case, these solutions are known as *domain walls*.

2. MATHEMATICAL ANALYSIS

The analysis of the ZE will be split into two cases. In the first case, the ZE will be studied in (1+1) dimensions while in the second case the study will be concentrated on the (1+2) dimensions case.

2.1. (1+1) dimensions

The (1+1) dimensions form of the Zakharov equations that are going to be studied in this paper is given by Abbasbandy et al. [1] and Javidi and Golbabai [4]:

$$iq_t + aq_{xx} + b\left|q\right|^{2m} q = qr, \tag{1}$$

$$r_{tt} - k^2 r_{xx} = (\left|q\right|^{2n})_{xx}. \tag{2}$$

In Eqs. (1) and (2), x and t represent the spatial and temporal variables respectively. In Eq. (1), a and b represent the coefficients of dispersion and nonlinearity. Also, the two dependent variables are q and r where q is a complex valued dependent variable while r is a real-valued dependent variable.

To obtain topological solution of Eqs. (1) and (2) we assume that

$$q(x,t) = A_1 \tanh^{p_1} \tau e^{i\phi}, \tag{3}$$

$$r(x,t) = A_2 \tanh^{p_2} \tau, \tag{4}$$

where A_1 and A_2 are free parameters of the kinks and represent the amplitudes of q and r topological solitons, respectively, and the variable τ is given by

$$\tau = B(x - vt), \tag{5}$$

where B is another parameter of the kink and v is the velocity of the topological soliton. In Eq. (3), ϕ represents the phase of the topological soliton that is defined as:

$$\phi = -\kappa x + \omega t + \theta . \tag{3}$$

Here, in Eq. (6), κ represents the wave-number, ω is the frequency and θ is the phase constant. The unknown exponents p_1 and p_2 will be determined, in terms of m during the course of derivation of the topological solution to Eqs. (1) and (2).

Substituting Eqs. (3), (4) into Eqs. (1) and (2) yield

$$\begin{aligned} &-ip_1 vA_1 B \tanh^{p_1-1} \tau + ip_1 vA_1 B \tanh^{p_1+1} \tau - \omega A_1 \tanh^{p_1} \tau + a\{p_1(p_1-1)A_1 B^2 \tanh^{p_1-2} \tau - \\ &-2ip_1 \kappa A_1 B \tanh^{p_1-1} \tau - (2p_1^2 B^2 + \kappa^2) A_1 \tanh^{p_1} \tau + 2ip_1 \kappa A_1 B \tanh^{p_1+1} \tau + \\ &+p_1(p_1+1)A_1 B^2 \tanh^{p_1+2} \tau\} + bA_1^{2m+1} \tanh^{(2m+1)p_1} \tau - A_1 A_2 \tanh^{p_1+p_2} \tau = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} &p_2(p_2-1)(v^2 - K^2)A_2 B^2 \tanh^{p_2-2} \tau - 2p_2^2(v^2 - K^2)A_2 B^2 \tanh^{p_2} \tau + \\ &+p_2(p_2+1)(v^2 - K^2)A_2 B^2 \tanh^{p_2+2} \tau - 2np_1(2np_1-1)A_1^{2n} B^2 \tanh^{2np_1-2} \tau + \\ &+8n^2 p_1^2 A_1^{2n} B^2 \tanh^{2np_1} \tau + 2np_1(2np_1+1)A_1^{2n} B^2 \tanh^{2np_1+2} \tau = 0 \end{aligned} \tag{8}$$

respectively. Decomposing (7) into two parts, the real and imaginary parts, we have

$$\begin{aligned} &-\omega_1 \tanh^{p_1} \tau + a\{p_1(p_1-1)B^2 \tanh^{p_1-2} \tau - (2p_1^2 B^2 + \kappa^2) \tanh^{p_1} \tau + \\ &+p_1(p_1+1)B^2 \tanh^{p_1+2} \tau\} + bA_1^{2m} \tanh^{(2m+1)p_1} \tau - A_2 \tanh^{p_1+p_2} \tau = 0, \end{aligned} \tag{9}$$

$$-v \tanh^{p_1-1} \tau + p_1 v \tanh^{p_1+1} \tau - 2ap_1 \kappa \tanh^{p_1-1} \tau + 2a\kappa \tanh^{p_1+1} \tau = 0. \tag{10}$$

From Eq. (9), equating the exponents $p_1 + p_2$ and $p_1 + 2$ gives

$$p_2 = 2 \tag{11}$$

and then equating $(2m+1)p_1$ with $p_1 + 2$ gives

$$p_1 = 1/m . \tag{12}$$

Finally, equating the exponents $(2n+1)p_1$ and $p_1 + p_2$ in Eq. (8) gives

$$p_2 = 2np_1 . \tag{13}$$

Now, from Eq. (12) we have

$$p_1 = 1/n . \tag{14}$$

From Eqs. (12) and (13) we have

$$n = m . \tag{15}$$

By virtue of (15), equation (2) modifies to

$$r_{tt} - k^2 r_{xx} = (|q|^{2m})_{xx}. \qquad (16)$$

Equation (10) gives the velocity of the topological soliton as

$$v = -2a\kappa. \qquad (17)$$

Setting the coefficients of the linearly independent functions $\tanh^{p_1+j}\tau$, $j=-2,0,2$ to zero in Eqs. (9) and (10) leads to

$$\omega = -\frac{a}{m^2}(2B^2 + m^2\kappa^2), \qquad (18)$$

$$a(m+1)B^2 + m^2 b A_1^{2m} - m^2 A_2 = 0. \qquad (19)$$

Similarly, equating the coefficients of $\tanh^{p_2+j}\tau$, $j=-2,0,2$ in 10) we get

$$(v^2 - k^2)A_2 - A_1^{2m} = 0. \qquad (20)$$

Thus, finally solving the coupled system (19) and (20) we obtain

$$A_1 = \left[\frac{a(m+1)(v^2-k^2)B^2}{m^2(1-b(v^2-k^2))}\right]^{\frac{1}{2m}}, \qquad (21)$$

$$A_2 = \frac{a(m+1)B^2}{m^2(1-b(v^2-k^2))}. \qquad (22)$$

It is necessary to observe that (21) and (22) satisfied the two constraints

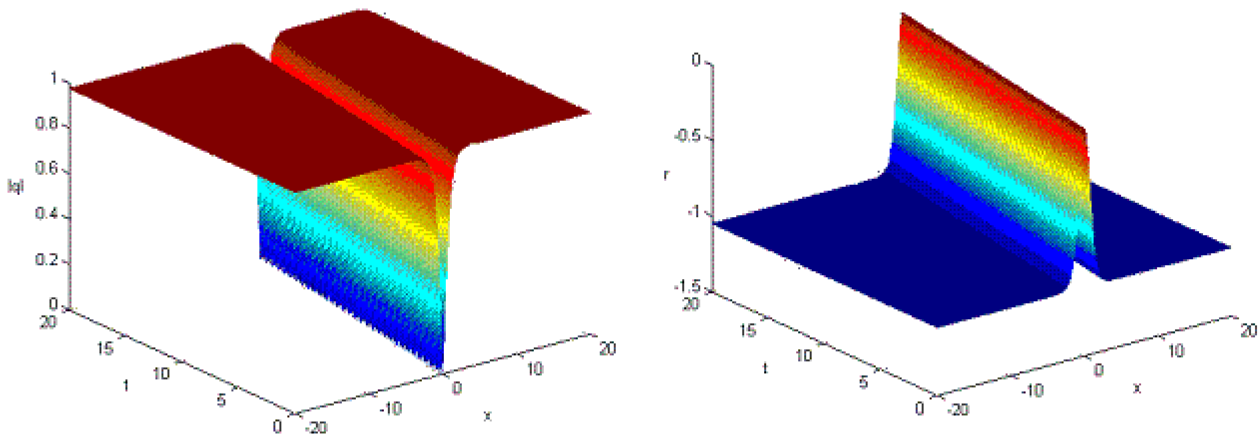
$$b(v^2 - k^2) \neq 1, \qquad (23)$$

$$a(v^2-k^2)\{1-b(v^2-k^2)\} > 0. \qquad (24)$$

Thus the topological solution of the q and r wave functions are given by

$$q(x,t) = A_1 \tanh^{\frac{1}{m}}(B(x-vt))e^{i(-\kappa x+\omega t+\theta)}, \qquad (25)$$

$$r(x,t) = A_2 \tanh^2(B(x-vt)) \;. \qquad (26)$$



(a) (b)

Fig. 1 – Topological solution for the Zakharov equations in (1+1) dimensions.

The solutions of $|q(x,t)|^2$ and $r(x,t)$ are plotted in Fig. 1a and b, respectively. For this example, the parameter values that are chosen are

$$a=-1,\ b=1,\ k=1,\ B=1,\ \kappa=0.15,\ m=1,\ -20\le x\le 20,\ 0\le t\le 20\,.$$

Although these values are chosen arbitrarily, the restriction given by (24) is valid for this set of parameter values.

2.2. (1+2) Dimensions

The (1+2) dimensions form of the Zakharov equations is given by

$$iq_t + a(q_{xx}+q_{yy}) + b|q|^{2m} q = qr\,, \tag{27}$$

$$r_{tt} - k^2(r_{xx}+r_{yy}) = (|q|^{2n})_{xx} + (|q|^{2n})_{yy}\,. \tag{28}$$

To obtain topological solutions of Eqs. (37) and (38) we assume that

$$q(x,y,t) = A_1 \tanh^{p_1}\tau e^{i\phi}\,, \tag{29}$$

$$r(x,y,t) = A_2 \tanh^{p_2}\tau\,, \tag{30}$$

where A_1 and A_2 are free parameters of the kinks and represent the amplitudes of q and r topological solitons, respectively, and τ is given by

$$\tau = B_1 x + B_2 y - vt\,, \tag{31}$$

where, in this case, B_1 and B_2 are the inverse widths of the topological solitons in the x- and y-directions, respectively, and v is the velocity of the topological soliton as before. In Eq. (29) ϕ represents the phase of the soliton that is defined as

$$\phi = -\kappa_1 x - \kappa_2 y + \omega t + \theta\,. \tag{32}$$

Here, in Eq. (32) κ_1 and κ_2 are the topological wave-numbers in the x- and y-directions, respectively, and ω is the topological frequency, while θ the phase constant.

Substituting (29), (30) into (26) and (27) yield

$$\begin{aligned}&-ip_1 v\tanh^{p_1-1}\tau + ip_1 v\tanh^{p_1+1}\tau - \omega\tanh^{p_1}\tau + a\{p_1(p_1-1)_1(B_1^2+B_2^2)\tanh^{p_1-2}\tau -\\ &-2ip_1(\kappa_1 B_1+\kappa_2 B_2)\tanh^{p_1-1}\tau - (2p_1^2(B_1^2+B_2^2)+\kappa_1^2+\kappa_2^2)_1\tanh^{p_1}\tau +\\ &+2ip_1(\kappa_1 B_1+\kappa_2 B_2)_1\tanh^{p_1+1}\tau + p_1(p_1+1)(B_1^2+B_2^2)\tanh^{p_1+2}\tau\} +\\ &+bA_1^{2m}\tanh^{(2m+1)p_1}\tau - A_2\tanh^{p_1+p_2}\tau = 0,\end{aligned} \tag{33}$$

$$\begin{aligned}&A_2(v^2 - K^2(B_1^2+B_2^2))\{p_2(p_2-1)\tanh^{p_2-2}\tau - 2p_2^2\tanh^{p_2}\tau + p_2(p_2+1)\tanh^{p_2+2}\tau\} -\\ &-A_1^{2n}(B_1^2+B_2^2)\{2np_1(2np_1-1)\tanh^{2np_1-2}\tau + 8n^2p_1^2\tanh^{2np_1}\tau -\\ &-2np_1(2np_1+1)\tanh^{2np_1+2}\tau\} = 0.\end{aligned} \tag{34}$$

Decomposing (33) into two parts, the real and imaginary parts, we have

$$\begin{aligned}&-\omega A_1\tanh^{p_1}\tau + a\{p_1(p_1-1)A_1(B_1^2+B_2^2)\tanh^{p_1-2}\tau - (2p_1^2(B_1^2+B_2^2)+\kappa_1^2+\kappa_2^2)\\ &A_1\tanh^{p_1}\tau + p_1(p_1+1)A_1(B_1^2+B_2^2)\tanh^{p_1+2}\tau\} + bA_1^{2m+1}\tanh^{(2m+1)p_1}\tau -\\ &-A_1A_2\tanh^{p_1+p_2}\tau = 0,\end{aligned} \tag{35}$$

$$-v\tanh^{p_1-1}\tau + v\tanh^{p_1+1}\tau - 2a(\kappa_1 B_1+\kappa_2 B_2)\tanh^{p_1-1}\tau + 2a(\kappa_1 B_1+\kappa_2 B_2)\tanh^{p_1+1}\tau = 0\,. \tag{36}$$

In this case the same value of the exponents p_1 and p_2 are obtained as in the (1+1) dimension case that are given by (11) and (12). Now from (36) we get

$$v = -2a(\kappa_1 B_1 + \kappa_2 B_2)$$

Setting the coefficients of $\tanh^{p_1+j}\tau,\ j=-2,0,2$ to zero gives

$$\omega = -\frac{a}{m^2}(2(B_1^2+B_2^2)+m^2(\kappa_1^2+\kappa_2^2))\,, \tag{37}$$

$$A_2(v^2 - K^2(B_1^2+B_2^2)) = A_1^{2m}(B_1^2+B_2^2)\,. \tag{38}$$

Finally, we get

$$a(m+1)(B_1^2+B_2^2)+bm^2A_1^{2m} - m^2A_2 = 0\,. \tag{39}$$

Solving Eqs. (38) and (39) we have

$$A_1 = \left[\frac{a(m+1)(B_1^2+B_2^2)(v^2-k^2(B_1^2+B_2^2))}{m^2((B_1^2+B_2^2)-b(v^2-k^2(B_1^2+B_2^2)))}\right]^{\frac{1}{2m}}, \tag{40}$$

$$A_2 = \frac{a(m+1)(B_1^2+B_2^2)^2}{m^2((B_1^2+B_2^2)-b(v^2-k^2(B_1^2+B_2^2)))}\,. \tag{41}$$

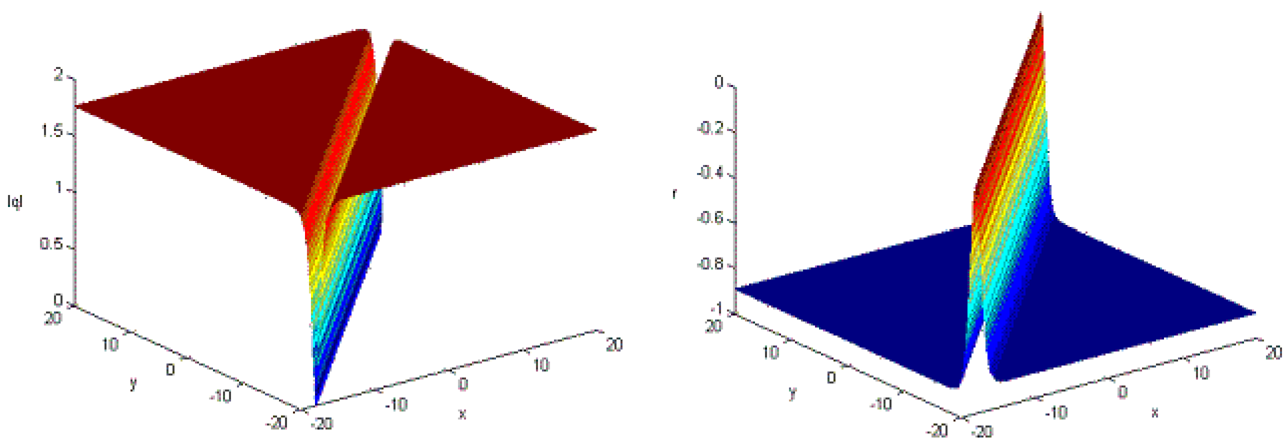
where

$$a(v^2-k^2(B_1^2+B_2^2))\{(B_1^2+B_2^2)-b(v^2-k^2(B_1^2+B_2^2))\} > 0\,. \tag{42}$$

Thus the solutions of (27) and (28) are given by

$$q(x,y,t) = A_1\tanh^{\frac{1}{m}}(B_1x+B_2y-vt)e^{i(-\kappa_1x-\kappa_2y+\omega t)}, \tag{43}$$

$$r(x,y,t) = A_1\tanh^2(B_1x+B_2y-vt)\,. \tag{44}$$



(a) (b)

Fig. 2 – Topological solution for the Zakharov equations in (1+2) dimensions.

The solutions of $|q(x,y,t)|^2$ and $r(x,y,t)$ are plotted in Fig. 2a and b, respectively. For this example, the parameter values that are chosen are

$$a=-1,\ b=1,\ k=1,\ B_1=1,\ B_2=-1,\ \kappa_1=1.15,\ ,\kappa_2=1,$$
$$m=1,\ -20\le x\le 20,\ -20\le y\le 20,\ t=2.$$

Although these values are chosen arbitrarily, the restriction given by (42) is valid for these set of parameter values.

3. CONCLUSIONS

This paper studied the Zakharov equation with power law nonlinearity in (1+1) dimensions as well as in (1+2) dimensions. The topological 1-soliton solutions were obtained. There were a couple of constraint conditions that were revealed during the course of derivation of the soliton solution. The conditions must remain valid in order for the soliton solutions to exist. These results are going to be of great importance in the context of Plasma Physics where Zakharov equation is analyzed in detail. This study holds a strong future. The conservation laws will be determined in future by the aid of Lie symmetry analysis. Additionally, the cnoidal waves and other solutions will be retrieved for this equation. The perturbation terms will be added and the adiabatic parameter dynamics will be obtained for these equations. The quasi-stationary soliton solutions will also be given by the aid of multiple-scale perturbation analysis. These open issues just form the tip of the iceberg.

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