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EQUILIBRIUM PROBLEMS OVER PRODUCT SETS

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Some types of equilibrium problems and systems of equilibrium problems on cones are studied. For these, we obtain equivalence results and prove, using a fixed - point theorem of Chowdury and Tan, existence results.

Key words: equilibrium problems, product sets, hemicontinuity, convexity, generalized pseudomonotonicity.

1. INTRODUCTION

The equilibrium problems were introduced in [7] and [11]. After that, systems of equilibrium problems were first considered in [3]. Since then, many different classes of such systems were studied [1, 2, 4, 5]. Here, we present some new classes of equilibrium problem systems over product set. These results represent, in a certain sense, a generalization of those obtained in [6] for variational inequalities over product sets.

2. FORMULATION OF EQUILIBRIUM MODELS AND SOME PRELIMINARY RESULTS

Let $I = \{1,2,\ldots,m\}$ be a finite index set and X_i, for each $i \in I$, be a real topological vector space, with K_i a nonempty convex subset. We put $X = \prod_{i\in I} X_i$ and $K = \prod_{i\in I} K_i$. For $x_i \in X_i$, $i \in I$, we denote $x = (x_i)_{i\in I} \in X$. For a real topological vector space Y, let C be a proper, closed and convex cone with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ denotes the topological interior of C in Y. Thus we consider Y to be a partial order wrt cone C.

Let for each $i \in I$, an arbitrary set Y_i and $f_i : K \to Y_i$. Also we define for each $i \in I$, a map $\Psi_i : Y_i \times K_i \times K_i \to Y$ and $A_i : K \to 2^{K_i}$ be a multivalued map with nonempty convex values. We put $f = (f_i)_{i\in I}$, $\Psi = (\Psi_i)_{i\in I}$ and define a multivalued map $A(x) = \prod_{i\in I} A_i(x)$.

Now we consider the following vector equilibrium problems over the product set K:

(Ψ - VEP) find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and

$$\sum_{i\in I} \Psi_i\left(f_i(\bar{x}), \bar{x}_i; y_i\right) \notin -\operatorname{int} C, \ \forall y_i \in A_i(\bar{x}),\ i \in I;$$

and the Minty type vector equilibrium problem

(Ψ - MVEP) find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and

$$\sum_{i\in I} \Psi_i\left(f_i(y), \bar{x}_i; y_i\right) \notin -\operatorname{int} C, \ \forall y_i \in A_i(\bar{x}),\ i \in I.$$

Also, we consider the Stampacchia type system of vector equilibrium problems:

(Ψ - SVEP) find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and

$$\Psi_i\left(f_i\left(\bar{x}\right), \bar{x}_i; y_i\right) \notin -\operatorname{int} C, \ \forall y_i \in A_i\left(\bar{x}\right), \ i \in I.$$

We denote by K_s, K_s^m and K_{ss} the solution sets of (Ψ - VEP), (Ψ - MVEP) and (Ψ - SVEP), respectively.

In the following we introduce some classes of mappings which extend the ones of relatively pseudomonotony, relatively maximal pseudomonotony and hemicontinuity. Further, some basic results relatively to this classes are stated.

Definition 2.1. The family $\{f_i\}_{i \in I}$ is

(i) relatively pseudomonotone wrt Ψ if for all $x, y \in K$ we have

$$\sum_{i \in I} \Psi_i\left(f_i(x), x_i; y_i\right) \notin -\operatorname{int} C \Rightarrow \sum_{i \in I} \Psi_i\left(f_i(y), x_i; y_i\right) \notin -\operatorname{int} C;$$

(ii) relatively maximal pseudomonotone wrt Ψ if it is relatively pseudomonotone wrt Ψ and for all $x, y \in K$ we have

$$\sum_{i \in I} \Psi_i\left(f_i(z), x_i; z_i\right) \notin -\operatorname{int} C, \ \forall z \in (x, y] \Rightarrow \sum_{i \in I} \Psi_i\left(f_i(x), x_i; y_i\right) \notin -\operatorname{int} C,$$

where $(x, y] = \prod_{i \in I} (x_i, y_i]$.

Definition 2.2. The family $\{f_i\}_{i \in I}$ is hemicontinuous wrt Ψ if for all $x, y \in K$ and $\lambda \in [0,1]$, the mapping $\lambda \mapsto \sum_{i \in I} \Psi_i\left(f_i\left(x + \lambda(y - x)\right), x_i; y_i\right)$ is continuous.

Now we consider some relation between the sets K_s, K_{ss} and K_s^m.

LEMMA 2.1. *We suppose the family* $\{f_i\}_{i \in I}$ *is hemicontinuous wrt* Ψ *and for each* $i \in I$, $\Psi_i\left(f_i(x), x_i; x_i\right) = 0$ *for any* $x \in K$. *Then* $K_s \subseteq K_{ss}$.

Proof. Let $\bar{x} \in K_s$. Now we see that $x_i \in A_i(\bar{x})$ for all $i \in I$, then $x = (x_i)_{i \in I} \in A(\bar{x})$. Since $\bar{x} \in A(\bar{x})$ we have that also y defined by $y_i = x_i$ with arbitrarily fixed $i \in I$ and $y_j = \bar{x}_j$ for each $j \neq i$ is an element of K_s. Using hemicontinuity and sequentially substituting y in (Ψ - VEP), with $i = 1, 2, \ldots, n$, we get that $\bar{x}$ is a solution of (Ψ - SVEP), i.e, $\bar{x} \in K_{ss}$ and lemma is proved.

LEMMA 2.2. *We suppose*

(i) *the family* $\{f_i\}_{i \in I}$ *is relatively maximal pseudomonotone wrt* Ψ;

(ii) *for each* $i \in I$, A_i *is nonempty and convex-valued map.*

Then $K_s = K_s^m$.

Proof. Using the assumption of relatively pseudomonotonicity wrt Ψ, we get easily that $K_s \subseteq K_s^m$.

Now let $\bar{x} \in K_s^m$. Then $\bar{x} \in A(\bar{x})$ and

$$\sum_{i \in I} \Psi_i\left(f_i(y), \bar{x}_i; y_i\right) \notin -\operatorname{int} C, \ \forall y_i \in A_i(x). \tag{2.1}$$

But we have that $\left(\bar{x}_i, y_i\right] \subset A_i\left(\bar{x}\right)$ for any $i \in I$. Hence by (2.1) we obtain

$$\sum_{i \in I} \Psi_i\left(f_i(z), \bar{x}_i; z_i\right) \notin -\operatorname{int} C, \ \forall z_i \in \left(\bar{x}_i, y_i\right], \ i \in I$$

Now using again relatively pseudomonotonicity, we obtain

$$\sum_{i\in I}\Psi_i\left(f_i(\overline{x}),\overline{x}_i;y_i\right)\notin -\operatorname{int}C,\ \forall y_i\in A(\overline{x}),\ i\in I,$$

i.e., $\overline{x}\in K_s$ or $K_s^m\subseteq K_s$. Thus the proof is complete.

Definition 2.3 [8]. A subset B of a topological space E is said to be compactly open (respectively, compactly closed) in E if, for any nonempty compact subset D of E, $B\cap D$ is open (respectively, closed) in D.

THEOREM 2.1 [8]. *Let K be a nonempty convex subset of a topological vector space (not necessarily Hausdorff) E and let $S,T:K\to 2^K$ be multivalued maps. Assume the following conditions hold:*

(a_1) For all $x\in K$, $S(x)\subseteq T(x)$;

(b_1) For all $x\in K$, $T(x)$ is convex and $S(x)$ is nonempty;

(c_1) For all $y\in K$, $S^{-1}(y)=\{x\in K\mid y\in S(x)\}$ is compactly open (i.e. for any nonempty compact subset D of E, $S^{-1}(y)\cap D$ is open in D);

(d_1) There exist a nonempty closed compact (not necessarily convex) subset D of K and a $\widetilde{y}\in D$ such that $K\setminus D\subset T^{-1}(y)$.

Then, there exists $\hat{x}\in K$ such that $\hat{x}\in T(\hat{x})$.

3. MAIN RESULTS

Let for each $i\in I$, X_i be a real topological vector space, Y, C, K_i, A_i for $i\in I$, K and A defined as in Section 2. Further we assume that for each $i\in I$ and for all $y_i\in K_i$, $A_i^{-1}(y_i)$ is compactly open in K, and the set $\mathcal{F}=\{x\in K\mid x\in A(x)\}$ is compactly closed.

THEOREM 3.1. *We assume*

(i_1) *the family $\{f_i\}_{i\in I}$ is relatively maximal pseudomonotone wrt Ψ;*

(i_2) *there exist a nonempty closed and compact set D of K and $\widetilde{y}\in D$ such that*

$$\sum_{i\in I}\Psi_i\left(f_i(x),x_i;\widetilde{y}_i\right)\in -\operatorname{int}C,\ \textit{for } x\in K\setminus D \textit{ with } \widetilde{y}\in A(x);$$

(i_3) *the mapping $y\mapsto \sum_{i\in I}\Psi_i\left(f_i(x),x_i;y_i\right)$ is quasi convex on K for any $x\in K$;*

(i_4) $\sum_{i\in I}\Psi_i\left(f_i(x),x_i;x_i\right)=0$, $\forall x\in K$.

Then $K_s\neq\emptyset$ and $K_{ss}\neq\emptyset$.

Proof. The proof of this theorem is based on Theorem 2.1. In order to do this, we construct two applications S and T that satisfy the hypotheses of the above mentioned theorem.

Let the multivalued maps $S,T:K\to 2^K$ given by

$$S(x)=\begin{cases} A(x)\cap\left\{y\in K\,\middle|\,\sum_{i\in I}\Psi_i\left(f_i(y),x_i;y_i\right)\in -\operatorname{int}C\right\} & \text{if } x\in\mathcal{F}\\ A(x) & \text{if } x\in K\setminus\mathcal{F}\end{cases}$$

and

$$T(x)=\begin{cases} A(x)\cap\left\{y\in K \middle| \sum_{i\in I}\Psi_i\left(f_i(x),x_i;y_i\right)\in-\operatorname{int}C\right\} & \text{if } x\in\mathcal{F} \\ A(x) \ \text{ if } x\in K\setminus\mathcal{F} \end{cases}.$$

If $P,Q:K\to 2^K$ are given by

$$P(x)=\left\{y\in K \middle| \sum_{i\in I}\Psi_i\left(f_i(y),x_i;y_i\right)\in-\operatorname{int}C\right\}$$

and

$$Q(x)=\left\{y\in K \middle| \sum_{i\in I}\Psi_i\left(f_i(x),x_i;y_i\right)\in-\operatorname{int}C\right\}$$

then we have

$$S(x)=\begin{cases} A(x)\cap P(x) & \text{if } x\in\mathcal{F} \\ A(x) & \text{if } x\in K\setminus\mathcal{F}\end{cases} \quad\text{and}\quad T(x)=\begin{cases} A(x)\cap Q(x) & \text{if } x\in\mathcal{F} \\ A(x) & \text{if } x\in K\setminus\mathcal{F}\end{cases}.$$

By (i_3) we get that for each $x\in K$, $Q(x)$ is convex and then by (i_1) we obtain $P(x)\subseteq Q(x)$ for any $x\in K$.

Since for each $y\in K$ the complement of $P^{-1}(y)$ in K is given by

$$\left[P^{-1}(y)\right]^c=\left\{x\in K \middle| \sum_{i\in I}\Psi_i\left(f_i(y),x_i;y_i\right)\notin-\operatorname{int}C\right\}$$

is a closed set in K, we have that the set $P^{-1}(y)$ is an open set in K. Thus, the set $P^{-1}(y)$, for any $y\in K$, is a compactly open set.

Also we see that $A(x)$ is a nonempty convex set. Since for any $i\in I$ and $y_i\in K_i$, $A_i^{-1}(y_i)$ is compactly open set, then $A^{-1}(y)=\bigcap_{i\in I}A_i^{-1}(y_i)$ is a compactly open set in K for all $y\in K$.

Thus, for all $x\in K$, $T(x)$ is a convex set with $S(x)\subseteq T(x)$, i.e., (a_1) and the first condition of (b_1) from

THEOREM 2.1. According to [9], Lemma 2.3, we have

$$S^{-1}(y)=\left(A^{-1}(y)\cap P^{-1}(y)\right)\cup\left((K\setminus\mathcal{F})\cap A^{-1}(y)\right).$$

Now, since for each $y\in K$, $A^{-1}(y)$, $P^{-1}(y)$ and $K\setminus\mathcal{F}$ are compactly open sets, then the set $S^{-1}(y)$ is compactly open (see [10]), i. e. (c_1) from Theorem 2.1 hold.

Now we prove that there exists $\bar{x}\in\mathcal{F}$, $A(\bar{x})\cap P(\bar{x})=\emptyset$. In order to do this, we suppose that $A(x)\cap P(x)\neq\emptyset$ for all $x\in\mathcal{F}$. Hence $S(x)\neq\emptyset$ for all $x\in K$, i.e., (b_1) from Theorem 2.1 is true. Finally, we observe that (i_2) is equivalent with (d_2) from the same theorem. Therefore we can apply this result. Thus, there exists $x^0\in K$ such that $x^0\in T(x^0)$. Because $\{x\in K | x\in T(x)\}\subseteq\{x\in K | x\in A(x)\}=\mathcal{F}$, we get $x^0\in\mathcal{F}$ and $x^0\in A(x^0)\cap Q(x^0)$. But $x^0\in Q(x^0)$ implies that

$$\sum_{i\in I}\Psi_i\left(f_i\left(x^0\right),x^0;x^0\right)\in -\operatorname{int}C,$$

which contradicts (i_4).

Therefore, there exists $\bar{x}\in\mathcal{F}$ with $A(\bar{x})\cap P(\bar{x})=\varnothing$. This statement is equivalent with $\bar{x}\in A(\bar{x})$ and $\sum_{i\in I}\Psi_i\left(f_i(y),x_i;y_i\right)\notin -\operatorname{int}C$, $\forall y_i\in A_i(\bar{x})$, $i\in I$, i.e., $\bar{x}\in K_s^m$ and by Lemma 2.2 we get $\bar{x}\in K_s$.

Finally, from the above result and from Lemma 2.1, we obtain $\bar{x}\in K_{ss}\neq\varnothing$, and the theorem is proved.

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