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ON A NEW SPACE OF INFINITE MATRICES

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In this paper we introduce and study some properties for a new class of linear operators namely $B_w^v(\ell^2)$. We characterize some special classes of this kind of matrices and we prove some new results concerning Schur multipliers. In particular, we prove that the space of Schur multipliers from $B_w^v(\ell^2)$ to $B_w^v(\ell^2)$ contains all matrices which represent bounded operators from ℓ^2 into ℓ^∞.

Key words: infinite matrices, Toeplitz matrices, Schur multipliers, diagonal matrices.

1. INTRODUCTION

In this paper we introduce a new class of Banach space of infinite matrices and we state and prove some properties of this space. On one hand we introduce this space motivated by the previous papers [3, 4, [5, 9, 10], on the other hand for the potential to apply some results of positive operators on cones in economy (see e.g. [1, 2]).

The space $B_w(\ell^2)$ consisting of infinite matrices A such that $A(x)\in\ell^2$ for every $x=(x_n)_n\in\ell^2$ with $|x_n|\downarrow 0$ has been studied in [9] and also in [10]. This space can be regarded as a "weak" variant of the classic space $B(\ell^2)$ since consists in those matrices which apply the decreasing sequences in absolute value from ℓ^2 in ℓ^2. More precisely, this space was introduced by N. Popa and has been appeared in the study of matricial analogue of Fejer's theory. The analogue of Fejer's theory in the framework of infinite matrices can be found in [4]. In our present paper we generalize this space and this represents another motivation to study this space.

Let $(v_n)_{n\geq 1}$ be a sequence of nonnegative real numbers. We define a space of infinite matrices denoted by

$$B_w^v(\ell^2)=\{A \text{ infinite matrix}; Ax\in\ell^2 \text{ for every } x=(x_n)_n\in\ell^2,\ \text{with } \frac{|x_n|}{v_n}\downarrow 0\}.$$

On this space we consider the following norm

$$\|A\|_{B_w^v(\ell^2)}=\sup_{(|x_n|/v_n)\downarrow}\frac{\|Ax\|_{\ell^2}}{\|x\|_{\ell^2}}.$$

It is clear that the space $B_w^v(\ell^2)$ is a Banach space with the above norm. Moreover $B_w^v(\ell^2)=B_w(\ell^2)$, when $v_n=1$ for every $n\geq 1$.

We define now the Schur product of two matrices (finite or infinite)

$$A * B = \left(a_{ij} \cdot b_{ij}\right)_{i,j\geq 1},$$

where $A = \left(a_{ij}\right)_{i,j\geq 1}, B = \left(b_{ij}\right)_{i,j\geq 1}$. We denote by

$$M\left(\ell^2\right) = \left\{ M : M * A \in B\left(\ell^2\right) \text{for every } A \in B\left(\ell^2\right) \right\}$$

the space of Schur multipliers which is a Banach space with the norm

$$\|M\| = \sup_{\|A\|_{B(\ell^2)} \leq 1} \|M * A\|_{B(\ell^2)}.$$

For an infinite matrix $A = (a_{ij})$ and an integer k, we denote by $A_k = (a'_{ij})$, where

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } j - i = k, \\ 0 & \text{otherwise,} \end{cases}$$

A_k is called Fourier coefficient of kth order associated to matrix A (see e.g. [4] and [3]).

For the convenience of the reader we present a theorem which can be found e.g. in [7]. The theorem is the analogy for $0 < p \leq 1$ of Sawyer's duality principle (see [6] and [11]).

THEOREM 1.1. *Let* $w = (w(n))_n$, $v = (v(n))_n$ *be two weights in* $\mathbb{N}^*$ *and let*

$$S = \sup_{f \downarrow 0} \frac{\sum_{n=0}^{\infty} f(n) v(n)}{\left(\sum_{n=0}^{\infty} f(n)^p w(n)\right)^{\frac{1}{p}}}.$$

If $0 < p \leq 1$, *then*

$$S = \sup_{n \geq 0} \frac{V(n)}{W^{\frac{1}{p}}(n)},$$

with W *defined by* $W(n) = \sum_{k=0}^{n} w(k)$, $n = 0,1,2,...$ *and* V *defined in the same way.*

The paper is organized as follows. In Section 2, the main result is a characterization of diagonal matrices from $B_w^v(\ell^2)$. Another result is the coincidence of the spaces $B_w^v(\ell^2)$ and $B(\ell^2)$ in the case of Toeplitz matrices. Finally, in the last Section, we state and prove some results concerning Schur multipliers. For instance, we prove that matrices that represents boundend operators from ℓ^2 into ℓ^2 are Schur multipliers from $B_w^v(\ell^2)$ to $B_w^v(\ell^2)$.

2. PARTICULAR CASES OF INFINITE MATRICES

We start this Section by giving a characterization of diagonal matrices in $B_w^v(\ell^2)$.

THEOREM 2.1. *Let* $v = (v_n)_n$ *be a monotone weight and the matrix* $A = A_0$ *given by the sequence* $a = (a_n)_n$. *Then*

$$A \in B_w^v(\ell^2) \textit{ if and only if } \sup_{n\geq 1} \frac{\sum_{k=1}^{n} |a_k|^2 v_k^2}{\sum_{k=1}^{n} v_k^2} < \infty.$$

Moreover the norm is

$$\|A\|_{B_w^v(\ell^2)} = \sup_{n\geq 1} \left(\frac{\sum_{k=1}^{n} |a_k|^2 v_k^2}{\sum_{k=1}^{n} v_k^2} \right)^{\frac{1}{2}}.$$

Proof. We will compute the term $\sup_{\frac{|x_n|}{v_n}\downarrow} \frac{\|Ax\|_{\ell^2}}{\|x\|_{\ell^2}}$. We have that

$$\sup_{\frac{|x_n|}{v_n}\downarrow} \frac{\|Ax\|_{\ell^2}}{\|x\|_{\ell^2}} = \sup_{\frac{|x_n|}{v_n}\downarrow} \frac{\left(\sum_{n=1}^{\infty} |a_n|^2 |x_n|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}} = \sup_{|y_n|\downarrow} \frac{\left(\sum_{n=1}^{\infty} |a_n y_n v_n|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=1}^{\infty} |y_n v_n|^2\right)^{\frac{1}{2}}},$$

where $y_n = \dfrac{x_n}{v_n}$, for all $n \geq 1$.

Let us denote

$$S := \sup_{|y_n|\downarrow} \frac{\left(\sum_{n=1}^{\infty} |a_n y_n v_n|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=1}^{\infty} |y_n v_n|^2\right)^{\frac{1}{2}}}.$$

Applying Theorem 1.1 with $p = 1$, $f(n) = |y_n|^2$, $v(n) = v_n^2 |a_n|^2$, $w(n) = v_n^2$ we obtain for every n,

$$V(n) = \sum_{k=1}^{n} v(k) = \sum_{k=1}^{n} v_k^2 |a_k|^2 \quad \text{and} \quad W(n) = \sum_{k=1}^{n} w(k) = \sum_{k=1}^{n} v_k^2.$$

It implies that

$$S = \sup_{n\geq 1} \left(\frac{\sum_{k=1}^{n} v_k^2 |a_k|^2}{\sum_{k=1}^{n} v_k^2} \right)^{\frac{1}{2}} \quad \text{and} \quad \|A\|_{B_w^v(\ell^2)} = \sup_{n\geq 1} \left(\frac{\sum_{k=1}^{n} v_k^2 |a_k|^2}{\sum_{k=1}^{n} v_k^2} \right)^{\frac{1}{2}}.$$

The proof is complete.

Remark 2.2. We remark that for every weight $v=(v_n)_n$ the following inclusion holds

$$B(\ell^2)\subseteq B_w^v(\ell^2). \tag{1}$$

When $v=(v_n)_n$ is bounded, it is clear that the inclusion (1) is proper. Next we give an example of unbounded weight such that the inclusion is also proper. For instance, if we take $v_n=n^{\alpha}$ with $\alpha>0$ and consider the matrix $A=A_0$ given by the sequence $a=(a_k)_k$ where

$$a_k=\begin{cases}2^{\frac{p-1}{2}-\alpha}, & \text{if } k=2^p\\ 0, & \text{otherwise,}\end{cases}$$

making easy calculations we get that $\|A\|_{B_w^v(\ell^2)}<\infty$ but the sequence $a=(a_k)_k$ is unbounded. This means that $A\notin B(\ell^2)$.

Although we remarked that in general, the spaces $B(\ell^2)$ and $B_w^v(\ell^2)$ are different, in the case of nondecreasing sequences these spaces coincide if we restrict to Toeplitz matrices.

THEOREM 2.3. *Let $v=(v_n)_n$ be a nondecreasing weight. Then*

$$B(\ell^2)\cap\mathfrak{T}=B_w^v(\ell^2)\cap\mathfrak{T},$$

where $B(\ell^2)\cap\mathfrak{T}$ and $B_w^v(\ell^2)\cap\mathfrak{T}$ represent the sets of Toeplitz matrices from $B(\ell^2)$ respectively $B_w^v(\ell^2)$.

Proof. By definition we have the following inclusions:

$$B(\ell^2)\subseteq B_w^v(\ell^2)\subseteq B_w(\ell^2). \tag{2}$$

It has been proved that in the case of Toeplitz matrices $B(\ell^2)$ and $B_w(\ell^2)$ coincide (see Theorem 9 from [9]).
Thus, using the Theorem mentioned before and the inclusions (2) it follows that $B(\ell^2)$ and $B_w^v(\ell^2)$ coincide. The proof is complete.

3. SCHUR MULTIPLIERS

It is well known that the classical space $B(\ell^2)$ is closed under Schur multiplication (see e.g. [5]), although $B_w(\ell^2)$ in not (see e.g. [9]). It is easy to see that $B_w^v(\ell^2)$ is also not closed under Schur multiplication. For example, we can use the matrix $A=A_0$ from Remark 2.2. Using Theorem 2.1 and easy computations we can observe that $A*A$ is not in $B_w^v(\ell^2)$, when $v_n=n^{\alpha}$ with $\alpha>0$. However, all infinite matrices from $B(\ell^2)$ are Schur multipliers from $B_w^v(\ell^2)$ in $B_w^v(\ell^2)$.

THEOREM 3.1. *Let $M(B_w^v(\ell^2),B_w^v(\ell^2))$ be the space of Schur multipliers from $B_w^v(\ell^2)$ to $B_w^v(\ell^2)$. Then we have that:*

$$B(\ell^2,\ell^{\infty})\subset M(B_w^v(\ell^2),B_w^v(\ell^2)).$$

Proof. Let us take arbitrary $A \in B(\ell^2, \ell^\infty)$ and $B \in B_w^v(\ell^2)$. Then the following inequalities hold:

$$\sum_j |\sum_k a_{jk} b_{jk} x_k|^2 \le \sum_j \left(\sum_k |a_{jk}||b_{jk}||x_k| \right)^2 \le \sum_j \left(\sum_k |a_{jk}|^2 \right) \sum_k \left(|b_{jk}|^2 |x_k|^2 \right) \le$$
$$\le \sup_j \left(\sum_k |a_{jk}|^2 \right) \sum_j \left(\sum_k |b_{jk}|^2 |x_k|^2 \right). \tag{3}$$

For estimating the last term from (3) we will use Rademacher functions $r_k(t) = sgn \sin(2^n \pi t)$ on $[0,1]$, for $k \ge 1$ (see e.g. [8] p. 126)

$$\sum_k |z_k|^2 = \int_0^1 |\sum_k z_k r_k(t)|^2 \, dt.$$

It follows that

$$\sum_j \left(\sum_k |b_{jk}|^2 |x_k|^2 \right) = \sum_j \int_0^1 |\sum_k b_{jk} x_k r_k(t)|^2 \, dt \le \operatorname{esssup}_{t \in [0,1]} \sum_j |\sum_k b_{jk} x_k r_k(t)|^2 \le \|B\|^2_{B_w^v(\ell^2)} \|x\|_2^2 .$$

Thus we have that

$$\|A * B\|_{B_w^v(\ell^2)} \le \|A\|_{2,\infty} \cdot \|B\|_{B_w^v(\ell^2)},$$

the required inclusion holds and the proof is complete.

COROLLARY 3.2. *Let* $M(B_w^v(\ell^2), B_w^v(\ell^2))$ *be the space of Schur multipliers from* Theorem 3.1. *Then we have that* $B(\ell^2) \subset M(B_w^v(\ell^2), B_w^v(\ell^2))$.

Proof. The inclusion results immediately from the above theorem and from inequality $\|A\|_{2,\infty} \le \|A\|_{B(\ell^2)}$.

The following result is in fact a characterization of diagonal matrices which are multipliers from $B_w^v(\ell^2)$ in $B_w^v(\ell^2)$.

THEOREM 3.3. *Let us take* $B = B_0$ *given by the sequence* $b = (b_k)_{k \ge 1}$. *Then we have that* $B \in M(B_w^v(\ell^2), B_w^v(\ell^2))$ *if and only if* $b = (b_k)_{k \ge 1}$ *is bounded.*

Proof. First we prove for matrices with positive entries.

Let $B \in M(B_w^v(\ell^2), B_w^v(\ell^2))$, then $B * A \in B_w^v(\ell^2)$, for every $A \in B_w^v(\ell^2)$. In particular, $B * A_0 \in B_w^v(\ell^2)$, where A_0 represents a diagonal matrix given by $a = (a_k)_{k \ge 1}$. Thus, we have that $A_0 x \in \ell^2$ for every $x \in \ell^2$ such that $\frac{|x_k|}{v_k} \downarrow 0$. Since $(B * A_0)x \in \ell^2$ it follows $b = (b_k)_{k \ge 1} \in \ell^\infty$.

For sufficiency we assume that $b = (b_k)_{k \ge 1}$ is a bounded sequence of real numbers. We claim that

$\|A_0\|_{B_w^v(\ell^2)} \le \|A\|_{B_w^v(\ell^2)}$, (4)

for every matrix $A \in B_w^v(\ell^2)$.

Then we have

$$\|(B * A)x\|_2^2 = \|(B * A_0)x\|_2^2 = \|BA_0 x\|_2^2 \le \|b\|_\infty^2 \cdot \|A_0 x\|_2^2 \le \|b\|_\infty^2 \cdot \|A_0\|^2_{B_w^v(\ell^2)} \cdot \|x\|_2^2 \le \|b\|_\infty^2 \cdot \|A\|^2_{B_w^v(\ell^2)} \cdot \|x\|_2^2, \tag{4}$$

for every $x=(x_k)_{k\geq 1}\in \ell^2$ such that $\frac{|x_k|}{v_k}\downarrow 0$.

In the case of positive matrices the proof is complete since the inequality (4) is trivial in these settings. In the general case, for proving (4) we can use Rademacher functions. Using the same arguments as in the proof of Theorem 3.1, where r_k for $k\geq 1$, are Rademacher functions, we have that

$$\|A_0x\|_2^2=\sum_k |a_{kk}x_k|^2\leq \sum_j\sum_k |a_{jk}x_k|^2=\sum_j\int_0^1\left|\sum_k a_{jk}x_kr_k(t)\right|^2 \mathrm{d}t\leq \operatorname{esssup}_{t\in[0,1]}\sum_j\left|\sum_k a_{jk}x_kr_k(t)\right|^2\leq \|A\|^2_{B_w^v(\ell^2)}\|x\|_2^2,$$

for every $x=(x_k)_{k\geq 1}\in \ell^2$ with $\frac{|x_k|}{v_k}\downarrow 0$. Thus the inequality (4) is proved and the proof is complete.

COROLLARY 3.4. *In the case of diagonals $M(\ell^2)$ and $M(B_w^v(\ell^2))$ coincide.*

Proof. The result follows from the characterization of diagonals from $M(\ell^2)$ and the previous theorem.

In the last theorem of this Section we prove that in the case of Toeplitz matrices, the multipliers from $B_w^v(\ell^2)$ into $B_w^v(\ell^2)$ are Schur multipliers on $B(\ell^2)$.

THEOREM 3.5. *For all Toeplitz matrices the following inclusion holds*:

$$M(B_w^v(\ell^2))=M(B_w^v(\ell^2),B_w^v(\ell^2))\subseteq M(\ell^2).$$

Proof. The proof is based on the Theorem 8.1 from [5]. Let $M=(m_{jk})_{jk}$ be a Toeplitz matrix of the form

$$m_{jk}=c_{j-k},\ j,k=0,1,2,\cdots \tag{5}$$

such that $M\in M(B_w^v(\ell^2))$. Using Theorem 2.3 it follows that there exists a complex Borel measure μ on the unit circle such that

$$\hat{\mu}(n)=c_n \quad \text{for } n=0,\ \pm 1,\ \pm 2,\cdots.$$

Moreover,

$$\|\mu\|\leq \|M\|_{M(B_w^v(\ell^2))}.$$

Applying Theorem 8.1 from [5] it follows that Toeplitz matrix (5) is in $M(\ell^2)$. The proof is complete.

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