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DARK VORTEX SOLITONS IN DEFOCUSING KERR MEDIA MODULATED BY A FINITE RADIAL LATTICE

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We study the existence and stability of dark fundamental and vortex solitons in defocusing Kerr media with an imprinted finite radial lattice. The size of dark core of solitons is determined by the scale of lattice. The existence domains of both types of solitons expand with the growth of lattice size. When the lattice region is large, solitons are restricted in a hoof-like area. While fundamental solitons are always stable, dark vortices with unit charge can propagate stably under appropriate conditions. In addition, we reveal that the finite radial lattice plays an important role for the stabilization of higher-charged vortex solitons. This is in sharp contrast to the higher-charged vortices in defocusing bulk media, where they unavoidably break into a set of single-charged vortices. Thus, we propose the first theoretical model which allows the existence of stable higher-charged dark vortex solitons.

Key words: dark solitons, finite lattices, defocusing nonlinearity.

1. INTRODUCTION

Vortex solitons are self-localized subjects in nonlinear systems [1, 2]. They are characterized by a phase singularity, at which the field amplitude of beam is strictly zero, and the phase becomes undetermined. Around the pivotal point, the phase charge of a simple closed curve is equal to $2m\pi$, where m is the so-called topological charge. In the context of superfluids, a two-dimensional nonlinear Schrödinger equation with defocusing nonlinearity was shown to support vortex solitons whose intensity asymptotically approaches a constant value at infinity [3]. In contrast to the vortex solitons with finite size in focusing media, such delocalized nonlinear modes are usually defined as dark vortex solitons [2, 4, 5].

Experimentally, dark vortex solitons were observed in a bulk self-defocusing optical medium [6]. Dark vortices with unit charge are always stable in defocusing Kerr media, whereas higher-charged dark vortices are always unstable and will break into a set of single-charged dark vortices. Strong instabilities of higher-charged dark vortices can be triggered by different mechanisms such as dissipation, nonlinearity saturation, or anisotropy [7].

Thus far, bright vortex solitons have been investigated in a variety of schemes, see, e.g., [8–14]. Moreover, diverse settings were proposed to support stable localized vortex solitons with higher charges [15–17]. In comparison with this, the existence of stable higher-charged dark vortex solitons is still an open problem.

Recently, various types of 2D solitons in Bose-Einstein condensates with self-attraction or self-repulsion, trapped in an axially symmetric lattice periodic along the radius were reported in [18]. In Ref. [18], the optical-lattice potential is of infinite size. On the other hand, 1D optical lattices with finite sizes were shown to support stable symmetric and antisymmetric solitons with nonvanishing intensities at infinity [19]. When the lattice site number is large, the properties of solitons in finite lattices approach to those in infinite lattices. For example, the existence domain of solitons in finite lattices is restricted by the bandgap of the infinite periodic lattice. It indicates that one can construct a radial ring potential with finite size to support radially symmetric nonlinear modes with nonvanishing intensities at infinity, i.e., dark solitons with a cylindrical symmetry.

The subject of the present paper is to find a theoretical model to support stable dark vortex solitons with higher topological charges. We reveal that the defocusing Kerr media with an imprinted radial lattice with finite size admit delocalized fundamental and vortex soliton solutions. The size of dark core of soliton is determined by the scale of lattice. With the growth of lattice size, the existence domains of both types of solitons expand. Although the introduction of finite ring lattice may result in the instability of vortices with unit charge, it can improve the stability of dark vortex solitons with higher topological charges.

2. THEORETICAL MODEL

We describe the propagation of a light beam in a defocusing Kerr medium with a transverse refractive index modulation by a nonlinear Schrödinger equation for the field amplitude A. Namely,

$$i\frac{\partial A}{\partial z}=-\frac{1}{2}\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}\right)+|A|^2 A-pR(x,y)A, \tag{1}$$

here x, y and z are the normalized transverse and longitudinal coordinates, respectively; p is the depth of refractive index modulation; the refractive index profile is given by $R(x,y)=\cos^2(\Omega r)$ for $r\le(2N-1)\pi/(2\Omega)$ and $R(x,y)=0$ otherwise, where $r=(x^2+y^2)^{1/2}$ is the radial distance, Ω is the frequency, and $N=1,2...$ is the number of rings. Obviously, the finite radial lattice includes a set of concentric rings. An example of such refractive-index landscapes is shown in Fig. 1(a).

Stationary solutions of Eq. (1) are searched by assuming $A(x,y,z)=q(r)\exp(ibz+im\phi)$, where q is a r-dependent real function describing the profile of stationary solution, b is a propagation constant, and m is the topological charge of vortex solitons. The nonlinear mode degenerates to a fundamental radially symmetric mode when $m=0$. Substitution of the light field into Eq. (1) leads to:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}q}{\mathrm{d}r}-\frac{m^2}{r^2}q-2bq-2q^3+2pRq=0, \tag{2}$$

which can be solved numerically by a relaxation iterative method. We apply the boundary conditions $q|_{r=0}=0, \mathrm{d}q/\mathrm{d}r|_{r\to\infty}=0$ for fundamental solitons and $\mathrm{d}q/\mathrm{d}r|_{r=0}=0, \mathrm{d}q/\mathrm{d}r|_{r\to\infty}=0$ for vortex solitons. Families of stationary solutions are determined by the propagation constant b, modulation depth p, modulation frequency Ω, and scale N. We vary b, p, N and fix $\Omega\equiv 2$ in following discussions unless stated otherwise. To quantitatively depict the energy flow of the vortex solitons trapped in the lattice region, we define the concept of "renormalized energy flow" based on the shapes of dark solitons as: $U_r=2\pi\int_{-\infty}^{\infty}\left[q(r)-|b|^{1/2}H(r)\right]^2 r\mathrm{d}r$, where $H(r)=1$ for $r\ge(2N-1)\pi/(2\Omega)$ and $H(r)=0$ otherwise.

To understand the stability of solitons, following the standard procedure, we perturb the stationary solution as: $A(x,y,z)=\left[q(r)+u(r)\exp(\lambda z+\mathrm{i}n\phi)+v^*(r)\exp(\lambda^* z-\mathrm{i}n\phi)\right]\exp(\mathrm{i}bz+\mathrm{i}m\phi)$, here the perturbation components $u,v<<1$ are normal-mode perturbations and can grow with a complex rate λ upon propagation and the superscript * represents complex conjugation. Here λ is the eigenvalue of the normal mode. Inserting this perturbed solution into Eq. (1) and dropping the higher-order terms in u,v, a linear eigenvalue problem is obtained:

$$\begin{aligned}
\mathrm{i}\lambda u&=-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{(m+n)^2}{r^2}\right)u+bu+(v+2u)q^2-pRu,\\
-\mathrm{i}\lambda v&=-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{(m-n)^2}{r^2}\right)v+bv+(u+2v)q^2-pRv.
\end{aligned} \tag{3}$$

The coupled equations can be solved by a finite-difference method. Linear stability of solitons is determined by the spectrum of the above linearization operator, and the existence of any eigenvalue with a positive real part implies the linear instability of solitons. Stationary solutions can be completely stable only when all real parts of eigenvalues equal zero.

3. NUMERICAL RESULTS

First, we address the properties of fundamental radially symmetric solitons in a defocusing medium with a built finite radial lattice. For nonlinear modes with $m=0$, it follows from Eq. (2) that $q(r\to\infty)=\sqrt{-b}$, which means that delocalized solitons can be found only when $b<0$. Due to the defocusing nonlinearity, the renormalized energy flow decreases monotonically with the growth of propagation constant (Fig. 1b). At fixed b, the renormalized energy flow decreases with the growth of p. The existence domain expands rapidly with the increase of N firstly and reaches a maximum expansion rate, afterwards, it expands slowly and approaches to a maximum ultimately (Fig. 1c).

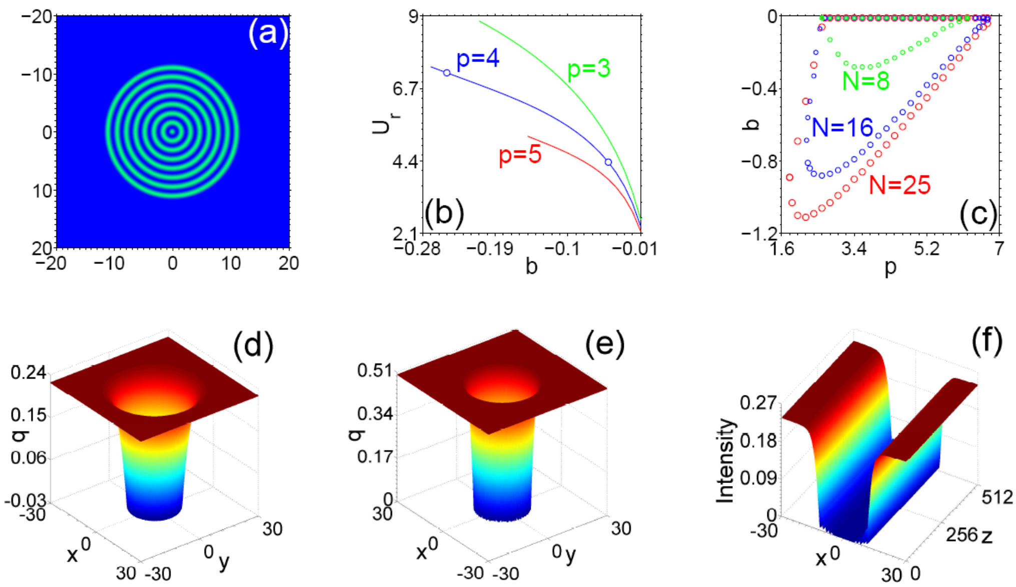

Fig. 1 – a) Typical example of refractive index modulation; b) renormalized energy flow of fundamental solitons versus propagation constant; c) existence domains of fundamental solitons for different N; d, e) stationary solutions of dark solitons marked in (b) $b = -0.04$ and -0.25, respectively; f) stable propagation of dark soliton shown in (e). Intensity distribution at $y = 0$ is shown in panel (f). Here $N = 8$ except for (c).

Although the radial ring lattice is of finite size, it will approaches to a radially periodic distribution if N is large. Since the term $1/r\mathrm{d}/\mathrm{d}r$ in Laplacian of Eq. (2) can be neglected at $r\to\infty$, the band-gap structure of a radially symmetric lattice is slightly different from that of 1D periodic lattice [20]. Thus, for large N, the existence domain of nonlinear modes will be approximately restricted into the first finite bandgap of the corresponding 1D periodic lattice. That is to say, the left and right boundaries of the existence domain approach to the upper and lower edges of the first bandgap Fig. 1c). It is the restriction of the first gap which accounts for the hoof-like existence domain.

Figures 1d and 1e show two profiles of fundamental solitons. Such soliton exhibits a pedestal in the form of a plane wave with an embedded dark hole. The height of the pedestal decreases monotonically with the growth of propagation constant since $q(r\to\infty)=(-b)^{1/2}$. In contrast to the bulk nonlinear medium without modulation, the size of the dark hole of soliton is determined by the radius of the outmost lattice ring. In other words, the distribution of soliton can be controlled by changing the size of finite lattice. Linear-stability analysis on the stationary solutions reveals that fundamental solitons are completely stable in their entire existence domain. A representative propagation example is shown in Fig. 1f. Note that due to the cylindrical symmetry of the soliton, intensity distribution of soliton at $y=0$ is displayed in Fig. 1f.

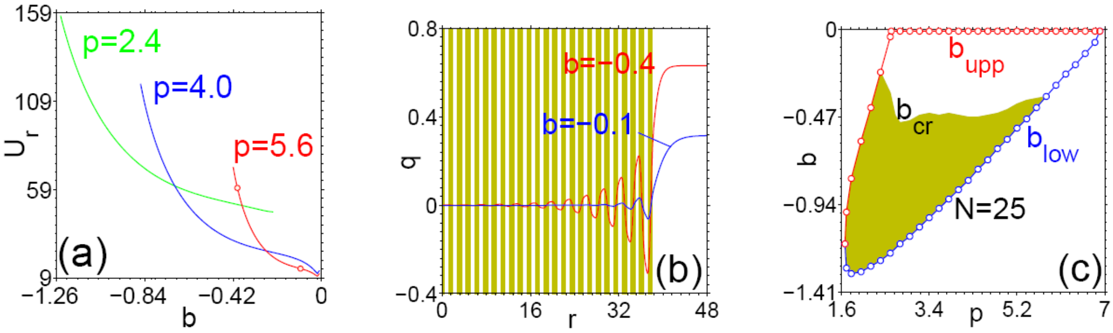

Fig. 2 – a) Renormalized energy flow of dark vortex solitons with unit charge versus propagation constant; b) profiles of vortices marked in (a) at different values of *b*. Bar areas denote the modulation of refractive index; c) areas of existence and instability (patched) on the (*p*, *b*) plane. In all panels $N = 25$.

Next, we explore the properties of dark vortex solitons with unit charge. In shallow lattice, there exists a nonzero upper cutoff of propagation constant. The renormalized energy flow of vortex solitons in deep lattices is a nonmonotonic function of propagation constant. The slope of renormalized energy flow changes its sign near the upper cutoff of propagation constant (Fig. 2a). Typical profiles of vortex solitons at different propagation constants are illustrated in Fig. 2b. As the propagation constant decreases, the height of pedestal increases with a faster speed for $b > -1$. In shallow lattices, the possibility of $b < -1$ results in that the growth rate of pedestal height is slower than the decrease rate of propagation constant. The amplitudes of oscillatory tails in the lattice region increase rapidly with the decrease of b, which accounts for the rapid growth of renormalized energy flow.

Vortex soliton exists only when the lattice depth exceeds a critical value. For example, as shown in Fig. 2c, the threshold value of lattice depth for the existence of vortices with unit charge is $p_{th} \approx 1.62$, below which no dark vortex solutions can be found. For $p \leq 2.45$, the existence domain expands with the lattice depth. It shrinks with the growth of lattice depth if $p \in (2.45, 6.94]$, due to the restriction of the ascending lower edge of the first bandgap. We should note that dark vortex solutions can be found if there is no lattice.

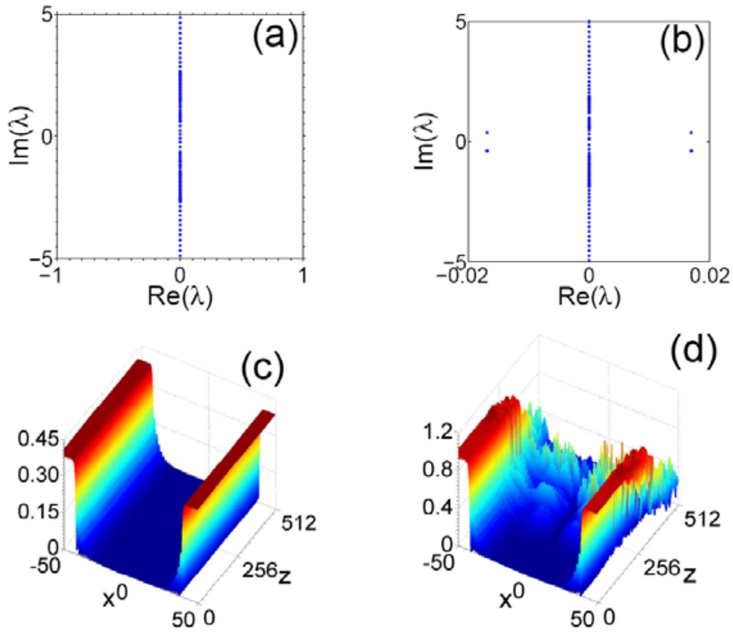

Fig. 3 – Spectra of the linearization operator (a, b) and the corresponding stable and unstable propagations (c, d) of dark vortex solitons. Intensity distributions at $y = 0$ are shown. Here $b = -0.4$ in (a, c) and -1 in (b, d); $p = 3$ in all panels.

Dark vortex soliton can be seen as a combination of a plane wave outside the lattice and a dark hole inside the lattice region, with a global screw phase dislocation. Since the plane wave always preserves its shape during propagation, the instability of dark vortex soliton is solely induced by its inner part (oscillatory tails) in the lattice region. To examine the stability of vortex solitons with unit charge, we conduct a linear-stability analysis on the stationary solutions according to Eqs. (3). The numerical results are summarized in Fig. 2c. Vortex with $m = 1$ is stable when its propagation constant exceeds a critical value b_{cr}. The precise structure of instability regions (patched) is rather complicated. There exist multiple narrow stability windows. By comparing the completely stable dark vortex solitons with unit charge in defocusing bulk media, one finds that the instability is induced by the introduction of external potential. To illustrate the details of instability, we show two examples of spectra of the linearization operator for vortices at $b = -0.4$ and -1 in lattice with $p = 3$ in Figs. 3a and 3b, respectively. The corresponding stable and unstable propagations are illustrated in Figs. 3c and 3d.

Besides the dark vortices with unit charge, we also find vortex solitons with higher charges. The properties of higher-charged dark vortices are similar to those of vortices with unit charge. For example, the renormalized energy flow of vortices with $m = 3$ is also a decreasing function of propagation constant (Fig. 4a). The existence domain still exhibits a hoof-like shape (Fig. 4b). Several profiles of dark vortices with $m = 3$ are displayed in Fig. 4c. The amplitudes of oscillatory tails in the lattice region decrease with the growth of propagation constant. The pedestal height still satisfies the relation $q(r \to \infty) = (-b)^{1/2}$. For comparison, we plot the profiles of dark vortices with $m = 1$ and 9 at $b = -0.5$ in Fig. 4d. The variation of topological charge almost has no influence on the distribution of soliton. It indicates that the renormalized energy flow of vortices decreases slightly with the growth of topological charge.

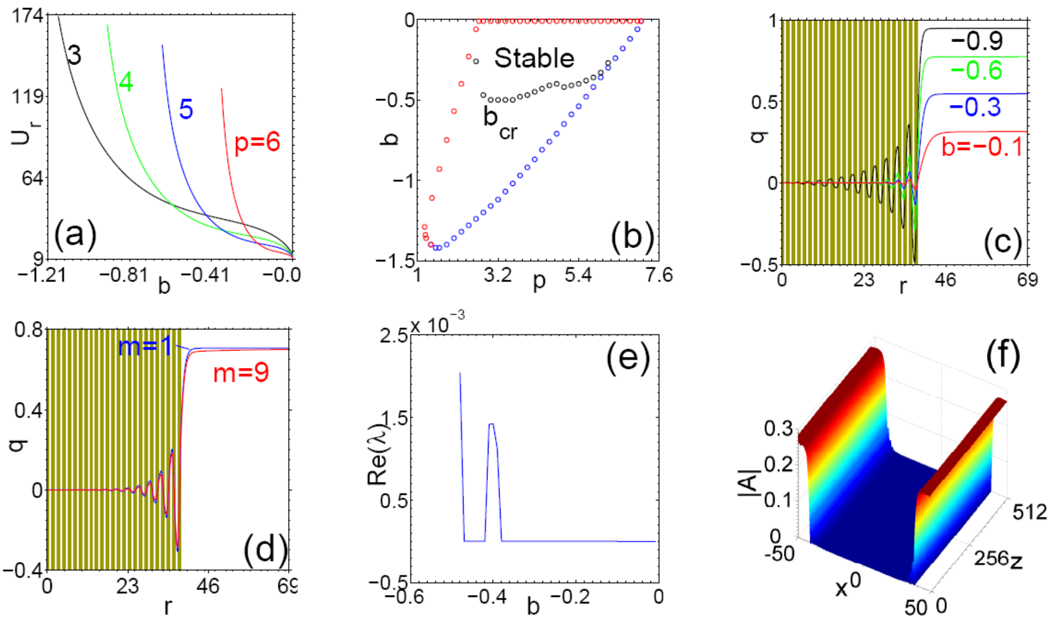

Fig. 4 – a) Renormalized energy flow of dark vortex solitons versus propagation constant; b) existence and stability domains of vortices with m=3 on the (p, b) plane; c) profiles of dark vortices at p=3; d) profiles of dark vortex solitons with m = 1 and 9 at b = –0.5; e) instability growth rate of dark vortices in lattice with p = 5.6; f) stable propagation of vortex soliton at b = –0.07, p = 5.6. N = 25 in all panels and m = 3 except for panel (d).

The stability of higher-charged dark vortex solitons is rather complicated. There exist numerous tiny stability windows below the critical propagation constant. The instability region is also similar to that of single-charged vortices (Fig. 4e). Near the upper cutoffs of propagation constant, a broad stability area allows dark vortices to propagate stably without any distortion. This is in sharp contrast to the higher-charged dark vortex solitons in defocusing bulk Kerr media, where the vortices will break into a set of vortices with unit charge. It is the confining potential which plays an important role for the stabilization of vortex solitons with higher charges. A stable propagation example of vortex soliton with $m = 3$ at $b = -0.07$, $p = 5.6$ is displayed in Fig. 4f.

4. CONCLUSIONS

In summary, we investigated the dynamics of dark fundamental and vortex solitons in defocusing Kerr media modulated by a radial ring lattice with finite size. The lattice can be utilized to control the size of the dark hole of solitons. The existence domains of both types of solitons expand with the growth of lattice size. When the lattice region is broad, solitons will be restricted in a hoof-like domain, which is confined by the edges of bandgap of the corresponding 1D periodic lattice. While fundamental solitons are always stable, dark vortices with unit charge can propagate stably in a broad region. Although the introduction of finite ring lattice may lead to the instability of vortices with unit charge, it can dramatically improve the stability of dark vortex solitons with higher topological charges, which is in sharp contrast to the higher-charged dark vortices in defocusing Kerr media, where they unavoidably break into several single-charged vortices. Our results may provide a helpful hint for the observation of dark vortex solitons with higher charges.

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