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ON ABELIAN HOPFION OF THE $\mathbf{CP}^2$ MODEL ON $\mathbf{R}^5$

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We propose a generalization of the three dimensional Skyrme-Fadde'ev model consisting in Abelian Hopfions defined on $\mathbf{R}^5$. An explicit ansatz is presented and the reduced action of the model is computed. We also evaluate the topological charge of the configurations.

Key words: Skyrme-Fadee'ev model, Abelian Hopfions, topological charge.

1. INTRODUCTION

The usual solitons, including instantons, and monopoles and vortices as well as their gauge decoupled versions (when these exist in the model at hand) are stabilised by topological charges which with the appropriate normalisations take on integer values [1]. The field theoretic models which support instantons are gauge field theories typified by asymptotically *pure gauge* connection, while the monopoles are (non-Abelian) gauged Higgs theories typified by asymptotically *half pure gauge* connection resulting in Dirac-Yang fields in that domain. Vortices are also supported by Higgs theories, but with Abelian connection. Both vortices and monopoles can have gauge decoupling limits [2, 3]. Apart from vortices, which are by necessity defined on $\mathbf{R}^2$ only, monopoles and instantons can be defined on all $\mathbf{R}^D$ (see [4]). These topological charges are Chern-Pontryagin (CP) charges or their descendents, or simply winding numbers.

What distinguishes Hopfions from the above mentioned usual solitons is, that they are **not** stabilised by CP charges, but rather by Chern--Simons (CS) charges, namely the volume integral of the CS density in the given dimensions. These are solutions to the $O(3)$ (or $\mathbf{CP}^1$) sigma model on $\mathbf{R}^3$, which were thoroughly investigated in the literature starting with the pioneering work [5]. These are topological solitons in systems involving a field $\Phi: R^2 \to S^2$. Such a field configuration is classified topologically by its Hopf number $N \in \pi_3(S^2)$. It is important to distinguish the CS densities in play here, from what one might call dynamical CS densities as those in [6], which appear as part of a Lagrangian density on a Minkowski space. The latter are defined in terms of Yang-Mills (YM) fields, while the CS densities used to stabilise Hopfions are defined instead in terms of *composite connections* (and their curvatures), constructed from a nonlinear sigma model of scalar fields on a Euclidean space. Of course, both dynamical CS densities [6] and those pertaining to Hopfions, can be defined only on odd dimensional spaces.

The familiar Hopfions mentioned above are defined on $\mathbf{R}^3$, but since CS densities can be defined just as well on any $\mathbf{R}^{2n+1}$, it may be worth considering such solutions. To date, models supporting such solutions have not been considered. This is essentially a problem of academic interest, although *knots* in 5 dimensions have been considered recently [7]. Nevertheless, relatively little is known about higher dimensional generalisations of Hopfions. It is our intention here to investigate this question in the simplest next case, namely that of an Abelian Hopfion on $\mathbf{R}^5$. To this end, one can employ *either* the $O(5)$ sigma model, or the $\mathbf{CP}^2$ sigma model, on $\mathbf{R}^5$. We have chosen to work with the second of these, but the equivalent analysis can be readily carried out in the first case also.

2. THE $\mathbf{CP}^n$ MODELS ON $\mathbf{R}^{2n+1}$: THE CASE n=2

We start with the generic structure of models that can support Abelian Hopfion on $\mathbf{R}^{2n+1}$. These are described by complex n-tuplets

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ .. \\ .. \\ z_{n+1} \end{bmatrix} \equiv z_a \ ; \ \overline{Z} = \begin{bmatrix} \overline{z}^1 \\ \overline{z}^2 \\ .. \\ .. \\ \overline{z}^{n+1} \end{bmatrix} \equiv \overline{z}^a \quad a = 1, 2, ..., n+1, \tag{1}$$

subject to the constraint

$$Z^\dagger Z \equiv \overline{z}^a z^a = 1, \tag{2}$$

taking their values in $\frac{U(n+1)}{U(n)\times U(1)}$, such they are described by $2n+1$ real parameters that parametrise $\mathbf{R}^{2n+1}$. In (1), $\overline{z}^a$ is the complex conjugate of z_a, transforming with an index that is *contravariant* to the *covariant* index of z_a, and $Z^\dagger$ in (2) is the transpose of $\overline{Z}$. This leads to the definition of the *projection operator*

$$P = \left(\hat{1} - ZZ^\dagger\right) \equiv \left(\delta_a^b - z_a \overline{z}^b\right). \tag{3}$$

The most interesting feature of these models is that when the field Z is subjected to a *local* $U(1)$ gauge transformation $g = e^{i\Lambda(x)}$, then the quantity defined as

$$B_i = Z^\dagger \partial_i Z \ , \ i = 1, 2, ..., 2n+1 \tag{4}$$

transforms like an *Abelian composite connection* under $g(\Lambda)$, which leads to the definition of the covariant derivative of Z and the *Abelian curvature* of this connection,

$$D_i Z = \partial_i Z - B_i Z, \tag{5}$$

$$G_{ij} = \partial_i B_j - \partial_j B_i. \tag{6}$$

The Abelian CS density on $\mathbf{R}^{2n+1}$ is then readily defined in terms of the quantities (6) and (5). This is what makes these models well suited to describing Abelian Hopfions in all odd dimensions.

$$\Omega_{CS} \cong \varepsilon_{i_1 i_2 \dots i_{2n+1}} B_{2n+1} G_{i_1 i_2} G_{i_3 i_4} \dots G_{i_{2n-1} i_{2n}}. \tag{7}$$

Let us first dispose of the well studied case $n = 1$, namely the $\mathbf{CP}^1$ models on $\mathbf{R}^3$. The most general model supporting finite energy solutions, consistent with the Derrick scaling requirement is

$$H_3 = \kappa_0^0 V + \frac{1}{2}\kappa_1^2 D_i Z^\dagger D_i Z + \frac{1}{4}\kappa_2^4 G_{ij}^2, \tag{8}$$

with $D_i Z$ and G_{ij} given by (5) and (6). The constants κ_0, κ_1, and κ_2 each have the dimension of length, and V is some pion mass type potential, which can most naturally be chosen to be

$$V = 1 + Z^\dagger \sigma_3 Z. \tag{9}$$

In the special case with $\kappa_0 = 0$, (8) reduces to the Skyrme-Fadde'ev model [5].

We proceed to the problem at hand, namely the study of the Hopfions of $\mathbf{CP}^2$ models on $\mathbf{R}^5$. The most general model supporting finite energy solutions, consistent with the Derrick scaling requirement is

$$H_5 = \kappa_0^0 V + \frac{1}{2}\kappa_1^2 D_i Z^\dagger D_i Z + \frac{1}{4}\kappa_2^4 G_{ij}^2 + \frac{1}{8}\kappa_3^6 (G_{[ij}D_{k]}Z)^\dagger (G_{[ij}D_{k]}Z) + \frac{1}{16}\kappa_4^8 G_{ijkl}^2, \qquad (10)$$

with $D_i Z$ and G_{ij} given by (5) and (6) , and the $4-$ form G_{ijkl} being the totally antisymmetrised product of this curvature. The constants κ_0 , κ_1 , κ_2 , κ_3 and κ_4 each have the dimension of length, and V is some pion mass type potential. According to the scaling requirement for finite energy, it is necessary to retain *at least one* of the constants $(\kappa_0, \kappa_1, \kappa_2)$ and *at least one* of the constants (κ_3, κ_4) , with the option of setting the rest equal to zero.

The virial identity resulting from the usual Derrick-type scaling requirement [8] that must be satisfied is

$$5\,\|V\| + 3\,\|D_i Z\|^2 + \|G_{ij}\|^2 - \|(G_{[ij}D_{k]}Z)\|^2 - 3\,\|G_{ijkl}\|^2 = 0, \qquad (11)$$

where the dimensional constants and the detailed normalisations have been suppressed, and where each of the quantities $\|.\|^2$ is the positive definite integral of the correspoding density in (10) .

The simplest truncation of the model (10), consistent with finite energy and the definition of a knot-charge, is

$$H_{(2,4)} = \frac{1}{4}\kappa_2^4 G_{ij}^2 + \frac{1}{16}\kappa_4^8 G_{ijkl}^2, \qquad (12)$$

featuring the *quartic* and *octic* terms multiplying the constants κ_2^4 and κ_4^8, respectively. It is however reasonable to include the usual quadratic term multiplying the constant κ_1^2, such that the system to be studied is

$$H_{(1,2,4)} = \frac{1}{2}\kappa_1^2 D_i Z^\dagger D_i Z + \frac{1}{4}\kappa_2^4 G_{ij}^2 + \frac{1}{16}\kappa_4^8 G_{ijkl}^2. \qquad (13)$$

3. AN HOPFION ANSATZ. IMPOSITION OF BI-AZIMUTHAL SYMMETRY

The most general Hopfion Ansatz would depend on five independent functions, which will solve a complicated set of second order partial differential equations (PDE's). To facilitate the construction of the solutions and the evaluation of the topological charge it is desirable to reduce this number. Just as the Hopfion solutions of the $\mathbf{CP}^1$ model on $\mathbf{R}^3$ can be constructed by subjecting the field Z to azimuthal symmetry in the (x_1, x_2) plane, here the corresponding imposition of symmetry on the $\mathbf{CP}^2$ field Z is bi-azimuthal symmetry, namely to two azimuthal symmetries in the (x_1, x_2) and (x_3, x_4) planes separately. This will reduce the 5$^{\text{th}}$ order field equations to 3$^{\text{rd}}$ order PDE's, just as in the Skyrme-Fadde'ev case 3$^{\text{rd}}$ order field equations are reduced to 2$^{\text{nd}}$ order PDE's. Moreover, this corresponds to the simplest ansatz leading to a nonvanishing topological charge. The choice of the particular symmetry imposition in both cases is predicated on the resulting demonstration that the Hopf charge can be defined as a topological charge integral in the residual dimensions. This also yields a verification of the validity of the Ansatz used.

The resulting Anstaz used for the field (1) on $\mathbf{R}^5$ is

$$Z = \begin{bmatrix} a(\rho,\sigma,z) + \mathrm{i}b(\rho,\sigma,z) \\ c(\rho,\sigma,z)e^{in\varphi} \\ d(\rho,\sigma,z)e^{im\chi} \end{bmatrix} \equiv \begin{bmatrix} \sin\frac{1}{2}f(\rho,\sigma,z)e^{i\alpha(\rho,\sigma,z)} \\ \cos\frac{1}{2}f(\rho,\sigma,z)\sin g(\rho,\sigma,z)e^{in\varphi} \\ \cos\frac{1}{2}f(\rho,\sigma,z)\cos g(\rho,\sigma,z)e^{im\chi} \end{bmatrix}, \qquad (14)$$

in terms of the variables $\rho=\sqrt{|x_\alpha|^2}$, $\sigma=\sqrt{|x_A|^2}$ with $\alpha=1,2$, $A=3,4$ and $z\equiv x_5$. φ and χ are the azimuthal angles in the (x_1,x_2) and (x_3,x_4) planes, respectively, (n,m) being the winding (vortex) numbers of planes, respectively (therefore the line element on $\mathbf{R}^5$ reads $\mathrm{d}s^2=\mathrm{d}z^2+\mathrm{d}\rho^2+\rho^2\mathrm{d}\varphi^2+\mathrm{d}\sigma^2+\sigma^2\mathrm{d}\chi^2$).

The resulting composite Abelian connections descending from (4) are (with $\hat{x}_\alpha=x_\alpha/\rho$, $\hat{x}_A=x_A/\sigma$)

$$\begin{gathered}B_\alpha=\mathrm{i}\left[(ab_\rho-ba_\rho)\hat{x}_\alpha+\frac{n}{\rho}c^2\ \ (\hat{x}\varepsilon)_\alpha\right],\\ B_A=\mathrm{i}\left[(ab_\sigma-ba_\sigma)\hat{x}_A+\frac{m}{\sigma}d^2\ \ (\hat{x}\varepsilon)_A\right],\\ B_z=\mathrm{i}(ab_z-ba_z),\end{gathered}\tag{15}$$

leading to the following components of the Abelian curvature

$$\begin{gathered}G_{\alpha\beta}=2i\frac{n}{\rho}cc_\rho\varepsilon_{\alpha\beta},\\ G_{AB}=-2\frac{m}{\sigma}dd_\sigma\varepsilon_{AB},\\ G_{\alpha A}=2i\left[a_{[\rho}b_{\sigma]}\hat{x}_\alpha\hat{x}_A-\frac{n}{\rho}cc_\sigma\hat{x}_A(\hat{x}\varepsilon)_\alpha+\frac{m}{\sigma}dd_\rho\hat{x}_\alpha(\hat{x}\varepsilon)_A\right],\\ G_{\alpha z}=2i\left[a_{[\rho}b_{z]}\hat{x}_\alpha-\frac{n}{\rho}cc_z(\hat{x}\varepsilon)_\alpha\right],\\ G_{Az}=2i\left[a_{[\sigma}b_{z]}\hat{x}_A-\frac{m}{\sigma}dd_z(\hat{x}\varepsilon)_A\right].\end{gathered}\tag{16}$$

Note that these Abelian connections and curvatures are pur imaginary, as *per* the definition (4) of the composite connection B_i. Once the connection and the curvature are known, we can evaluate the individual pieces which enter (10). First, we give the quadratic kinetic term multiplying κ_1^2

$$D_iZ^\dagger D_iZ=\partial_iZ^\dagger\partial_iZ-B_i^2=(\partial_\alpha Z^\dagger\partial_\alpha Z-B_\alpha^2)+(\partial_AZ^\dagger\partial_AZ-B_A^2)+(\partial_zZ^\dagger\partial_zZ-B_z^2),$$

yielding

$$\begin{aligned}D_iZ^\dagger D_iZ=&(a_\rho^2+b_\rho^2+c_\rho^2+d_\rho^2)+(a_\sigma^2+b_\sigma^2+c_\sigma^2+d_\sigma^2)+(a_z^2+b_z^2+c_z^2+d_z^2)\\ &-(ab_\rho-ba_\rho)^2-(ab_\sigma-ba_\sigma)^2-(ab_z-ba_z)^2\\ &+\frac{n^2}{\rho^2}c^2(1-c^2)+\frac{m^2}{\sigma^2}d^2(1-d^2).\end{aligned}\tag{17}$$

Next, the term with coupling strength κ_2^4 in (10), namely

$$G_{ij}^2=G_{\alpha\beta}^2+G_{AB}^2+2G_{\alpha A}^2+2G_{\alpha z}^2+2G_{Az}^2,$$

can be calculated immediately to yield

$$\frac{1}{8}G_{ij}^2=\left[\left(a_{[\rho}b_{\sigma]}\right)^2+\left(a_{[\rho}b_{z]}\right)^2+\left(a_{[\sigma}b_{z]}\right)^2\right]+\frac{n^2}{\rho^2}c^2\left(c_\rho^2+c_\sigma^2+c_z^2\right)+\frac{m^2}{\sigma^2}d^2\left(d_\rho^2+d_\sigma^2+d_z^2\right).\tag{18}$$

To calculate the term with coupling strength κ_4^8 we first evaluate the three distinct components of G_{ijkl}, namely

$$G_{ijkl} = (G_{\alpha\beta AB}, G_{\alpha\beta Az}, G_{AB\alpha z}),$$

with

$$\begin{aligned} G_{\alpha\beta AB} &= 4\frac{nm}{\rho\sigma} cdc_{[\rho}d_{\sigma]}\varepsilon_{\alpha\beta}\varepsilon_{AB}, \\ G_{\alpha\beta Az} &= 4\frac{n}{\rho}c\left[(c_z a_{[\rho}b_{\sigma]} + c_\rho a_{[\sigma}b_{z]} + c_\sigma a_{[z}b_{\rho]})\hat{x}_A - \frac{m}{\sigma}dc_{[\rho}d_{z]}(\hat{x}\varepsilon)_A\right], \\ G_{AB\alpha z} &= 4\frac{m}{\sigma}d\left[-(d_z a_{[\rho}b_{\sigma]} + d_\rho a_{[\sigma}b_{z]} + d_\sigma a_{[z}b_{\rho]})\hat{x}_\alpha + \frac{n}{\rho}cc_{[\sigma}d_{z]}(\hat{x}\varepsilon)_\alpha\right]. \end{aligned} \tag{19}$$

Substituting these into

$$\frac{1}{2}G_{ijkl}^2 = 3G_{\alpha\beta AB}^2 + 5G_{\alpha\beta Az} + 5G_{AB\alpha z},$$

we find the simple compact expression

$$\begin{aligned} \frac{1}{3\cdot 2^7}G_{ijkl}^2 &= \left[\frac{n^2}{\rho^2}c^2(c_{[z}a_{[\rho}b_{\sigma]]})^2 + \frac{m^2}{\sigma^2}d^2(d_{[z}a_{[\rho}b_{\sigma]]})^2\right] + \\ &+ \frac{n^2m^2}{\rho^2\sigma^2}c^2d^2\left[(c_{[\rho}d_{\sigma]})^2 + (c_{[\rho}d_{z]})^2 + (c_{[\sigma}d_{z]})^2\right]. \end{aligned} \tag{20}$$

With these relations, the derivation of the corresponding equations of motion for the scalars a, b, c and d within the generic model (10) is straightforward. These equations are very complicated and we shall not present them here.

4. CHERN-SIMONS DENSITY ON R^5 AND THE TOPOLOGICAL CHARGE

As mentioned in the previous section, imposition of the appropriate symmetry is crucial in ensuring the existence of a knotted (nontrivial) Hopfion. In this context, it is clear that the imposition of spherical symmetry is inappropriate since this leaves no room for any useful winding. In our imposition of bi-azimuthal symmetry on the $\mathbf{CP}^2$ system on $\mathbf{R}^5$ here, we are guided by the application of axial (mono-azimuthal) symmetry on the corresponding $\mathbf{CP}^1$ system on $\mathbf{R}^3$ [9].

The Chern-Simons density on $\mathbf{R}^5$, denoting the coordinates $x_i = (x_\mu, x_5)$, is given by the following expression

$$\begin{aligned} \Omega_{CS}^{(5)} &= \varepsilon_{mijkl}B_m G_{ij}G_{kl} = \varepsilon_{\mu\nu\rho\sigma}\left(B_5 G_{\mu\nu}G_{\rho\sigma} + 4B_\mu G_{\nu 5}G_{\rho\sigma}\right) = \\ &= 2\varepsilon_{\alpha\beta}\varepsilon_{AB}\left\{B_5\left(G_{\alpha\beta}G_{AB} - 2G_{\alpha A}G_{\beta B}\right) + \right. \\ &\left. +2\left[B_\alpha\left(G_{\beta 5}G_{AB} - 2G_{A5}G_{\beta B}\right) + B_A\left(G_{B5}G_{\alpha\beta} - 2G_{\alpha 5}G_{B\beta}\right)\right]\right\}. \end{aligned} \tag{21}$$

Substituting the bi-azimuthally symmetric Ansatz (14) yields the following simple expression of (21)

$$\frac{1}{2}\Omega_{CS}^{(5)} = \det\begin{vmatrix} a & b & c & d \\ a_\rho & b_\rho & c_\rho & d_\rho \\ a_\sigma & b_\sigma & c_\sigma & d_\sigma \\ a_z & b_z & c_z & d_z \end{vmatrix}. \tag{22}$$

Then it is clear that if any one of the functions a, b, c and d vanishes, $\Omega_{CS}^{(5)}$ vanishes.

As usual, the topological charge $Q_{CS}^{(5)}$ of the solutions is the integral of the Chern-Simons density. In evaluating $Q_{CS}^{(5)}$, it is convenient to work with the trigonometric parametrisation in (14), namely the parametrisation in which the sigma model constraint is already imposed. Then the integral of (22) reduces to the simple expression

$$Q_{CS}^{(5)} = 4\cdot(2\pi)^2 n_1 n_2 \int \left[\partial_\rho(\cos f)\partial_\sigma g \partial_z \alpha + cycl.(\rho,\sigma,z)\right] \mathrm{d}\rho \mathrm{d}\sigma \mathrm{d}z. \quad (23)$$

Denoting the coordinates $(\rho,\sigma,z)=\xi_i$, $i=1,2,3$, *i.e.*,

$$\xi_i = \begin{pmatrix} r\sin\psi\sin\theta \\ r\sin\psi\cos\theta \\ r\cos\psi \end{pmatrix}, \quad (24)$$

with $0\le r<\infty$, $0\le\psi\le\pi$ and $0\le\theta\le\frac{\pi}{2}$, the charge (23) can be re-expresed as

$$\begin{aligned} Q_{CS}^{(5)} &= 4\cdot(2\pi)^2 n_1 n_2 \int \varepsilon_{ijk}\partial_i(\cos f)\partial_j g\partial_k \alpha \mathrm{d}^3\xi = \\ &= 4\cdot(2\pi)^2 n_1 n_2 \int \varepsilon_{ijk}\left((\cos f)\partial_j g \partial_k\alpha\right)|_{r\to\infty}\, \hat{\xi}_i \mathrm{d}S \end{aligned} \quad (25)$$

in an obvious notation, where $\mathrm{d}S = r^2\sin\psi \mathrm{d}\psi \mathrm{d}\theta$, and where we have applied Gauss' Theorem.

The result of the integration is

$$Q_{CS}^{(5)} = 4\cdot(2\pi)^2 n_1 n_2 \int_{\psi=0}^{\pi}\int_{\theta=0}^{\frac{\pi}{2}} \cos f\left(\partial_\psi g\partial_\theta\alpha - \partial_\psi\alpha\partial_\theta g\right)|_{r\to\infty} \mathrm{d}\psi \mathrm{d}\theta. \quad (26)$$

Finally, requiring the boundary values (with m an arbitrary integer)

$$\lim_{r\to\infty} g = \theta, \quad \lim_{r\to\infty}\alpha = m\pi, \quad (27)$$

the relation (25) yields the following simple expression for the charge of the Abelian Hopfion on $\mathbf{R}^5$

$$Q_{CS}^{(5)} = -32\pi^3 n_1 n_2 m. \quad (28)$$

5. CONCLUSIONS

The purpose of this work was to propose a generalization of the well-known Skyrme-Fadde'ev Abelian Hopf model in $\mathbf{R}^3$ to the case of five dimensional space. These configurations are stabilized by the Abelian CS term. An explicit ansatz subject to bi-azimuthal symmetry in a four dimensional subspace has also been proposed. This ansatz contains three independent functions with dependence of three variables. Finally, we have evaluated the topological charge of the Hopfions and show that it can be written as a product of three winding numbers which enters the Hopfions' Ansatz and the boundary conditions.

For the case of the $D=3$ Skyrme-Fadde'ev Abelian Hopf model, there are explicit constructions of solutions of the equations of motion corresponding to global energy minima [10, 11]. (Note that all these solutions were found numerically, no exact results existing so far, see the review work [12]). The model discussed in this work should also possess finite energy solutions, those within the Ansatz (1) subject to the bi-azimuthal symmetry being the simplest class. Although our preliminary attempts to construct them have met with severe difficulties, we think this is a numerical problem only, and such solutions should exist.

Finally, let us remark that the model in this work together with Skyrme-Fadde'ev one can be viewed as the first two members of an hierarchy of Abelian Hopfions of the $\mathbf{CP}^n$ sigma model, on $\mathbf{R}^{2n+1}$. Indeed, a natural generalisation of the Ansätze in the $n=1,2$ cases here exists. This is *n-fold-azimuthal* symmetry on $\mathbf{R}^{2n+1}$. The only crucial check is that the CP density reduces to a total divergence subject to that symmetry, about which one may be quite confident.

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