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FRACTIONAL SYNCHRONIZATION OF CHAOTIC SYSTEMS WITH DIFFERENT ORDERS

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In this paper, we consider two chaotic systems with different orders. First, we consider the case when one of them is fractional order (master system) and another one is integer order (slave system). Second, we consider the case when both of them are fractional order but the orders are different. Using a fractional synchronization scheme in the presence of discrepancy between initial conditions of these systems for both cases the trajectories of the slave system are forced to track the master system trajectories. The effectiveness of the proposed technique is verified by numerical simulations for Chen systems.

Key words: chaotic system, fractional order synchronizer, Chen system.

1. INTRODUCTION

Chaos theory, as a new branch of physics, has provided us a new way of viewing the universe and is an important tool to understand the world we live in. To avoid troubles arising from unusual behaviors of a chaotic system, chaos control has gained increasing attention in recent years. An important objective of a chaos controller is to suppress the chaotic oscillations completely or reduce them to the regular oscillations [1]. Many control techniques such as open-loop control methods, adaptive control methods, and fuzzy control methods have been implemented in the control of chaotic systems [2]. The subject of fractional calculus has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering [3–5]. Moreover, fractional order dynamic systems have been studied in the design and practice of control systems [6]. Studies have shown that a fractional order controller can provide better performances than an integer order one and lead to more robust control performance [7].

In this work we consider two chaotic Chen systems. One of them with incommensurate fractional order is selected as master system and the other one is selected as slave system. As will be shown in the following Sections of this paper, using a simple fractional synchronization scheme we force the slave trajectories to track the master trajectories.

This paper is organized as follows. Some backgrounds form fractional calculus is presented in Section 2. In Section 3 we give a fractional synchronization algorithm for synchronizing the well-known Chen systems. Numerical simulations are presented in Section 4 to illustrate the ability of the proposed method. Finally, the conclusions are given in Section 5.

2. PRELIMINARIES FROM FRACTIONAL CALCULUS

In this subsection some mathematical backgrounds are presented.

Definition 1 [8]. The fractional order integral operator of a Lebesgue integrable function $x(t)$ is defined as follows:

$$ {}_aD_t^{-q}x(t) := \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}x(s)\mathrm{d}s, \quad q\in\Re^+. \tag{1}$$

Definition 2 [9].The left fractional order derivative operator in the sense of Riemann-Liouville (LRL) is defined as follows in which $m-1<q<m\in Z^+$:

$$ {}_a^{RL}D_t^q x(t) := D^m\, {}_aD_t^{-(m-q)}x(t) = \frac{1}{\Gamma(m-q)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_a^t (t-s)^{m-q-1}x(s)\mathrm{d}s. \tag{2}$$

***Remark* 1** [10]. For fractional derivative and integral RL operators we have: $L\left\{{}_aD_t^{-q}x(t)\right\}=s^{-q}X(s)$, $\lim_{q\to m} {}_0D_t^{-q}x(t)={}_0D_t^{-m}x(t), \quad q>0, \quad m\in Z^+$, ${}_0^{RL}D_t^q c=\frac{ct^{q-1}}{\Gamma(1-q)}$.

Definition 3 [3]. The left fractional order derivative operator in the sense of Caputo is defined as follows:

$$ {}_a^CD_t^q x(t) := {}_a^{RL}D_t^{-(m-q)}D^m x(t) = \frac{1}{\Gamma(m-q)}\int_a^t (t-s)^{m-q-1}x^{(m)}(s)\mathrm{d}s. \tag{3}$$

***Remark* 2** [10]. For fractional Caputo derivative operator we have: ${}_0^CD_t^q c=0$, ${}_0^CD_t^q\, {}_0D_t^{-q}x(t) = {}_0^{RL}D_t^q\, {}_0D_t^{-q}x(t) = x(t), \quad 0<q<1$.

THEOREM 1 [6]. *The commensurate order system*: ${}_0^CD_t^q x(t)=Ax(t), \quad x(0)=x_0$ *with* $0<q\le 1, \quad x\in\Re^n$ *and* $A\in\Re^{n\times n}$ *is asymptotically stable if and only if* $|\arg\ (\lambda)|>q\frac{\pi}{2}$ *is satisfied for all eigenvalues* λ *of* A. *Also, this system is stable if and only if* $|\arg\ (\lambda)|\ge q\frac{\pi}{2}$ *is satisfied for all eigenvalues* λ of A *with those critical eigenvalues satisfying* $|\arg\ (\lambda)|=q\frac{\pi}{2}$ *have geometric multiplicity of one*.

THEOREM 2 [11]. *Consider the linear fractional order system*: ${}_0^CD_t^q x(t)=Ax(t), \quad x(0)=x_0$ *with* $x\in\Re^n$ *and* $A\in\Re^{n\times n}$ *and* $q=\left(q_1\ q_2\ \cdots\ q_n\right)^T$, $0<q_i\le 1$. *Also* $q_i=\frac{n_i}{d_i}$, $\gcd(n_i,d_i)=1$. *Let M be the lowest common multiple of the denominators* d_i's. *The zero solution of the system is globally asymptotically stable in the Lyapunov sense if all roots* λ's *of the equation* $\Delta(\lambda)=\det\ \left(diag\ (\lambda^{Mq_i})-A\right)=0$ *satisfy* $|\arg\ (\lambda)|>\frac{\pi}{2M}$.

3. FRACTIONAL SYNCHRONIZATION TECHNIQUE

Consider the 3-D autonomous system $\dot{x}=f(x)$. Let Q be equilibrium of the system: Q: (x_1^*,x_2^*,x_3^*). Q is called a saddle point if the eigenvalues of the Jacobian matrix $J=\frac{\partial f}{\partial x}$ evaluated at Q are a and $b\pm jc$, where $ab<0$ and $c\ne 0$. The saddle point Q is called a saddle point of index 1 if $a>0$ and $b<0$, and it is called a saddle point of index 2 if $a<0$ and $b>0$. It is known that scrolls in a chaotic attractor are generated only around the saddle points of index 2 in chaotic systems of Shil'nikov type. Moreover, saddle points of index 1 are responsible only for connecting scrolls [12].

Assume that the 3-D chaotic system ${}_0^C D_t^q x = f(x)$ displays a chaotic attractor. As was told for every scroll existing in the chaotic attractor, this system has a saddle point of index 2 encircled by its respective scroll. Suppose that **Ω** is the set of equilibrium points of the system surrounded by scrolls. We know that system ${}_0^C D_t^q x = f(x)$ with $q = (q_1, q_2, q_3)^T$ and system $\dot{x} = f(x)$ have the same equilibrium points. Hence, a necessary condition for fractional order system ${}_0^C D_t^q x = f(x)$ to exhibit the chaotic attractor similar to its integer order counterpart is instability of the equilibrium points in **Ω**. Otherwise, one of these equilibrium points becomes asymptotically stable and attracts the nearby trajectories. This necessary condition is mathematically equivalent to [13]: $\frac{\pi}{2M} - \min_i \{|\arg\ (\lambda_i)|\} \geq 0$, where λ_i's are roots of

$$\det\ \left(diag\ \left(\lambda^{Mq_1} \quad \lambda^{Mq_2} \quad \lambda^{Mq_3} \right) - J\big|_Q \right) = 0, \quad \forall Q \in \Omega. \tag{4}$$

Consider the master-slave synchronization scheme of two autonomous different fractional order chaotic systems:

$$\begin{aligned} &\text{Master: } {}_a^C D_t^q x(t) = f(x) \\ &\text{Slave: } {}_a^C D_t^q y(t) = g(y) + u, \end{aligned} \tag{5}$$

where q is the fractional order, $x, y \in R^n$ represent the states of the drive and response systems, respectively, $f: R^n \to R^n$, $g: R^n \to R^n$ are the vector fields of the drive and response systems, respectively. The aim is to choose a suitable control function $u = (u_1,\ u_2,\ \cdots,\ u_n)^T$ such that the states of the drive and response systems are synchronized, i.e. $\lim_{t\to\infty} \|y(t) - x(t)\| = 0$. In what follows we proceed to design a new synchronization scheme for two chaotic systems which are different in order and initial conditions.

Case I. Consider the following chaotic incommensurate fractional order system as the master system:

$${}_{t_0}^C D_t^{q_i} x_i(t) = f_i(x_1, x_2, x_3), \tag{6}$$

with the initial conditions $(x_1(t_0), x_2(t_0), x_3(t_0)) = (x_{10}, x_{20}, x_{30}) \in \Re^3$, and f_i, $i = 1,2,3$ are nonlinear functions. Also suppose that the structure of the slave system is as follows:

$$\dot{y}_i = f_i(y_1, y_2, y_3) + u_i, \tag{7}$$

with the initial conditions $(y_1(t_0), y_2(t_0), y_3(t_0)) = (y_{10}, y_{20}, y_{30}) \in \Re^3$, and u_i, $i = 1,2,3$ are the control signals. Note that in spite of discrepancy between initial conditions and difference between orders of the systems, we want to synchronize the systems. Now if we consider the controller structure as follows:

$$u_i = v_i + \dot{y}_i - {}_{t_0}^C D_t^{q_i} y_i(t), \quad i = 1,2,3, \tag{8}$$

then the slave system (7) reduces to:

$${}_{t_0}^C D_t^{q_i} y_i(t) = f_i(y_1, y_2, y_3) + v_i, \quad i = 1,2,3. \tag{9}$$

Defining errors as: $e_i = y_i - x_i$, $i = 1,2,3$ by subtracting (6) from (9) we have:

$${}_{t_0}^C D_t^{q_i} e_i(t) = f_i(y_1, y_2, y_3) - f_i(x_1, x_2, x_3) + v_i, \quad i = 1,2,3. \tag{10}$$

Based on active control methodology choosing:

$$v_i = f_i(x_1, x_2, x_3) - f_i(y_1, y_2, y_3) + a_i e_1 + b_i e_2 + c_i e_3, \quad i = 1,2,3 \tag{11}$$

Therefore (10) reduces to:

$$\begin{cases} {}_{t_0}^{C}D_t^{q_1} e_1(t) = a_1 e_1 + b_1 e_2 + c_1 e_3 \\ {}_{t_0}^{C}D_t^{q_2} e_2(t) = a_2 e_1 + b_2 e_2 + c_2 e_3 \\ {}_{t_0}^{C}D_t^{q_3} e_3(t) = a_3 e_1 + b_3 e_2 + c_3 e_3. \end{cases} \tag{12}$$

Now by choosing appropriate constants a_i, b_i, c_i, $i = 1, 2, 3$, we can design a stabilizing controller for our synchronization goal. Note that for checking the stability of (12) we must use THEOREM 2**.** The details of design procedure can be seen in the numerical simulations in the next section.

Case II. Using similar methodology we can synchronize two fractional order chaotic systems. Indeed if the master system is selected as (6) and the slave systems is as follows:

$${}_{t_o}^{C}D_t^{q_i'} y_i = f_i(y_1, y_2, y_3) + u_i, \tag{13}$$

with the initial conditions $\left(y_1(t_0), y_2(t_0), y_3(t_0)\right) = \left(y_{10}, y_{20}, y_{30}\right) \in \Re^3$, and u_i, $i = 1,2,3$ are the control signals. Now if we consider the controller structure as follows:

$$u_i = v_i + {}_{t_o}^{C}D_t^{q_i'} y_i - {}_{t_0}^{C}D_t^{q_i} y_i(t), \tag{14}$$

then the slave system (13) reduces to:

$${}_{t_0}^{C}D_t^{q_i} y_i(t) = f_i(y_1, y_2, y_3) + v_i. \tag{15}$$

Now continuing the argument given above we can design an appropriate controller for synchronizing (6) and (13).

4. NUMERICAL SIMULATION

Consider integer order Chen system [14]:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 35(x_2 - x_1) \\ -7x_1 - x_1 x_3 + 28x_2 \\ x_1 x_2 - 3x_3 \end{pmatrix}. \tag{16}$$

Numerical simulations of system (16) are brought in Fig. 1. Now let's consider the fractional order version of Chen system with incommensurate orders as follows:

$$\begin{pmatrix} {}_{t_0}^{C}D_t^{q_1} x_1 \\ {}_{t_0}^{C}D_t^{q_2} x_2 \\ {}_{t_0}^{C}D_t^{q_3} x_3 \end{pmatrix} = \begin{pmatrix} 35(x_2 - x_1) \\ -7x_1 - x_1 x_3 + 28x_2 \\ x_1 x_2 - 3x_3 \end{pmatrix}. \tag{17}$$

The equilibria of (16) and (17) are the same: $Q_1 : (0,0,0)$, $Q_2 : (7.94, 7.94, 21)$, $Q_3 : (-7.94, -7.94, 21)$. Computing the Jacobian matrix for (16) and (17) at the equilibrium point $Q : (x_1^*, x_2^*, x_3^*)$:

$$J = \begin{pmatrix} -35 & 35 & 0 \\ -7 - x_3^* & 28 & -x_1^* \\ x_2^* & x_1^* & -3 \end{pmatrix}. \tag{18}$$

Now for each equilibrium points, we determine associated eigenvalues:

$$\begin{aligned}
\Lambda_1 &= (\lambda_1, \lambda_2, \lambda_3) = \left(-30.86, 23.66, -3\right) \\
\Lambda_2 &= (\lambda_1', \lambda_2', \lambda_3') = (-18.43, 4.21 + j14.88, 4.21 - j14.88) \\
\Lambda_3 &= (\lambda_1'', \lambda_2'', \lambda_3'') = (-18.43, 4.21 + j14.88, 4.21 - j14.88).
\end{aligned} \tag{19}$$

As can be seen from the eigenvalues, Q_2 and Q_3 are saddle point of type 2 and Q_1 is a saddle point of type 1. So if there is chaos in (17), we have 2-scroll in the phase space. Numerical simulations for (17) in which the order is $\left(q_1, q_2, q_3\right) = \left(0.8, 1, 0.9\right)$ are depicted in Fig. 2 which shows chaos.

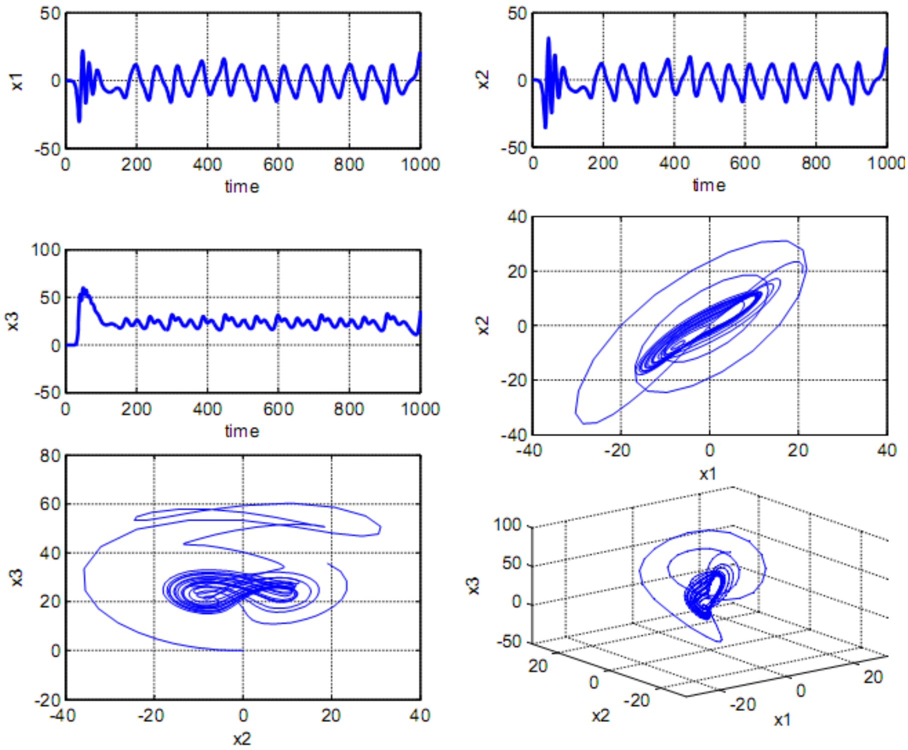

Fig. 1 – Numerical results of Chen system (16).

Let us consider the following master-slave synchronization scheme:

$$\text{master system}: \begin{pmatrix} {}_{t_0}^{C}D_t^{q_1} x_1 \\ {}_{t_0}^{C}D_t^{q_2} x_2 \\ {}_{t_0}^{C}D_t^{q_3} x_3 \end{pmatrix} = \begin{pmatrix} 35(x_2 - x_1) \\ -7x_1 - x_1 x_3 + 28x_2 \\ x_1 x_2 - 3x_3 \end{pmatrix}, \tag{20}$$

with order $\left(q_1, q_2, q_3\right) = \left(0.8, 1, 0.9\right)$ which exhibit chaos according to the simulations and satisfying the necessary condition obtained above and initial conditions $\left(x_1(t_0), x_2(t_0), x_3(t_0)\right) = \left(x_{10}, x_{20}, x_{30}\right) \in \Re^3$, and:

$$\text{slave system}: \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 35(y_2 - y_1) + u_1 \\ -7y_1 - y_1 y_3 + 28y_2 + u_2 \\ y_1 y_2 - 3y_3 + u_3 \end{pmatrix}, \tag{21}$$

with initial conditions $\left(y_1(t_0), y_2(t_0), y_3(t_0)\right) = \left(y_{10}, y_{20}, y_{30}\right) \in \Re^3$.

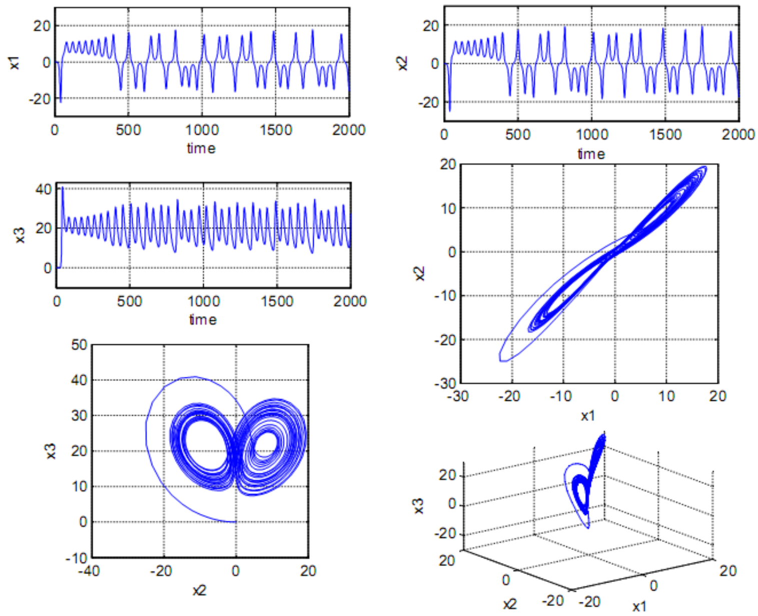

Fig. 2 – Numerical results for fractional order Chen system (17) when the order of master system is $(q_1, q_2, q_3) = (0.8, 1, 0.9)$.

Choosing:

$$u_i = v_i + \dot{y}_i - {}_{t_0}^{C}D_t^{q_i} y_i(t). \tag{22}$$

Thus (21) reduces to:

$$\begin{pmatrix} {}_{t_o}^{C}D_t^{q_1} y_1(t) \\ {}_{t_o}^{C}D_t^{q_2} y_2(t) \\ {}_{t_o}^{C}D_t^{q_3} y_3(t) \end{pmatrix} = \begin{pmatrix} 35(y_2 - y_1) + v_1 \\ -7y_1 - y_1y_3 + 28y_2 + v_2 \\ y_1y_2 - 3y_3 + v_3 \end{pmatrix}. \tag{23}$$

Subtracting (20) form (23) and considering errors $e_i = y_i - x_i, \quad i = 1,2,3$

$$\begin{pmatrix} {}_{t_o}^{C}D_t^{q_1} e_1(t) \\ {}_{t_o}^{C}D_t^{q_2} e_2(t) \\ {}_{t_o}^{C}D_t^{q_3} e_3(t) \end{pmatrix} = \begin{pmatrix} 35(e_2 - e_1) + v_1 \\ -7e_1 - y_1y_3 + x_1x_3 + 28e_2 + v_2 \\ y_1y_2 - x_1x_2 - 3e_3 + v_3 \end{pmatrix}. \tag{24}$$

Selecting:

$$\begin{cases} v_1 = 35e_1 - 34e_2 \\ v_2 = y_1y_3 - x_1x_3 + 7e_1 - 28e_2 + e_3 \\ v_3 = x_1x_2 - y_1y_2 - k_1e_1 - k_2e_2 - (k_3 - 3)e_3 \end{cases}. \tag{25}$$

Then (24) reduces to:

$$\begin{pmatrix} {}_{t_o}^{C}D_t^{q_1} e_1(t) \\ {}_{t_o}^{C}D_t^{q_2} e_2(t) \\ {}_{t_o}^{C}D_t^{q_3} e_3(t) \end{pmatrix} = \begin{pmatrix} e_2 \\ e_3 \\ -k_1e_1 - k_2e_2 - k_3e_3 \end{pmatrix}. \tag{26}$$

Thus by choosing appropriate k_1, k_2, k_3 we can stabilize the error vector. If we choose $k_1 = 48$, $k_2 = 44$, and $k_3 = 12$, we see that the eigenvalues of (26) are: –2, –4, and –6. Let's determine the stability of (26) for these k_i's. According to THEOREM 2 we constitute $\Delta(\lambda)$ for (26) as follows:

$$\Delta(\lambda) = \det\left(diag(\lambda^9, \lambda^8, \lambda^7) - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{pmatrix} \right) = 0. \tag{27}$$

After expanding one can write: $\lambda^{24} + 12\lambda^{17} + 44\lambda^9 + 48 = 0$. Solving this equation for λ, we see that $\min_i \left(|\arg(\lambda_i)| \right) = 0.3196$ which is greater than $\frac{\pi}{2M} = 0.1571$. Therefore based on THEOREM 2, we conclude the stability of (26). Fig. 3 shows the numerical results for this synchronization scheme in which the controller is turn on at $t = 4$s.

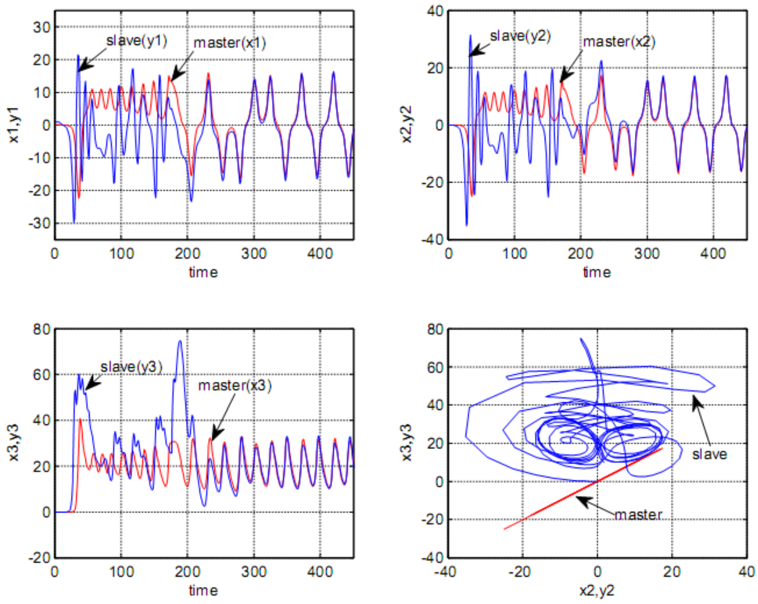

Fig. 3 – Numerical results for fractional synchronization between (20) and (21) when order of (20) is $(q_1, q_2, q_3) = (0.8, 1, 0.9)$.

In Fig. 4. we illustrate our fractional synchronization method for two fractional Chen systems; master system is the same as (17) with order $(q_1, q_2, q_3) = (0.8, 1, 0.9)$ which exhibit chaos and initial conditions $(x_1(t_0), x_2(t_0), x_3(t_0)) = (x_{10}, x_{20}, x_{30}) \in \Re^3$ and slave system is as follows:

$$\begin{pmatrix} {}_{t_0}^{C}D_t^{q_1} x_1 \\ {}_{t_0}^{C}D_t^{q_2} x_2 \\ {}_{t_0}^{C}D_t^{q_3} x_3 \end{pmatrix} = \begin{pmatrix} 35(x_2 - x_1) \\ -7x_1 - x_1 x_3 + 28 x_2 \\ x_1 x_2 - 3x_3 \end{pmatrix}, \tag{28}$$

with orders $(q_1, q_2, q_3) = (0.6, 1, 0.7)$. Note that system (28) is not chaotic.

5. CONCLUSIONS

In this paper we proposed a simple active synchronization method that synchronizes two different chaotic systems. The differences are the initial conditions and orders. The master system was considered a fractional order and the slave system was taken both an integer order and fractional order. In two separate steps we designed an active controller capable to force the trajectories of the slave system to track the master trajectories. Analytical and numerical investigations clarified the effectiveness of the proposed method.

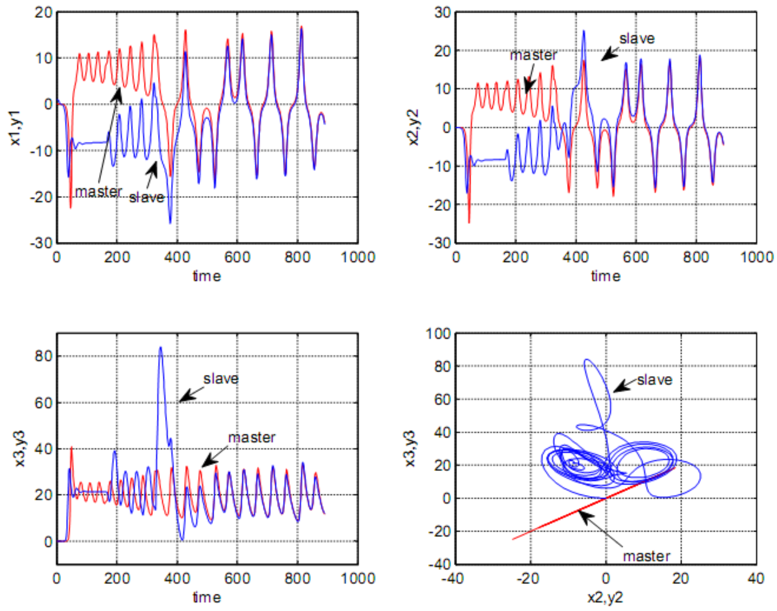

Fig. 4 – Numerical results for fractional synchronization between (20) and (28) when order of (28) is $(q_1, q_2, q_3) = (0.6, 1, 0.7)$.

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