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WEAK AMENABILITY OF TENSOR PRODUCT OF BANACH ALGEBRAS

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Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. We show that when $\mathcal{B}$ is commutative or character space of $\mathcal{B}$ is nonempty and $\mathcal{A}\hat{\otimes}\mathcal{B}$ is weakly amenable, then $\mathcal{A}$ is weakly amenable too.

Key words: Banach algebra, weak amenability, tensor product.

1. INTRODUCTION

Let $\mathcal{X}$ be a Banach space. We denote the dual space of $\mathcal{X}$ by $\mathcal{X}^*$; the action of $x^* \in \mathcal{X}^*$ on element $x \in \mathcal{X}$ is written as $<x, x^*>$. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then $\mathcal{X}^*$ is also a Banach $\mathcal{A}$-bimodule with the following module actions

$$<x, x^*a> = <ax, x^*>, \quad <x, ax^*> = <xa, x^*> \quad (x \in \mathcal{X}),$$

for all $a \in \mathcal{A}$ and $x^* \in \mathcal{X}^*$. The space $\mathcal{X}^*$ with these actions is the dual module of $\mathcal{X}$. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A bounded linear map D from $\mathcal{A}$ into $\mathcal{X}$ is a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b,$$

for all $a, b \in \mathcal{A}$.

For each $x \in \mathcal{X}$, the mapping $\delta_x : \mathcal{A} \to \mathcal{X}$ defined by $\delta_x(a) = a \cdot x - x \cdot a$ is a derivation. Derivations of this form are called inner derivations. The cohomology space $\mathcal{H}^1(\mathcal{A}, \mathcal{X})$ is the quotient of the space of derivations by the inner derivations, and triviality of this space is important. In particular, Banach algebra $\mathcal{A}$ is contractible if for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, $\mathcal{H}^1(\mathcal{A}, \mathcal{X}) = \{0\}$, amenable if for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, $\mathcal{H}^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$, and weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$ [1, 4, 5, 7, 8, 9].

Let $\mathcal{A}$ be a Banach algebra. Then the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule, where the module actions are specified by

$$a \cdot (b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c) \cdot a = b \otimes ca,$$

for all $a, b, c \in \mathcal{A}$. We define the multiplication map π: $\mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ by

$$\pi(a \otimes b) = ab \quad (a, b \in \mathcal{A}).$$

Then π becomes a bounded $\mathcal{A}$-bimodule homomorphism. The Banach algebra $\mathcal{A}$ is said to be biprojective if π has a bounded right inverse which is an $\mathcal{A}$-bimodule homomorphism, i.e. there is a bounded $\mathcal{A}$-bimodule homomorphism ρ: $\mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $\pi\,\rho = I_{\mathcal{A}}$. A-Banach algebra $\mathcal{A}$ is contractible if and only if it is unital and biprojective. The dual map $\phi *$ is also a $\mathcal{A}$-bimodule homomorphism. A Banach algebra $\mathcal{A}$ is said to be biflat if π^* has a left inverse as a bounded $\mathcal{A}$-bimodule homomorphism. It is easy to see that a biprhojective Banach algebra is biflat. A Banach algebra $\mathcal{A}$ is amenable if and only if it is biflat and has a bounded approximate identity. Every biflat Banach algebra is weakly amenable [6].

An approximate diagonal for Banach algebra $\mathcal{A}$ is a bounded net $\{m_\alpha\}$ in $\mathcal{A}\otimes\mathcal{A}$ such that

$$\lim_\alpha a\cdot m_\alpha - m_\alpha\cdot a = 0 \quad \text{and} \quad \lim_\alpha \pi(m_\alpha)\cdot a = a$$

for each $a\in\mathcal{A}$.

A virtual diagonal for $\mathcal{A}$ is an element M of $(\mathcal{A}\hat{\otimes}\mathcal{B})^{**}$ such that

$$M\cdot a = a\cdot M \quad \text{and} \quad \pi^{**}(M)\cdot a = \hat{a}$$

for each $a\in\mathcal{A}$. It is well known that Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ has an approximate diagonal if and only if $\mathcal{A}$ has a virtual diagonal [3].

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then the space $\mathcal{A}\hat{\otimes}\mathcal{B}$ becomes a Banach algebra with the multiplication given by

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2) = a_1a_2\otimes b_1b_2 \quad (a_1,a_2\in\mathcal{A},\ b_1,b_2\in\mathcal{B}).$$

A Banach algebra $\mathcal{A}$ is said to be essential if $\mathcal{A}^2$ is dense in $\mathcal{A}$, where

$$\mathcal{A}^2 = lin\{a\cdot b\ ;\ a,b\in\mathcal{A}\}.$$

2. MAIN RESULTS

THEOREM 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. If $\mathcal{A}\hat{\otimes}\mathcal{B}$ is weakly amenable, then $\mathcal{A}$ and $\mathcal{B}$ are essential.*

Proof. Suppose that $\mathcal{A}$ is not essential. By Hahn-Banach theorem there is a non-zero $a^*\in\mathcal{A}^*$ such that $<aa',a^*>=0$ for each $a,a'\in\mathcal{A}$. Let b^* is a non-zero element of $\mathcal{B}^*$. The map $a^*\otimes b^*$ is a bounded linear functional on $\mathcal{A}\hat{\otimes}\mathcal{B}$ such that $<a\otimes b,a^*\otimes b^*>=<a,a^*><b,b^*>$. The map D: $(\mathcal{A}\hat{\otimes}\mathcal{B})\rightarrow(\mathcal{A}\hat{\otimes}\mathcal{B})^*$ defined by

$$D(m) = <m,a^*\otimes b^*> a^*\otimes b^*$$

is a bounded linear map and for $a,a'\in A$ and $b,b'\in B$ we have

$$D(a\otimes b\cdot a'\otimes b') = <aa'\otimes bb',a^*\otimes b^*> = <aa',a^*><bb',b^*> a^*\otimes b^* = 0,$$

on the other hand,

$$\begin{aligned}&a\otimes b\cdot D(a'\otimes b') + D(a\otimes b)\cdot a'\otimes b' =\\ &=<a'\otimes b',a^*\otimes b^*> a\otimes b\cdot a^*\otimes b^* + <a\otimes b,a^*\otimes b^*> a^*\otimes b^*\cdot a'\otimes b' =\\ &=<a'\otimes b',a^*\otimes b^*> aa^*\otimes bb^* + <a\otimes b,a^*\otimes b^*> a^*a'\otimes b^*b' = 0.\end{aligned}$$

Hence D is a derivation, so there is ϕ in $(\mathcal{A}\hat{\otimes}\mathcal{B})^*$ such that $D=\delta_\phi$. For each $a\in\mathcal{A}$ and $b\in\mathcal{B}$ we have

$$\begin{aligned}0 &=<a\otimes b, a\otimes b\cdot\phi - \phi\cdot a\otimes b>\\ &=<a\otimes b, D(a\otimes b)> = (<a,a^*><b,b^*>)^2.\end{aligned}$$

This is a contradiction. Similarly $\mathcal{B}$ is essential.

THEOREM 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Suppose that $\mathcal{B}$ is commutative and $\mathcal{A}\hat{\otimes}\mathcal{B}$ is weakly amenable, then $\mathcal{A}$ is weakly amenable.*

Proof. Let b^* be a non-zero element of $\mathcal{B}^*$. By Theorem 2.1., there are $c,d\in\mathcal{B}$ such that $<cd,b^*>=1$. Now let $d:\mathcal{A}\to\mathcal{A}^*$ be a derivation. Consider the map D: $(\mathcal{A}\hat{\otimes}\mathcal{B})\to(\mathcal{A}\hat{\otimes}\mathcal{B})^*$ defined by $<a'\otimes b',D(a\otimes b)>=<a',d(a)><b',bb^*>$. The map D is a bounded linear map and for each $a',a_1,a_2\in\mathcal{A}$ and $b',b_1,b_2\in\mathcal{B}$ we have

$$
\begin{aligned}
&<a'\otimes b',D(a_1\otimes b_1\cdot a_2\otimes b_2)>=<a'd(a_1a_2)><b',b_1b_2b^*>\\
&=<a'a_1,d(a_2)><b'b_1,b_2b^*>+<a_2a',d(a_1)>+<b_2b',b_1b^*>\\
&=<a'\otimes b',a_1\otimes b_1D(a_2\otimes b_2)>+<a'\otimes b',D(a_1\otimes b_1)a_2\otimes b_2>.
\end{aligned}
$$

Therefore D is a derivation and so there is ϕ in $(\mathcal{A}\hat{\otimes}\mathcal{B})^*$ such that $D=\delta_\phi$.

We define a^* on $\mathcal{A}$ by $a^*(a)=\phi(a\otimes cd)$ for all $a\in\mathcal{A}$. The map a^* is a bounded linear functional and for each $a,a'\in\mathcal{A}$ we have

$$
\begin{aligned}
&<a',d(a)=<a',d(a)><c,db^*>=\\
&=<a'\otimes c,D(a\otimes d)>=<a'\otimes c,a\otimes d\cdot\phi-\phi\cdot a\otimes d>=\\
&=<(a'a-aa')\otimes cd,\phi>=<a'a-aa',a^*>=<a',\delta_{a^*}(a)>.
\end{aligned}
$$

Therefore d is an inner derivation.

THEOREM 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. Then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is weakly amenable if and only $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable.*

Proof. Suppose that $\mathcal{A}\hat{\otimes}\mathcal{B}$ is weakly amenable. By Theorem 2.2., $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable. Conversely let $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable. By [3, Theorem 2.8.71] $\mathcal{A}\hat{\otimes}\mathcal{B}$ is weakly amenable.

We recall that a character ϕ on Banach algebra $\mathcal{B}$ is a non-zero linear functional on $\mathcal{B}$ such that $\phi(bb')=\phi(b)\phi(b')$ for all $b,b'\in\mathcal{B}$. We write $\Phi_{\mathcal{B}}$ for the set of all characters on $\mathcal{B}$. It is well known that $\Phi_{\mathcal{B}}\subseteq\mathcal{B}^*$ [2, 16.3].

THEOREM 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $\Phi_{\mathcal{B}}\neq\varnothing$. If $\mathcal{A}\hat{\otimes}\mathcal{B}$ is weakly amenable, then $\mathcal{A}$ is weakly amenable too.*

Proof. Let $\phi\in\Phi_{\mathcal{B}}$. Choose $b_0\in\mathcal{B}$ such that $\phi(b_0)=1$. Now let d: $\mathcal{A}\to\mathcal{A}^*$ be a derivation. We define D: $(\mathcal{A}\hat{\otimes}\mathcal{B})\to(\mathcal{A}\hat{\otimes}\mathcal{B})^*$ by $<a'\otimes b',D(a\otimes b)>=<a',d(a)><\phi(bb')$. The map D is bounded linear map and for each $a',a_1,a_2\in A$ and $b',b_1,b_2\in\mathcal{B}$ we have

$$
\begin{aligned}
&<a'\otimes b',a_1\otimes b_1\cdot D(a_2\otimes b_2)+D(a_1\otimes b_1)\cdot a_2\otimes b_2>\\
&=<a'a_1\otimes b'b_1,D(a_2\otimes b_2)>+<a_2a'\otimes b_2b',D(a_1\otimes b_1)>=\\
&=(<a'a_1,d(a_2)>+<a_2a',d(a_1>)\phi(b'b_1b_2)=\\
&=<a'\otimes b',D(a_1\otimes b_1\cdot a_2\otimes b_2)>,
\end{aligned}
$$

and so D is a derivation. Thus there exists $\psi\in\left(\mathcal{A}\hat{\otimes}\mathcal{B}\right)^*$ such that $D=\delta_\psi$. Define a^*: $\mathcal{A}\to\mathbb{C}$ by $a^*(a)=\psi(a\otimes b_0^2)$. For each $a,a'\in\mathcal{A}$ we have

$$<a',d(a)=<a',d(a)>\phi(b_0^2)=<a'\otimes b_0,D(a\otimes b_0)>=<(a'a-aa')\otimes b_0^2,\psi>=<a',\delta_{a^*}(a).$$

Consequently $d = \delta_{a^*}$ is an inner derivation.

Example. Let $\mathcal{A}$ be a Banach algebra and let $0 \neq \phi \in Ball(\mathcal{A}^*)$. Then $\mathcal{A}$ with the product $a \cdot a' = \phi(a)a'$ for all $a, a' \in \mathcal{A}$, becomes a Banach algebra. We denote this algebra with ${}_\phi\mathcal{A}$. It is easy to see that $\Phi({}_\phi\mathcal{A}) = \{\phi\}$. Now let $\mathcal{B}$ be a Banach algebra. By Theorem 2.4., if ${}_\phi\mathcal{A} \hat{\otimes} \mathcal{B}$ is weakly amenable, then so is $\mathcal{B}$.

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