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φ-MEANS OF SOME BANACH SUBSPACES ON A BANACH ALGEBRA

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In this paper, among the other things, we study the concept of φ-amenability of a Banach algebra A, where φ is a nonzero multiplicative linear functional on A. We present a few results in the theory of φ-amenable Banach algebras, and we obtain necessary and sufficient conditions for A^{**} to have a left invariant φ-mean. Let $Wap(A)$ be the Banach space of all weakly almost periodic functionals on A. Our second purpose in this paper is to present several characterizations of the existence of a left (right) invariant φ-mean on $Wap(A)$. Other results in this direction are also obtained.

Key words: Banach algebra, φ-amenability, φ-means, weak almost periodic, weak* topology.

1. INTRODUCTION

The concept of amenability for Banach algebras was first introduced by Johnson in [14]. According to Johnson' s definition, a Banach algebra A is amenable if every derivation from A into the dual A-bimodule E^* is inner for all Banach A-bimodules E. This concept of amenability has occupied an important place in the research of Banach algebras, operator algebras and harmonic analysis.

In [17], Lau introduced and investigated a large class of Banach algebras which he called F-algebras. Later, F-algebras were termed Lau algebras. They are Banach algebras A such that the dual A^* is a von Neumann algebra and the identity of A^* is a multiplicative linear functional on A. The concept of left amenability for a Lau algebra has been extensively extended for an arbitrary Banach algebra by introducing the notion of φ-amenability. Let A be an arbitrary Banach algebra and φ be a character of A, that is a homomorphism from A onto C. A is called φ-amenable if there exists a bounded linear functional m on A^* satisfying $\langle m,\varphi\rangle = 1$ and $\langle m, f.a\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in A$ and $f \in A^*$. This concept considerably generalizes the notion of left amenability for Lau algebras. For more details on φ-amenability of a Banach algebra the interested reader is referred to [13], [16] and [18]. Recently the notion of α-amenable hypergroups was introduced and studied in [2], [3] and [8].

The main purpose of this paper is to investigate the φ-amenability for certain Banach subspaces of dual Banach algebras. We continue our work [10] in the study of amenability of a Banach algebra A defined with respect to a character φ of A. Various necessary and sufficient conditions are found for a Banach algebra to possess a left invariant φ-mean. We prove that $Wap(A)$ has a left (right) invariant φ-mean if and only if a certain compact semitopological semigroup $\overline{S}$ (to be defined) associated with $\{\lambda_a; a \in P_1(A,\varphi)\}$ contains a left (right) zero. Other results in this direction are also obtained. We obtain sufficient conditions and some necessary conditions about A to have a left invariant φ-mean.

2. NOTATION AND PRELIMINARY RESULTS

In this paper, the second dual A^{**} of a Banach algebra A will always be equipped with the first Arens product which is defined as follows. For $a,b \in A$, $f \in A^*$ and $m,n \in A^{**}$, the elements fa and mf of A^* and $mn \in A^{**}$ are defined by

$$\langle f\cdot a,b\rangle=\langle f,ab\rangle,\ \langle n\cdot f,a\rangle=\langle n,f\cdot a\rangle,\ \langle mn,f\rangle=\langle m,n\cdot f\rangle,$$

respectively. With this multiplication, A^{**} is a Banach algebra and A is a subalgebra of A^{**} [5]. We write A^*A for the closed linear span in A^* of $\{f\cdot a; f\in A^*, a\in A\}$. When A has a bounded right approximate identity the Cohen-Hewitt factorization theorem (Theorem 32.22 of [12]) shows that in fact $A^*A=\{f\cdot a; f\in A^*, a\in A\}$. A functional $f\in A^*$ for which $\{f\cdot a; \|a\|\le 1\}$ is relatively compact in the weak (norm) topology of A^* is said to be weakly almost periodic (almost periodic). The set of weakly almost periodic (almost periodic) functionals on A is denoted by $Wap(A)$ $(Ap(A))$ (see [7] and [11]). If A is a Banach algebra, we shall denote by $B(Wap(A))$ the usual Banach algebra of bounded linear operators on $Wap(A)$. $B(Wap(A))$ is a semitopological semigroup under operator multiplication and the weak operator topology. We shall speak of any subsemigroup of $B(Wap(A))$ as a semigroup of operators.

Finally, we say that an element a of A is φ-maximal if it satisfies $\|a\|=\varphi(a)=1$. Let $P_1(A,\varphi)$ denote the collection of all φ-maximal elements of A [15]. When A is an Lau algebra and φ is the identity of the von Neumann algebra A^*, the φ-maximal elements are precisely the positive linear functionals of norm 1 on A^* and hence span A. Let $X(A,\varphi)$ denote the closed linear span of $P_1(A,\varphi)$. If $f\in A^*$ and $a\in A$, we also consider $\lambda_a(f)=f\cdot a$. Throughout the paper, $\Delta(A)$ will denote the set of all homomorphisms from A onto C.

3. MAIN RESULTS

Let A be a Banach algebra and let X be a closed subspace of A^*. We say that X is invariant if $f\cdot a\in X$ whenever $f\in X$ and $a\in A$.

Definition 3.1. Let A be a Banach algebra and let X be a closed subspace of A^* with $\varphi\in X$ that is invariant. A continuous functional m on X is called a left invariant φ-mean on X if the following properties holds:

$$\langle m,\varphi\rangle=1,\quad \langle m,f\cdot a\rangle=\varphi(a)\langle m,f\rangle\qquad (f\in X, a\in A).$$

LEMMA 3.2. *Let A be a Banach algebra and let $\varphi\in\Delta(A)$. Suppose that A has a bounded approximate identity. Then A^{**} has a left invariant φ-mean if and only if $(A^*A)^*$ has a left invariant φ-mean.*

Proof. Observe first that invariance properties of φ-means are transmitted by heredity onto subspaces. So $(A^*A)^*$ has a left invariant φ-mean.

To prove the converse, assume that n is a left invariant φ-mean on A^*A. Let $m\in A^{**}$ such that m extends n and $\|m\|=\|n\|$. Choose $a\in A$ with $\varphi(a)=1$. Let $\{e_\alpha\}_{\alpha\in I}$ be a bounded approximate identity in A. For $f\in A^*$ and $b\in A$, we have

$$\langle am,f\cdot b\rangle=\langle m,f\cdot ba\rangle=\langle n,f\cdot ba\rangle=\lim_\alpha\langle n,f\cdot e_\alpha ba\rangle=\varphi(ba)\lim_\alpha\langle n,fe_\alpha\rangle$$
$$=\varphi(b)\lim_\alpha\langle n,f\cdot e_\alpha a\rangle=\varphi(b)\langle n,f\cdot a\rangle=\varphi(b)\langle am,f\rangle.$$

This shows that am is a left invariant φ-mean on A^*.

Let G be a locally compact group and let $LUC(G)$ denote the Banach space of all bounded left uniformly continuous functions on G. For $f\in L^\infty(G)$ and $\phi\in L^1(G)$ we know that $f\phi=\tilde\phi * f$ (where $\tilde\phi(x)=\Delta(x^{-1})\phi(x^{-1})$, Δ here being the modular function on G). Thus

$LUC(G) = L^1(G) * L^\infty(G) = L^\infty(G)L^1(G)$ [12]. Lemma 3.2 shows that $LUC(G)^*$ has a left invariant φ-mean if and only if $L^\infty(G)^*$ has a left invariant φ-mean.

***Remark* 3.3**. Let A be a Banach algebra with a bounded approximate identity and $\varphi \in \Delta(A)$. Suppose that A^{**} admits a left invariant φ-mean. By Theorem 1.4 in [16], there exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in A such that $\| aa_\alpha - \varphi(a)a_\alpha \| \to 0$ for all $a \in A$ and $\varphi(a_\alpha) = 1$ for all $\alpha \in I$. Therefore $\| baa_\alpha - \varphi(a)ba_\alpha \| \to 0$ for every $a, b \in A$. Conversely, suppose that there exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in A such that $\varphi(a_\alpha) = 1$ for all $\alpha \in I$ and $\| baa_\alpha - \varphi(a)ba_\alpha \| \to 0$ for all $a, b \in A$. Then every weak* cluster point m of $\{a_\alpha\}_{\alpha\in I}$ in A^{**} clearly satisfies $\langle m, \varphi\rangle = 1$ and $\langle m, f\cdot a\rangle = \varphi(a)\langle m, f\rangle$ for all $f \in A^*A$ and $a \in A$. By Lemma 3.2, A^{**} has a left invariant φ-mean.

Suppose that for each $f \in A^*$ there exists $n_f \in A^{**}$ such that $\| n_f \| = \langle n_f, \varphi\rangle = 1$ and $\langle n_f, f\cdot ab\rangle = \varphi(ab)\langle n_f, f\rangle$ for all $a, b \in A$. There exists a net $\{a_\alpha\}_{\alpha\in I}$ in A such that $a_\alpha \to n_f$ in the weak* topology. Let $m_f = n_f n_f$. Then $\langle m_f, \varphi\rangle = \langle n_f, \varphi\rangle^2 = 1$ and $\| m_f \| = 1$. Moreover,

$$\begin{aligned}\varphi(b)\langle m_f, f\cdot a\rangle = \varphi(b)\langle n_f n_f, f\cdot a\rangle &= \lim_\alpha \varphi(b)\langle n_f, f\cdot aa_\alpha\rangle \\ &= \lim_\alpha \varphi(b)\varphi(aa_\alpha)\langle n_f, f\rangle = \lim_\alpha \varphi(a)\langle n_f, f\cdot ba_\alpha\rangle \\ &= \varphi(a)\langle n_f n_f, f\cdot b\rangle = \varphi(a)\langle m_f, f\cdot b\rangle\end{aligned}$$

for every $a, b \in A$. By Theorem 2.1 in [10], A admits a left invariant φ-mean of norm 1. Thus we have shown that a Banach algebra A has a left invariant φ-mean of norm 1 if and only if for each $f \in A^*$ there exists $n_f \in A^{**}$ such that $\| n_f \| = \langle n_f, \varphi\rangle = 1$ and $\langle n_f, f\cdot ab\rangle = \varphi(ab)\langle n_f, f\rangle$ for all $a, b \in A$.

Proposition 3.4. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then the following statements are equivalent*:

(i) *There exists a left invariant φ-mean m with $\| m \| = 1$;*

(ii) *if $\varepsilon \in (0,1)$, $k \geq 1$, $f_1, \ldots, f_k \in A^*$ and $m_1, \ldots m_k, n_1, \ldots n_k \in A^{**}$, then*

$$\sup\{\textstyle\sum_{i=1}^k \mathrm{Re}\langle(\varphi(m_i)n_i - \varphi(n_i)m_i)p, f_i\rangle;\ p \in A^{**}, \langle p, \varphi\rangle = 1, \| p \| < 1+\varepsilon\} \geq 0;$$

(iii) *There exists a net $\{m_\alpha\}_{\alpha\in I}$ in A^{**} such that $\lim_\alpha \langle m_\alpha, \varphi\rangle = 1$ and $\| m_\alpha \| = 1$ for all α and for every $a \in A$, $am_\alpha - \varphi(a)m_\alpha \to 0$ in the weak* topology of A^{**}.*

Proof. (i) implies (ii). Suppose that A^{**} admits a left invariant φ-mean of norm 1, say m. Let $\varepsilon \in (0,1)$, $k \geq 1$, $f_1, \ldots, f_k \in A^*$ and $m_1, \ldots m_k, n_1, \ldots n_k \in A^{**}$. Since A is weak* dense in A^{**} [5], there are bounded nets $a_\alpha^i, b_\alpha^i \in A$, $\alpha \in I$, such that $a_\alpha^i \to m_i$, $b_\alpha^i \to n_i$ in the weak* topology of A^{**}. (Notice that we can assume that a_α^i, b_α^i have the same directed set I; otherwise take the product of their directed sets.) Obviously,

$$\begin{aligned}\textstyle\sum_{i=1}^k \langle(\varphi(m_i)n_i - \varphi(n_i)m_i)m, f_i\rangle &= \textstyle\sum_{i=1}^k \lim_\alpha \langle(\varphi(a_\alpha^i)b_\alpha^i - \varphi(b_\alpha^i)a_\alpha^i), m\cdot f_i\rangle \\ &= \textstyle\sum_{i=1}^k \lim_\alpha \varphi(a_\alpha^i)\langle m, f_i\cdot b_\alpha^i\rangle - \varphi(b_\alpha^i)\langle m, f_i\cdot a_\alpha^i\rangle \\ &= \textstyle\sum_{i=1}^k \lim_\alpha \varphi(a_\alpha^i)\varphi(b_\alpha^i)(\langle m, f_i\rangle - \langle m, f_i\rangle) = 0,\end{aligned}$$

we have (i) implies (ii).

Conversely, suppose that (ii) holds. It is sufficient to show that

$$\sup\{\sum_{i=1}^{k}\mathrm{Re}\langle(\varphi(a_i)b_i-\varphi(b_i)a_i)c,f_i\rangle;c\in A,1-\varepsilon<\varphi(c)<1+\varepsilon,\|c\|<1+\varepsilon\}\geq 0,$$

where $k\in N$, $\varepsilon\in(0,1)$, $f_1,\dots,f_k\in A^*$ and $a_1,\dots a_k,b_1,\dots b_k\in A$ (see Theorem 2.1 in [10] and its proof). Let us assume on the contrary that there exist $\varepsilon\in(0,1)$, $f_1,\dots,f_k\in A^*$ and $a_1,\dots a_k,b_1,\dots b_k\in A$ such that

$$\sup\{\sum_{i=1}^{k}\mathrm{Re}\langle(\varphi(a_i)b_i-\varphi(b_i)a_i)c,f_i\rangle;1-\varepsilon<\varphi(c)<1+\varepsilon,\|c\|<1+\varepsilon\}<\beta<0.$$

By assumption, we can choose $p\in A^{**}$ such that $\langle p,\varphi\rangle=1$, $\|p\|<1+\varepsilon$ and

$$\sum_{i=1}^{k}\mathrm{Re}\langle(\varphi(a_i)b_i-\varphi(b_i)a_i)p,f_i\rangle\;\geq\;\frac{\beta}{2}.$$

By the Goldstine' s theorem [21], there exists a net $\{c_\gamma'\}_{\gamma\in J}$ in A such that $c_\gamma'\to p$ in the weak* topology and $\|c_\gamma'\|\leq\|p\|<1+\varepsilon$ for all $\gamma\in J$. Put $c_\gamma=\overline{\varphi(c_\gamma')}c_\gamma'$. For some $\gamma\in J$, we have $\|c_\gamma\|<1+\varepsilon$, $1-\varepsilon<\varphi(c_\gamma)<1+\varepsilon$, $|\varphi(c_\gamma)-\langle p,\varphi\rangle|<\varepsilon$ and

$$|\langle(\varphi(a_i)b_i-\varphi(b_i)a_i)p,f_i\rangle-\langle(\varphi(a_i)b_i-\varphi(b_i)a_i)c_\gamma,f_i\rangle\;|<\frac{-\beta}{2k},$$

for all $1\leq i\leq k$. This shows that

$$\sum_{i=1}^{k}\mathrm{Re}\langle(\varphi(a_i)b_i-\varphi(b_i)a_i)p,f_i\rangle<\sum_{i=1}^{k}\mathrm{Re}\langle(\varphi(a_i)b_i-\varphi(b_i)a_i)c_\gamma,f_i\rangle+\frac{-\beta}{2}\;<\;\frac{\beta}{2},$$

which is a contradiction. So (ii) implies (i).

Clearly, (i) implies (iii). Suppose that (iii) holds. If such a net $\{m_\alpha\}_{\alpha\in I}$ exists, then every weak* cluster point m of it in A^{**} clearly satisfies $\langle m,\varphi\rangle=1$, $\|m\|=1$ and $\langle m,f\cdot a\rangle=\varphi(a)\langle m,f\rangle$ for all $f\in A^*$ and $a\in A$. This completes the proof.

If S is a semigroup of operators on $Wap(A)$, the orbit $O(f)$ of an element f of $Wap(A)$ is defined to be $\{T(f);T\in S\}$. S will be called weakly almost periodic if each orbit has compact closure in the weak topology of $Wap(A)$. In the following theorem, we obtain necessary and sufficient conditions for $Wap(A)^*$ to have a left invariant φ-mean. Various necessary and sufficient conditions found for weakly almost periodic functions on a locally compact topological semigroup to possess a left invariant mean (see [9], [20], [23] and [24]).

THEOREM 3.5. *Let A be a Banach algebra and $\varphi\in\Delta(A)$. The closure $\overline{S}$ of $S=\{\lambda_a;a\in P_1(A,\varphi)\}$ in the weak operator topology is a compact convex semitopological semigroup in the same topology. Moreover, among the following two properties, the implication* (i) $\to$ (ii) *hold. If $X(A,\varphi)=A$, then $(ii)\to(i)$.*

(i) *$Wap(A)^*$ has a left invariant φ-mean $m\in\overline{P_1(A,\varphi)}^{w^*}$;*

(ii) *The semigroup $\overline{S}$ has a left zero, that is, there exists some $S\in\overline{S}$ such that $SoT=S$ for any $T\in\overline{S}$.*

Proof. For $f\in Wap(A)$, $\{f\cdot a;a\in P_1(A,\varphi)\}$ has compact closure in the weak topology of A^*. It follows that $\{\lambda_a(f);a\in P_1(A,\varphi)\}$ also has compact closure in the same topology or $\{\lambda_a;a\in P_1(A,\varphi)\}$ is weakly almost periodic. It is known that if S is weakly almost periodic, then the weak operator closure, $\overline{S}$ of S in $B(Wap(A))$ is a compact semitopological semigroup with weak operator topology, see Theorem 3.1 in [6].

Next assume (i) holds and let $m \in \overline{P_1(A,\varphi)}^{w^*}$ be a left invariant φ-mean on $Wap(A)$. Choose a net $\{a_\alpha\}_{\alpha\in I}$ in $P_1(A,\varphi)$ with the property that $a_\alpha \to m$ and $aa_\alpha - a_\alpha \to 0$ in the weak* topology of $Wap(A)^*$ for any $a \in P_1(A,\varphi)$. By compactness of $\overline{S}$, we can assume that λ_{a_α} converges to some $L_m \in \overline{S}$ in the weak operator topology, passing to a subnet if necessary. We claim that $L_m oT = L_m$ for all $T \in \overline{S}$. Let $a \in P_1(A,\varphi)$, $f \in Wap(A)$ and p Є $Wap(A)^*$. Choose a net $\{b_\beta\}_{\beta\in J}$ in A with the property that $b_\beta \to p$ in the weak* topology of $Wap(A)^*$ and $\| b_\beta \| \le \| p \|$ for all $\beta \in J$ [5]. Hence $b_\beta f \to pf$ in the weak* topology of $Wap(A)^*$. Since f is in $Wap(A)$, $\{a\cdot f; \|a\| \le \|p\|\}$ is relatively weakly compact [7]. Now $b_\beta f$, pf all belong to the weakly compact set $\overline{\{a\cdot f; \|a\| \le \|p\|\}}$ on which the weak topology and the weak* topology coincide. Hence $pf \in Wap(A)$. Since $pf \in Wap(A)$, we have

$$\langle p, L_m o\lambda_a - L_m(f)\rangle = \lim_\alpha \langle p, \lambda_{a_\alpha} o\lambda_a - \lambda_{a_\alpha}(f)\rangle = \lim_\alpha \langle p, f\cdot(aa_\alpha - a_\alpha)\rangle$$

$$= \lim_\alpha \langle pf, aa_\alpha - a_\alpha\rangle = 0.$$

We conclude that $L_m o\lambda_a = L_m$. Since $\{\lambda_a; a \in P_1(A,\varphi)\}$ is dense in $\overline{S}$ in the weak operator topology, we must have $L_m oT = L_m$ for any $T \in \overline{S}$ and (i) implies (ii).

Conversely assume (ii) and let S be a left zero of $\overline{S}$, then $So\lambda_\alpha = S$ for any $a \in P_1(A,\varphi)$. There is a net $\{a_\alpha\}_{\alpha\in I}$ in $P_1(A,\varphi)$ such that $\lambda_{a_\alpha} \to S$ in the weak operator topology. By the Banach-Alaoglu' s theorem [21], without loss of generality we may assume that $a_\alpha \to m$ in the weak * topology of $Wap(A)^*$. We set out to prove that mm is a left invariant φ-mean on $Wap(A)$. Clearly $mm \in \overline{P_1(A,\varphi)}^{w^*}$. For every $f \in Wap(A)$ and $a \in P_1(A,\varphi)$, we have

$$\langle mm, f\cdot a\rangle - \langle mm, f\rangle = \langle m, m\cdot(f\cdot a)\rangle - \langle m, m\cdot f\rangle = \lim_\alpha \langle m, f\cdot aa_\alpha\rangle - \langle m, f\cdot a_\alpha\rangle =$$

$$= \lim_\alpha \langle m, \lambda_{a_\alpha} o\lambda_a(f)\rangle - \langle m, \lambda_{a_\alpha}(f)\rangle =$$

$$= \langle m, So\lambda_a(f)\rangle - \langle m, S(f)\rangle = 0.$$

This shows that mm is a left invariant φ-mean on $Wap(A)$.

We can replace the weak topology and the weak operator topology in the above Theorem by the strong topology and the strong operator topology, respectively, we have

THEOREM 3.6. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. The closure $\overline{S}$ of $S = \{\lambda_a; a \in P_1(A,\varphi)\}$ in the strong operator topology is a compact convex topological semigroup in the same topology. Moreover, among the following two properties, the implication* (i)$\to$(ii) hold. If $X(A,\varphi) = A$, *then* (ii)$\to$(i).

(i) $Ap(A)^*$ *has a left invariant* φ*-mean* $m \in \overline{P_1(A,\varphi)}^{w^*}$;

(ii) *The semigroup* $\overline{S}$ *has a left zero*.

We introduce some additional concepts. A linear functional $m \in Wap(A)^*$ is called a right invariant φ-mean on $Wap(A)$ if $\langle m, \varphi\rangle = 1$ and $\langle m, af\rangle = \varphi(a)\langle m, f\rangle$ whenever $f \in Wap(A)$ and $a \in A$. A left invariant and right invariant φ-mean on $Wap(A)$ is called invariant φ-mean. Kaniuth, Lau and Pym in [15] showed that an element m of $Wap(A)^*$ is an invariant φ-mean for $Wap(A)$ if and only if there exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in A such that $\varphi(a_\alpha) = 1$ for all α and for each $a \in A$,

$$\| aa_\alpha - \varphi(a)a_\alpha \| \to 0, \quad \| a_\alpha a - \varphi(a)a_\alpha \| \to 0.$$

THEOREM 3.7: *Let A be a Banach algebra and $\varphi \in \Delta(A)$. If $Wap(A)^*$ has a left invariant φ-mean $m \in \overline{P_1(A,\varphi)}^{w^*}$ and a right invariant φ-mean $n \in \overline{P_1(A,\varphi)}^{w^*}$, then the compact semitopological semigroup $\overline{S}$ contains a left zero and a right zero. Moreover, $m = n$ and it is the unique invariant φ-mean on $Wap(A)$.*

Proof. Assume that $Wap(A)^*$ has a right invariant φ-mean $n \in \overline{P_1(A,\varphi)}^{w^*}$. There is a net $\{b_\beta\}_{\beta\in J}$ in $P_1(A,\varphi)$ such that for all $a \in P_1(A,\varphi)$, $b_\beta a - b_\beta \to 0$ and also $b_\beta \to n$ in the weak* topology of $Wap(A)^*$. An argument similar to the proof of Theorem 3.5 shows that the semigroup $\overline{S}$ has a right zero, say R_n. Since L_m is a left zero of $\overline{S}$ and R_n is a right zero of $\overline{S}$, we have $L_m = L_m o R_n = R_n$. Let $\Phi : Wap(A)^* \to B(Wap(A))$ be defined by $\Phi(p)(f) = \langle p, f\rangle\varphi$. Clearly Φ is a bounded linear operator and an isometry. If $\Phi(n) \notin \overline{S}$, then Theorem 3.4 in [21] implies that there is a p in $Wap(A)^*$, an f in $Wap(A)$, an c in R, and $\varepsilon > 0$ such that

$$\mathrm{Re}\langle p, fb_\beta\rangle \le c < c+\varepsilon \le \mathrm{Re}\langle p, \Phi(n)(f)\rangle = \mathrm{Re}\langle n, f\rangle\langle p,\varphi\rangle.$$

for all $\beta \in J$. We conclude that

$$\mathrm{Re}\langle np, f\rangle \le c < c+\varepsilon \le \mathrm{Re}\langle n, f\rangle\langle p,\varphi\rangle$$

By Goldstin' s theorem [21], there exists a net $\{c_\gamma\}_{\gamma\in L}$ in A such that $c_\gamma \to p$ in the weak* topology of $Wap(A)^*$. Since $Wap(A)^*$ is Arens regular [7], we have

$$\mathrm{Re}\langle n, f\rangle\langle p,\varphi\rangle = \lim_\gamma \mathrm{Re}\langle n, f\rangle\varphi(c_\gamma) = \lim_\gamma \mathrm{Re}\langle n, c_\gamma f\rangle = \mathrm{Re}\langle np, f\rangle$$
$$< \mathrm{Re}\langle n, f\rangle\langle p,\varphi\rangle.$$

This is a contradiction. We conclude that $\Phi(n) \in \overline{S}$. Clearly $\Phi(n)$ is a right zero and so $\Phi(n) = R_n$.
For each $a \in A$, $f \in Wap(A)$ and $b \in A$,

$$\langle na\cdot f, b\rangle = \langle n, a\cdot(f\cdot b)\rangle = \varphi(a)\langle n, f\cdot b\rangle = \varphi(a)\langle n\cdot f, b\rangle.$$

Therefore $na\cdot f = \varphi(a)n\cdot f$. This shows that

$$\langle mn, a\cdot f\rangle = \langle m, n\cdot(a\cdot f)\rangle = \varphi(a)\langle m, n\cdot f\rangle = \varphi(a)\langle mn, f\rangle,$$

and so mn is a right invariant φ-mean in $\overline{P_1(A,\varphi)}^{w^*}$. Consequently

$$L_m(f) = \Phi(mn)(f) = \Phi(n)(f) = \langle n, f\rangle\varphi$$

for every $f \in Wap(A)$. Hence $m = mn = n$ is the unique invariant φ-mean on $Wap(A)$.

As a simple application of the preceding theorems, let G be a locally compact group. Consider the Banach space $W(G)$ of all weakly almost periodic functions on G with supremum norm [4]. It is known that $W(G) = Wap(L^1(G))$ [22]. By the Ryll-Nardzewski theorem, $W(G)$ has a left and right invariant mean [19]. By Theorem 3.7, there exists exactly one invariant mean on $W(G)$.

***Remark* 3.8**. (i). Let A be a Banach algebra and $\varphi \in \Delta(A)$. If $Wap(A)^*$ has a right invariant φ-mean $n \in \overline{P_1(A,\varphi)}^{w^*}$, then $\overline{S}$ has a right zero, say R_n. Let $C_b(\overline{S})$ be the space of bounded continuous functions on $\overline{S}$ with usual sup norm. It is easily checked that the map M, where $M(f) = f(R_n)$, is a left invariant mean on $C_b(\overline{S})$ [4]. The reader is referred to [4] for more information on amenability of semigroups. Conversely, let $C_b(\overline{S})$ have a left invariant mean. Then the map $(S,T) \to SoT$ of $\overline{S}\times\overline{S} \to \overline{S}$ is an A-representation (see [1] for details). By theorem 1 in [1], there exists some $T \in \overline{S}$ such that $SoT = S$ for every $S \in \overline{S}$. Thus $\overline{S}$ has a right zero and so $Wap(A)^*$ has a right invariant φ-mean.

(ii). Let A be a Banach algebra and $\varphi \in \Delta(A)$. Under the weak topology, $P_1(A,\varphi)$ is a semitopological semigroup. If $C_b(P_1(A,\varphi))$ has a left invariant mean, then $C_b(\overline{S})$ has a left invariant mean (see Lemma 2.1 in [6]). Therefore $Wap(A)^*$ has a right invariant φ-mean.

COROLLARY 3.9. *Let A be a reflexive Banach algebra and $\varphi \in \Delta(A)$. If A^{**} has a left invariant φ-mean $m \in \overline{P_1(A,\varphi)}^{w^*}$ and a right invariant φ-mean $n \in \overline{P_1(A,\varphi)}^{w^*}$, then $m = n$ and it is the unique invariant φ-mean on A^*.*

Proof. Assume that A is a reflexive Banach algebra. For $f \in A^*$, $\{f \cdot a; \|a\| \le 1\}$ has compcat closure in the weak* topology of A^*. As A is reflexive, $\{f \cdot a; \|a\| \le 1\}$ has compact closure in the weak topology [21]. Therefore $A^* = Wap(A)$. By Theorem 3.7, A has a unique invariant φ-mean.

THEOREM 3.10. *Let A be a Banach algebra and $\varphi \in \Delta(A)$.*

(i) *Let M be the set of all left invariant φ-means on $Ap(A)$. Then M is a compact convex topological semigroup under Arens product and the weak* topology of $Ap(A)^*$;*

(ii) *Let M be the set of all left invariant φ-means on $Wap(A)$. Then M is a compact convex semitopological semigroup under Arens product and the weak* topology of $Wap(A)^*$.*

Proof. (i) M is clearly a convex semigroup under Arens product and is weak* compact in $Ap(A)^*$. All we need is to show that the map $(m,n) \mapsto mn$ is continuous. Let (m_α, n_α) be a net in $M \times M$ converging to (m,n) in $M \times M$. Let f be any element in $Ap(A)$. Since $n_\alpha \to n$ weak *, we conclude that $n_\alpha f \to nf$ weak* and hence norm. Therefore

$$\begin{aligned} |\langle m_\alpha n_\alpha, f\rangle - \langle mn, f\rangle| &\le |\langle m_\alpha, n_\alpha \cdot f\rangle - \langle m_\alpha, n \cdot f\rangle| + |\langle m_\alpha, n \cdot f\rangle - \langle m, n \cdot f\rangle| \\ &\le \|m_\alpha\| \|n_\alpha \cdot f - n \cdot f\| + |\langle m_\alpha, n \cdot f\rangle - \langle m, n \cdot f\rangle| \to 0. \end{aligned}$$

(ii) A similar argument as above implies this part.

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