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INVERSE DYNAMICS OF THE SPATIAL 3-*RPS* PARALLEL ROBOT

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Recursive matrix relations for kinematics and dynamics of a spatial three-degrees-of-freedom 3-RPS parallel mechanism are established in this paper. Three identical legs connect the moving platform by revolute joints. Knowing the motion of the platform, we develop first the inverse kinematical problem and determine the positions, velocities and accelerations of the robot. Further, the inverse dynamics problem of the manipulator is formulated using either the principle of virtual work or the Lagrange equations with their multipliers. Compact matrix equations offer expressions and graphs for the power requirement comparison of each of three actuators in two different actuation schemes: revolute actuators and prismatic actuators.

Key words: kinematics, dynamics, parallel mechanism, platform, virtual work.

1. INTRODUCTION

Parallel manipulators consist, in general, of two main bodies coupled via numerous legs acting in parallel. The links of the robot are connected one to the other by spherical joints, universal joints, revolute joints or prismatic joints [1]. Compared with serial robots, the followings are the advantages of parallel manipulators: higher kinematical accuracy, lighter weight with better structural rigidity and stabile capacity.

Considerable efforts have been devoted to the kinematics and dynamic analysis of fully parallel robots. Among these, the class of manipulators known as Stewart-Gough platform focused great attention (Stewart [2]; Merlet [3]). They are used in flight simulators and more recently for Parallel Kinematics Machines. The prototype of Delta parallel robot (Clavel [4]; Tsai and Stamper [5]; Staicu [6]) developed by Clavel at the Federal Polytechnic Institute of Lausanne and by Tsai and Stamper at the University of Maryland as well as the Star parallel manipulator (Hervé and Sparacino [7]) are equipped with three motors, which train on the mobile platform in a three-degrees-of-freedom general translation motion. Angeles [8], Wang and Gosselin [9] analyzed the kinematics and dynamics of Agile Wrist spherical robot with three actuators.

The 3-RPS parallel mechanism consists of a moving platform, a fixed base and three legs of identical structure. The potential application as a positioning device of the tool in a new parallel kinematics machine for high precision blasting attracted a scientific and practical interest to this manipulator type. The robot motion in the translation and the orientation degrees of freedom is *interconnected*. This makes the investigations more complicated and this problem does not exist in the manipulators with a different structure. Shah and Lee [10] analyzed the inverse and forward kinematics equations for the case when the manipulator motion is determined by two degrees of orientation freedom and one degree of vertical translation. The inverse kinematics problem was considered by Song and Zhang [11]. Fang and Huang [12] revealed the relationships between output motions and three input parameters. A new solution of the inverse kinematics task for the manipulator with RPS joint structure is obtained by Sokolov and Xirouchakis [13].

2. KINEMATICS ANALYSIS

A recursive method is introduced in the present paper, to reduce significantly the number of equations and computation operations by using a set of matrices for kinematics and dynamics models of the 3-RPS spatial parallel mechanism. Having a closed-loop structures, the robot is a special symmetrical structure composed of three kinematical chains of variable length with identical topology, all connecting the fixed base to the moving platform. The extensible leg connects the moving platform by spherical joint and the base by means of revolute joint, axe of which being parallel to the opposite edge. Each leg is made up of a cylinder and a piston connected together by a prismatic joint. The actuated revolute joints or prismatic joints of the legs can drive the manipulator.

The manipulator consists of the base $A_1B_1C_1$ and the upper platform $A_3B_3C_3$ that are two equilateral triangles with $L = l_0\sqrt{3}$ and $l = r\sqrt{3}$ the lengths of the sides, respectively. Overall, there are seven moving links, three revolute joints, three prismatic joints and three spherical joints.

In a first kind of the robot (RPS) we consider the moving platform as the output link and the links A_1A_2, B_1B_2, C_1C_2 as the input links. Thus, all actuators can be installed on the fixed base. In the second configuration (RPS) each prismatic joint is an actively controlled prismatic cylinder. A Cartesian frame $Ox_0y_0z_0(T_0)$ is attached to the fixed base with its origin located at triangle centre O, the z_0 axis perpendicular to the base and the x_0 axis pointing towards A_1 from O. Another mobile reference frame $Gx_Gy_Gz_G$ is attached to the moving platform. The origin of this coordinate central system is located just at the centre G of the moving triangle (Fig. 1). It is noted that the relative rotation with $\varphi_{k,k-1}$ angle or relative translation with $\lambda_{k,k-1}$ displacement must be always pointing about or along the direction of z_k axis.

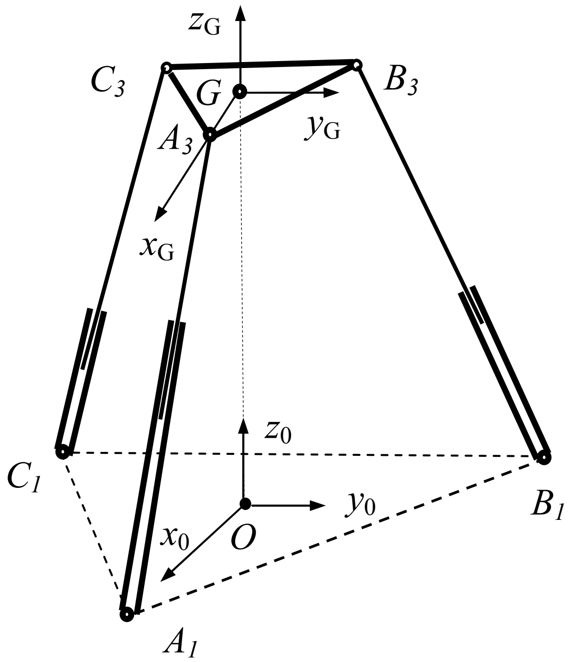

Fig. 1 – The 3-RPS parallel mechanism.

In what follows we consider that the moving platform is initially located at a *central configuration* and the mass centre is located at an elevation $OG = h$. A complete description of the position and orientation of the moving platform with respect to the reference frame requires six variables: the coordinates x_0^G, y_0^G, z_0^G of the mass centre G and three Euler's angles α_1, α_2 α_3 associated with three successive rotations. Since all rotations take place successively about the moving coordinate axes, the general rotation matrix $R = d_{32}d_{21}d_{10}$ is obtained by multiplying three transformation matrices

$$\mathrm{d}_{10} = a_1\theta_1,\ \mathrm{d}_{21} = a_2\theta_1\theta_2,\ \mathrm{d}_{32} = a_3\theta_2\theta_1\theta_2, \tag{1}$$

where

$$\theta_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \theta_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ a_k = \begin{bmatrix} \cos\alpha_k & \sin\alpha_k & 0 \\ -\sin\alpha_k & \cos\alpha_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (k = 1, 2, 3)\,. \tag{2}$$

Here, z_0^G, α_1 and α_2 are chosen as the independent variables and x_0^G, y_0^G, α_3 are parameters of *parasitic* motions. The three parasitic motions from the six commonly known motions of the moving platform are permanently dependent on three independent variables.

One of three active legs (for example leg A) consists of a fixed revolute joint, a moving cylinder **1** of length l_1, mass m_1 and tensor of inertia $\hat{J}_1$, which has rotation about z_1^A axis with the angle φ_{10}^A, the angular velocity $\omega_{10}^A = \dot{\varphi}_{10}^A$ and the angular acceleration $\varepsilon_{10}^A = \ddot{\varphi}_{10}^A$. A prismatic joint is as well as a piston **2** linked at the $A_2 x_2^A y_2^A z_2^A$ frame, having a relative motion with the displacement λ_{21}^A, the velocity $v_{21}^A = \dot{\lambda}_{21}^A$ and the acceleration $\gamma_{21}^A = \ddot{\lambda}_{21}^A$. It has the length l_2, mass m_2 and tensor of inertia $\hat{J}_2$. Finally, a spherical joint is introduced at a planar moving platform, which is schematised as an equilateral triangle with edge l, mass m_p and tensor of inertia $\hat{J}_p$ (Fig. 2).

At the central configuration, we also consider that all legs are initially extended at equal length $\lambda_0 + l_2 = h/\cos\beta$. Also, the angles of orientation of fixed pivots are given by

$$\alpha_A = 0, \alpha_B = \frac{2\pi}{3}, \alpha_C = -\frac{2\pi}{3}, \ \tan\beta = \frac{l_0 - r}{h}\,. \tag{3}$$

Pursuing the first leg A in the $OA_1A_2A_3$ way, for example, we obtain the following matrices of transformation:

$$a_{10} = a_{10}^{\varphi} a_{\beta} \theta_3 \, a_{\alpha}^A, \ a_{21} = \theta_3^T, \ a_{20} = a_{21} a_{10}, \tag{4}$$

where

$$a_{10}^{\varphi} = \begin{bmatrix} \cos\varphi_{10}^A & \sin\varphi_{10}^A & 0 \\ -\sin\varphi_{10}^A & \cos\varphi_{10}^A & 0 \\ 0 & 0 & 1 \end{bmatrix}, a_{\alpha}^A = \begin{bmatrix} \cos\alpha_A & \sin\alpha_A & 0 \\ -\sin\alpha_A & \cos\alpha_A & 0 \\ 0 & 0 & 1 \end{bmatrix}, a_{\beta} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\theta_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{5}$$

Analogous relations can be written for other two legs of the mechanism.

Three rotations angles $\varphi_{10}^A, \varphi_{10}^B, \varphi_{10}^C$ of the active links are the joint variables that give the input vector $\vec{\varphi}_{10} = [\varphi_{10}^A \quad \varphi_{10}^B \quad \varphi_{10}^C]^T$ of the instantaneous position of the mechanism in the first study configuration. In the inverse geometric problem, the position of the mechanism is completely given through the coordinate z_0^G of the mass centre G and two Euler angles α_1, α_2 of rotation about the axes x_G, y_G, respectively. For the 3-RPS mechanism, the z_0^G coordinate is the only completely independent variable with other five pose parameters.

We suppose that following analytical functions can describe the general motion of the moving platform

$$z_0^G = h + z_0^{G*}(1 - \cos\frac{\pi}{3}t), \ \alpha_1 = \alpha_1^*(1 - \cos\frac{\pi}{3}t), \ \alpha_2 = \alpha_2^*(1 - \cos\frac{\pi}{3}t)\,. \tag{6}$$

Six independent variables φ_{10}^A, $\lambda_{21}^A, \varphi_{10}^B$, $\lambda_{21}^B, \varphi_{10}^C$, λ_{21}^C and the parameters α_3, x_0^G, y_0^G will be determined by several vector-loop equations as follows

$$\vec{r}_{10}^{\,A} + \sum_{k=1}^{2} a_{k0}^{T}\vec{r}_{k+1,k}^{\,A} - R^{T}\vec{r}_{G}^{\,A_3} = \vec{r}_{10}^{\,B} + \sum_{k=1}^{2} b_{k0}^{T}\vec{r}_{k+1,k}^{\,B} - R^{T}\vec{r}_{G}^{\,B_3} = \vec{r}_{10}^{\,C} + \sum_{k=1}^{2} c_{k0}^{T}\vec{r}_{k+1,k}^{\,C} - R^{T}\vec{r}_{\mathrm{G}}^{\,C_3} = \vec{r}_{0}^{\,G}, \tag{7}$$

where one denoted

$$\vec{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ \vec{u}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix},\ \vec{u}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix},\ \tilde{u}_3 = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix} \tag{8}$$

$$\vec{r}_{10}^{\,i} = l_0 a_{\alpha}^{iT}\vec{u}_1,\ \vec{r}_{21}^{\,i} = (\lambda_0 + \lambda_{21}^{i})\vec{u}_2,\ \vec{r}_{32}^{\,i} = l_2\vec{u}_3,\ \vec{r}_{G}^{\,i_3} = r a_{\alpha}^{iT}\vec{u}_1 \quad (i = A, B, C).$$

These vector equations mean that there is only one inverse geometric solution for the manipulator:

$$\begin{aligned}
&-(\lambda_0 + l_2 + \lambda_{21}^{i})\cos\alpha_i \sin(\varphi_{10}^{i} + \beta) = x_0^G - x_{10}^{i} + q_1^{i},\\
&-(\lambda_0 + l_2 + \lambda_{21}^{i})\sin\alpha_j \sin(\varphi_{10}^{i} + \beta) = y_0^G - y_{10}^{i} + q_2^{i},\\
&(\lambda_0 + l_2 + \lambda_{21}^{i})\cos(\varphi_{10}^{i} + \beta) = z_0^G + q_3^{i} \quad (i = A,\ B,\ C),
\end{aligned} \tag{9}$$

with the notations

$$\begin{aligned}
&q_1^{i} = x_G^{i_3}\cos\alpha_2\cos\alpha_3 - y_G^{i_3}\cos\alpha_2\sin\alpha_3,\\
&q_2^{i} = x_G^{i}(\sin\alpha_1\sin\alpha_2\cos\alpha_3 + \cos\alpha_1\sin\alpha_3) + y_G^{i_3}(-\sin\alpha_1\sin\alpha_2\sin\alpha_3 + \cos\alpha_1\cos\alpha_3),\\
&q_3^{i} = x_G^{i_3}(-\cos\alpha_1\sin\alpha_2\cos\alpha_3 + \sin\alpha_1\sin\alpha_3) + y_G^{i_3}(\cos\alpha_1\sin\alpha_2\sin\alpha_3 + \sin\alpha_1\cos\alpha_3).
\end{aligned} \tag{10}$$

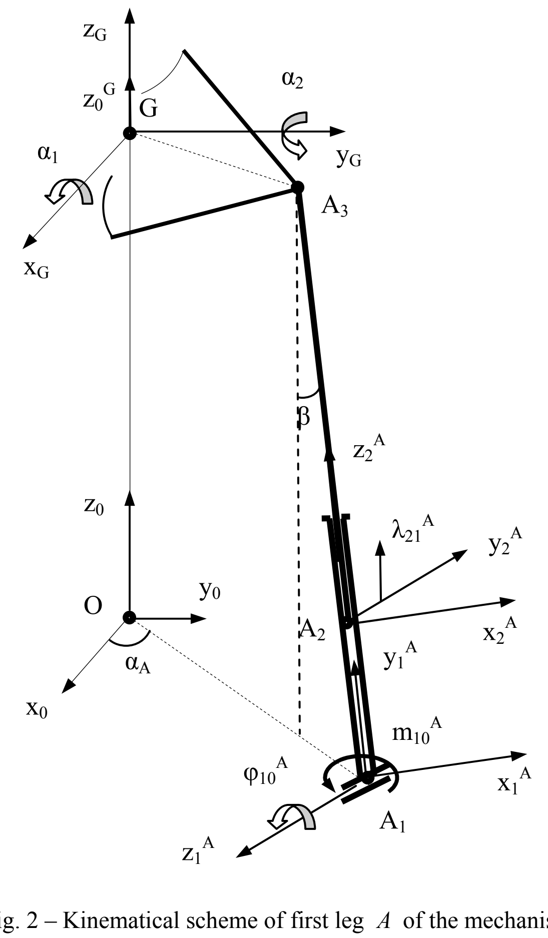

Fig. 2 – Kinematical scheme of first leg *A* of the mechanism.

We can obtain the following significant functions from Eqs. (9) with the notations

$$\tan\alpha_3 = -\frac{\sin\alpha_1 \sin\alpha_2}{\cos\alpha_1 + \cos\alpha_2}, \quad (11)$$
$$x_0^G = 0.5 l_0 (\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 - \cos\alpha_1 \cos\alpha_3 + \cos\alpha_2 \cos\alpha_3) \;\; ; \;\; y_0^G = l_0 \cos\alpha_2 \sin\alpha_3 .$$

We develop the inverse kinematics problem and determine the velocities and accelerations of the manipulator, supposing that the motion of the moving platform is known.

First, we compute in terms of the angular velocities $\dot\alpha_1, \dot\alpha_2$ and the vertical velocity $\dot z_0^G$ the angular velocities $\dot\alpha_3, \omega_{10}^A, \omega_{10}^B, \omega_{10}^C$ and the linear velocities $\dot x_0^G$, $\dot y_0^G$, $v_{21}^A, v_{21}^B, v_{21}^C$. The motion of the elements of each leg (e.g. leg A) are characterized by the following skew symmetric matrices *associated* to velocities $\vec\omega_{k0}^A$

$$\tilde\omega_{k0}^A = a_{k,k-1} \tilde\omega_{k-1,0}^A a_{k,k-1}^T + \omega_{k,k-1}^A \tilde u_3, \;\; \omega_{k,k-1}^A = \dot\varphi_{k,k-1}^A . \quad (12)$$

Following relations give the velocities $\vec v_{k0}^A$ of the joints A_k

$$\vec v_{k0}^A = a_{k,k-1} \vec v_{k-1,0}^A + a_{k,k-1} \tilde\omega_{k-1,0}^A \vec r_{k,k-1}^A + v_{k,k-1}^A \vec u_3 \;\; ; \;\; v_{10}^A = 0 \;\; ; \;\; \omega_{21}^A = 0 . \quad (13)$$

Equations of geometrical constraints (7) can be derivate with respect to time to obtain the following *matrix conditions of connectivity* (Staicu et al., [14])

$$\omega_{10}^A \vec u_j^T a_{10}^T \tilde u_3 \{\vec r_{21}^A + a_{21}^T \vec r_{32}^A\} + v_{21}^A \vec u_j^T a_{10}^T \vec u_2 = \vec u_j^T \dot{\vec r}_0^G + \vec u_j^T R^T \tilde\omega_p \vec r_G^{A_3} \quad (j = 1, 2, 3), \quad (14)$$

where

$$\tilde\omega_p = \dot\alpha_1 \, d_{32} \, d_{21} \, \tilde u_3 \, d_{21}^T \, d_{32}^T + \dot\alpha_2 \, d_{32} \, \tilde u_3 \, d_{32}^T + \dot\alpha_3 \tilde u_3 \quad (15)$$

denotes the skew-symmetric matrix associated to the angular velocity $\vec\omega_p$ of moving platform.

Three independent displacements $\lambda_{21}^A, \lambda_{21}^B, \lambda_{21}^C$ of the active links are the joint variables giving the input vector $\vec\lambda_{21} = [\lambda_{21}^A \;\; \lambda_{21}^B \;\; \lambda_{21}^C]^T$ of the instantaneous pose of the mechanism in the *second* study configuration.

Now, let us assume that the manipulator has successively two *virtual motions* determined by the angular velocities $\omega_{10a}^{Av} = 1, \omega_{10a}^{Bv} = 0, \omega_{10a}^{Cv} = 0$ or the linear velocities $v_{21a}^{Av} = 1$, $v_{21a}^{Bv} = 0$, $v_{21a}^{Cv} = 0$. The characteristic *virtual velocities* are expressed as functions of the position of the robot by the constraints equations (14).

The derivatives with respect to time of equations (14) give following conditions of connectivity for the accelerations $\ddot\alpha_3, \varepsilon_{10}^A, \varepsilon_{10}^B, \varepsilon_{10}^C, \ddot x_0^G, \ddot y_0^G, \gamma_{21}^A, \gamma_{21}^B, \gamma_{21}^C$ (Staicu and Zhang [15])

$$\begin{aligned}
&\varepsilon_{10}^A \vec u_j^T a_{10}^T \tilde u_3 \{\vec r_{21}^A + a_{21}^T \vec r_{32}^A\} + \gamma_{21}^A \vec u_j^T a_{10}^T \vec u_2 = \vec u_j^T \ddot{\vec r}_0^G + \vec u_j^T R^T \{\tilde\omega_p \tilde\omega_p + \dot{\tilde\omega}_p\} \vec r_G^{A_3} - \\
&- \omega_{10}^A \omega_{10}^A \vec u_j^T a_{10}^T \tilde u_3 \tilde u_3 a_{21}^T \{\vec r_{21}^A + a_{21}^T \vec r_{32}^A\} - 2\omega_{10}^A v_{21}^A \vec u_j^T a_{10}^T \tilde u_3 \vec u_2 \quad (j = 1, 2, 3),
\end{aligned} \quad (16)$$

where an useful square matrix is expressed

$$\begin{aligned}
\tilde\omega_p \tilde\omega_p + \dot{\tilde\omega}_p = {} & \ddot\alpha_1 d_{32} d_{21} \tilde u_3 d_{21}^T d_{32}^T + \ddot\alpha_2 d_{32} \tilde u_3 d_{32}^T + \ddot\alpha_3 \tilde u_3 + \dot\alpha_1^2 d_{32} d_{21} \tilde u_3 \tilde u_3 d_{21}^T d_{32}^T + \dot\alpha_2^2 d_{32} \tilde u_3 \tilde u_3 d_{32}^T + \dot\alpha_3^2 \tilde u_3 \tilde u_3 + \\
& + 2\dot\alpha_1 \dot\alpha_2 d_{32} d_{21} \tilde u_3 d_{21}^T \tilde u_3 d_{32}^T + 2\dot\alpha_2 \dot\alpha_3 d_{32} \tilde u_3 d_{32}^T \tilde u_3 + 2\dot\alpha_3 \dot\alpha_1 d_{32} d_{21} \tilde u_3 d_{21}^T d_{32}^T \tilde u_3 .
\end{aligned} \quad (17)$$

Following recursive relations give the angular accelerations $\vec\varepsilon_{k0}^A$ and the accelerations $\vec\gamma_{k0}^A$ of joints A_k

$$\begin{aligned}
&\vec\varepsilon_{k0}^A = a_{k,k-1} \vec\varepsilon_{k-1,0}^A + \varepsilon_{k,k-1}^A \vec u_3 + \omega_{k,k-1}^A a_{k,k-1} \tilde\omega_{k-1,0}^A a_{k,k-1}^T \vec u_3 \\
&\vec\gamma_{k0}^A = a_{k,k-1} \vec\gamma_{k-1,0}^A + a_{k,k-1} (\tilde\omega_{k-1,0}^{Ai} \tilde\omega_{k-1.0}^A + \tilde\varepsilon_{k-1,0}^A) \vec r_{k,k-1}^A + 2 v_{k,k-1}^A a_{k,k-1} \tilde\omega_{k-1,0}^A a_{k,k-1}^T \vec u_3 + \gamma_{k,k-1}^A \vec u_3 \quad (k = 1, 2, 3).
\end{aligned} \quad (18)$$

3. EQUATIONS OF MOTION

Difficulties encountered in dynamics modelling of parallel robots include problematic issues such as: spatial kinematical structure with possess of large number of passive DOF, dominance of inertial forces over the frictional and gravitational components and the problem linked to the inverse dynamics. Several methods have been applied to formulate the dynamics of parallel mechanisms, which could provide the same results concerning these actuating torques or forces. The first one is using the Newton-Euler classic procedure (Dasgupta and Mruthyunjaya [16]), the second one applies the Lagrange's equations and multipliers formalism (Geng et al. [17]) and the third approach is based on the principle of virtual work (Tsai [1]; Angeles [8]; Zhang and Song [18]; Sokolov and Xirouchakis [19]).

3.1. Principle of virtual work

Three electric motors A_1, B_1, C_1 that generate the torques $\vec{m}_{10}^A = m_{10}^A \vec{u}_3$, $\vec{m}_{10}^B = m_{10}^B \vec{u}_3$, $\vec{m}_{10}^C = m_{10}^C \vec{u}_3$ oriented about coplanar axes or three independent mechanical systems acting along the directions z_2^A, z_2^B, z_2^C with the forces $\vec{f}_{21}^A = f_{21}^A \vec{u}_3$, $\vec{f}_{21}^B = f_{21}^B \vec{u}_3$, $\vec{f}_{21}^C = f_{21}^C \vec{u}_3$ control the motion of the moving platform.

The force of inertia and the resulting moment of inertia forces of a rigid body T_k^A, for example,

$$\vec{f}_{k0}^{inA} = -m_k^A \left[\vec{\gamma}_{k0}^A + \left(\tilde{\omega}_{k0}^A \tilde{\omega}_{k0}^A + \tilde{\varepsilon}_{k0}^A \right) \vec{r}_k^{CA} \right] \; ; \; \vec{m}_{k0}^{inA} = -[m_k^A \tilde{r}_k^{CA} \vec{\gamma}_{k0}^A + \hat{J}_k^A \vec{\varepsilon}_{k0}^A + \tilde{\omega}_{k0}^A \hat{J}_k^A \vec{\omega}_{k0}^A] \tag{19}$$

are determined with respect to the centre of joint A_k. On the other hand, the wrench of two vectors $\vec{f}_k^{*A}$ and $\vec{m}_k^{*A}$ evaluates the influence of the action of the weight $m_k^A \vec{g}$ and of other external and internal forces applied to the same element T_k^A of the manipulator, for example:

$$\vec{f}_k^{*A} = m_k^A g a_{k0} \vec{u}_2 \; ; \; \vec{m}_k^{*A} = m_k^A g \tilde{r}_k^{CA} a_{k0} \vec{u}_2 \; ; \; (k = 1, \; 2) . \tag{20}$$

The fundamental principle of the virtual work states that a mechanism is under dynamic equilibrium if and only if the total virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism.

Applying *the fundamental equations of the parallel robots dynamics* (Staicu et al. [14]), compact matrix relations results for the torque of first active *revolute joint*

$$\begin{aligned} m_{10}^A = \vec{u}_3^T \{ &\vec{M}_1^A + v_{21a}^{Av} \vec{F}_2^A + v_{21a}^{Bv} \vec{F}_2^B + v_{21a}^{Cv} \vec{F}_2^C \} + v_{10a}^{Dv} \vec{F}_1^D + v_{21a}^{Dv} \vec{F}_2^D + \\ &+ v_{32a}^{Dv} \vec{F}_3^D + \omega_{10a}^{Dv} \vec{M}_1^D + \omega_{21a}^{Dv} \vec{M}_2^D + \omega_{32a}^{Dv} \vec{M}_3^D \} \end{aligned} \tag{21}$$

and for the force of first active *prismatic joint*.

$$\begin{aligned} f_{21}^A = \vec{u}_3^T \{ &\omega_{10a}^{Av} \vec{M}_1^A + \vec{F}_2^A + \omega_{21a}^{Bv} \vec{M}_2^B + \omega_{21a}^{Cv} \vec{M}_2^C \} + \\ &+ v_{10a}^{Dv} \vec{F}_1^D + v_{21a}^{Dv} \vec{F}_2^D + v_{32a}^{Dv} \vec{F}_3^D + \omega_{10a}^{Dv} \vec{M}_1^D + \omega_{21a}^{Dv} \vec{M}_2^D + \omega_{32a}^{Dv} \vec{M}_3^D . \end{aligned} \tag{22}$$

Two recursive relations generate the vectors

$$\vec{F}_k^A = \vec{F}_{k0}^A + a_{k+1,k}^T \vec{F}_{k+1}^A , \;\; \vec{M}_k^A = \vec{M}_{k0}^A + a_{k+1,k}^T \vec{M}_{k+1}^A + \tilde{r}_{k+1,k}^A a_{k+1,k}^T \vec{F}_{k+1}^A , \tag{23}$$

where one denoted

$$\vec{F}_{k0}^A = -\vec{f}_{k0}^{inA} - \vec{f}_k^{*A} , \; \vec{M}_{k0}^A = -\vec{m}_{k0}^{inA} - \vec{m}_k^{*A} \tag{24}$$

The relations (21), (22), (23) and (24) represent the *inverse dynamics model* of the 3-RPS parallel robot.

3.2. Equations of Lagrange

A solution of the dynamics problem of a 3-RPS parallel robot can be developed based on the Lagrange equations of second kind. The generalized coordinates of the robot are represented by 12 parameters

$q_1 = x_0^G, q_2 = y_0^G, q_3 = z_0^G, q_4 = \alpha_1, q_5 = \alpha_2, q_6 = \alpha_3, q_7 = \varphi_{10}^A, q_8 = \lambda_{21}^A, q_9 = \varphi_{10}^B, q_{10} = \lambda_{21}^B, q_{11} = \varphi_{10}^C, q_{12} = \lambda_{21}^C$.

The Lagrange's equations with their nine multipliers $\lambda_1, \lambda_2, \dots, \lambda_9$ will be expressed by 12 differential relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\frac{\partial L}{\partial \dot{q}_k}\} - \frac{\partial L}{\partial q_k} = Q_k + \sum_{s=1}^{9} \lambda_s c_{sk} \quad (k = 1, 2, ..., 12), \tag{25}$$

which contain following 12 generalized forces $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 0$, $Q_4 = 0$, $Q_5 = 0$, $Q_6 = 0$, $Q_7 = m_{10}^A$, $Q_8 = 0, Q_9 = m_{10}^B$, $Q_{10} = 0$, $Q_{11} = m_{10}^C$, $Q_{12} = 0$, for example. A number of nine kinematical conditions of constraint are given by the relations (14):

$$\sum_{k=1}^{12} c_{sk} \dot{q}_k = 0 \quad (s = 1,\ 2, ...,\ 9). \tag{26}$$

The components of the general expression of the Lagrange function $L = L_p + \sum_{\nu=1}^{2} (L_\nu^A + L_\nu^B + L_\nu^C)$ expressed as analytical functions of the generalized coordinates and their first derivatives with respect to time:

$$L_p = \frac{1}{2} m_p (\dot{x}_0^{G2} + \dot{y}_0^{G2} + \dot{z}_0^{G2}) + \frac{1}{2} \vec{\omega}_p^T \hat{J}_p \vec{\omega}_p - m_p g z_0^G,\ L_1^i = \frac{1}{2} \vec{\omega}_{10}^{iT} \hat{J}_1^i \vec{\omega}_{10}^i - m_1^i g \vec{u}_3^T \{\vec{r}_{10}^i + p_{10}^T \vec{r}_1^{Ci}\},$$

$$L_2^i = \frac{1}{2} m_2^i \vec{v}_{20}^{iT} \vec{v}_{20}^i + \frac{1}{2} \vec{\omega}_{20}^{iT} \hat{J}_2^i \vec{\omega}_{20}^i + m_2^i \vec{v}_{20}^{iT} \tilde{\omega}_{20}^i \vec{r}_2^{Ci} - m_2^i g \vec{u}_3^T \{\vec{r}_{10}^i + p_{10}^T \vec{r}_{21}^i + p_{20}^T \vec{r}_2^{Ci}\},\ i = (A,\ B,\ C),$$
$$p = (a,\ b,\ c). \tag{27}$$

Angular velocities, joint's velocities, skew-symmetric matrices associated to the angular velocities and first derivatives of orthogonal matrices $p_{k,k-1}$ are expressed as follows:

$$\vec{\omega}_{10}^i = \dot{\varphi}_{10}^i \vec{u}_3\ ;\ \vec{\omega}_{20}^i = p_{21} \vec{\omega}_{10}^i\ ;\ \tilde{\omega}_{10}^i = \dot{\varphi}_{10}^i \tilde{u}_3\ ;\ \tilde{\omega}_{20}^i = p_{21} \tilde{\omega}_{10}^i p_{21}^T\ ;\ \vec{v}_{10}^i = \vec{0}\ ;\ \vec{v}_{20}^i = p_{21} \tilde{\omega}_{10}^i \vec{r}_{21}^i + \dot{\lambda}_{21}^i \vec{u}_3,$$

$$\dot{p}_{k,k-1} = \dot{\varphi}_{k,k-1}^i \tilde{u}_3^T p_{k,k-1}\ ;\ \dot{p}_{k,k-1}^T = \dot{\varphi}_{k,k-1}^i p_{k,k-1}^T \tilde{u}_3\ ;\ \frac{\partial p_{k,k-1}}{\partial \varphi_{k,k-1}^i} = \tilde{u}_3^T p_{k,k-1}\ ;\ \frac{\partial p_{k,k-1}^T}{\partial \varphi_{k,k-1}^i} = p_{k,k-1}^T \tilde{u}_3, \tag{28}$$

$$i = (A, B, C)\ ;\ p = (a, b, c)$$

A long calculus of the derivatives with respect to time $\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\}$ $(k = 1,\ 2, ...,\ 12)$ of some above functions leads to a system of 12 relations. In the direct or inverse dynamics problem, after elimination of the nine multipliers, finally we obtain same expressions (21), (22) for the three input torques or input forces.

As application let us consider a manipulator which has the following characteristics

$$z_0^{G*} = 0.1\text{m},\ \alpha_1^* = \alpha_2^* = \frac{\pi}{18},\ l_0 = 1\text{m},\ r = 0.5\text{m},\ l = r\sqrt{3},\ \ h = 1.2\text{m}$$

$$\lambda_0 = 0.35\text{m},\ l_1 = 0.7\text{m},\ l_2 = \frac{h}{\cos\beta} - \lambda_0,\ m_1 = m_2 = 0.5\text{kg},\ m_p = 5\text{kg},\ \Delta t = 3s.$$

With MATLAB software, a program was developed for inverse dynamics. A comparative study in two configurations: revolute actuators (R̲PS) and prismatic actuators (RP̲S), is based on the computation of the power required by each actuator. First, the platform moves along the vertical z_0 direction. As can be seen from Fig. 3, it is proved to be true that all powers are permanently equal to one another in both actuation schemes. If the platform rotates about x_G axis, the powers required are again calculated and plotted as follows: p_{10}^A, p_{21}^A (Fig. 4), p_{10}^B, p_{21}^B (Fig. 5) and p_{10}^C, p_{21}^C (Fig. 6).

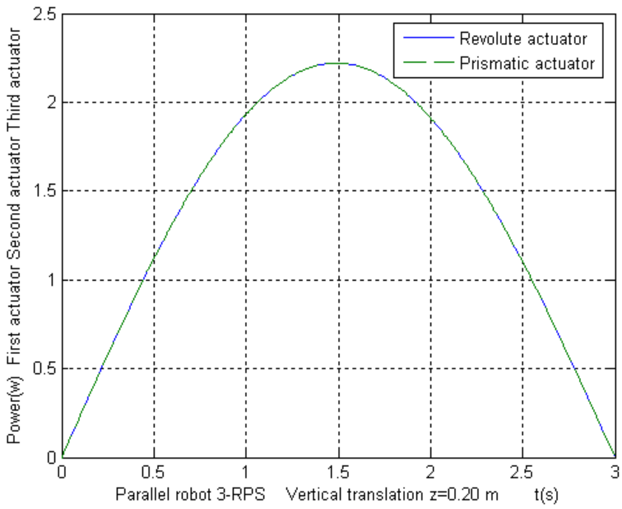

Fig. 3 – Powers of three actuators.

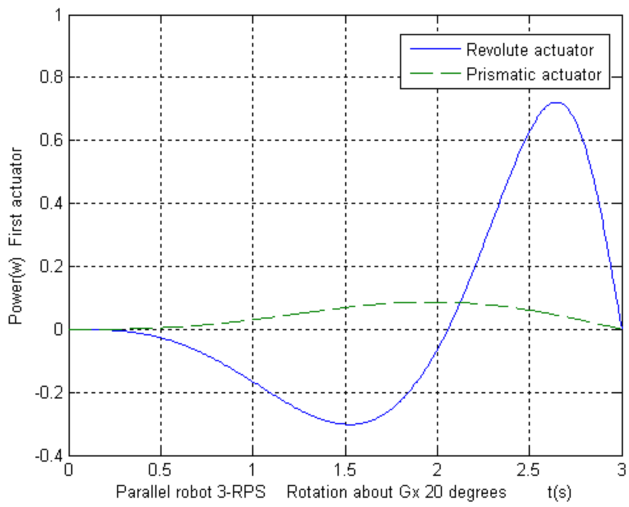

Fig. 4 – Powers p_{10}^{A}, p_{21}^{A} of first actuator.

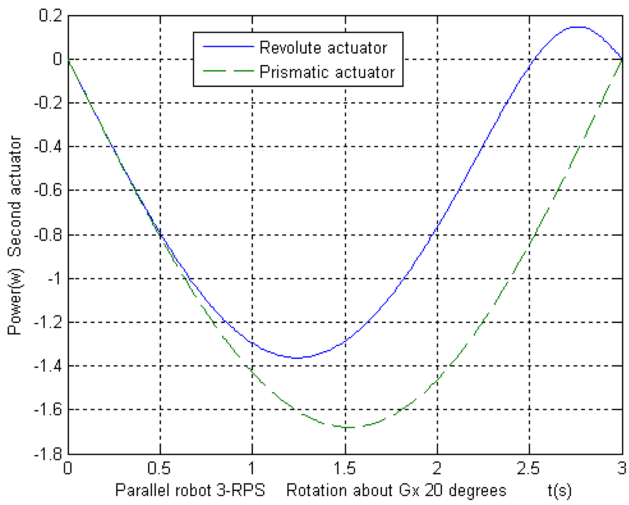

Fig. 5 – Powers p_{10}^{B}, p_{21}^{B} of second actuator.

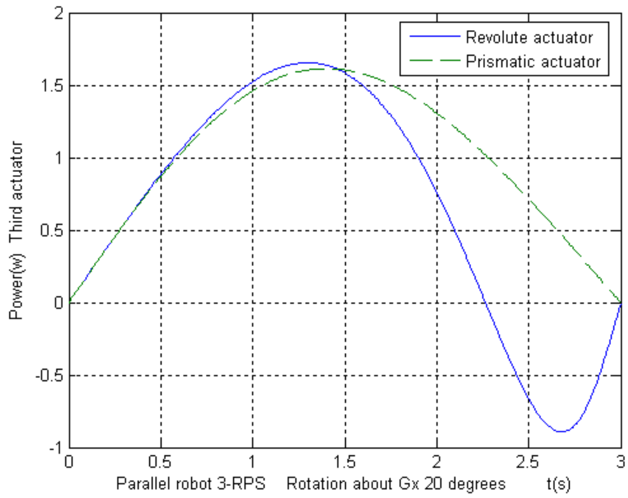

Fig. 6 – Powers p_{10}^{C}, p_{21}^{C} of third actuator.

3. CONCLUSIONS

In the inverse kinematics analysis some exact relations that give in real-time the position, velocity and acceleration of each element of the parallel robot have been established in present paper. The dynamics model takes into consideration the masses and forces of inertia introduced by all component elements of the parallel mechanism. The new approach based on the principle of virtual work can eliminate all forces of internal joints and establishes a direct determination of the time-history evolution of powers required by the actuators. Choosing appropriate serial kinematical circuits connecting many moving platforms, the present method can easily be applied in forward and inverse mechanics of various types of parallel mechanisms, complex manipulators of higher degrees of freedom and particularly *hybrid structures*, when the number of components of the mechanisms is increased.

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