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SOLITON PERTURBATION THEORY FOR THE GENERALIZED KLEIN-GORDON EQUATION WITH FULL NONLINEARITY

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This paper studies the perturbation theory of the generalized Klein-Gordon equation in presence of perturbation terms that appear with full nonlinearity. There are five forms of nonlinearity that are considered for the Klein-Gordon equation. It is observed that all five forms of nonlinearity lead to the same structure of the adiabatic dynamics of the soliton velocity.

Key words: solitons, perturbation, adiabaticity.

1. INTRODUCTION

The Klein-Gordon equation (KGE) appears in Theoretical Physics. This equation has been in existence for the past few decades [1–15]. In fact, in Quantum Mechanics there are at least a couple of versions of this equation that has been studied. They are Φ^4 model and the Φ^6 model [7]. The generalized form of the KGE with full nonlinearity has been recently integrated [10]. This paper is going to carry out the study of adiabatic dynamics of the velocity of the soliton in presence of fully nonlinear perturbation terms. The soliton perturbation theory will be adopted to carry out this study.

2. MATHEMATICAL ANALYSIS

The generalized KGE (gKGE) is modeled by the equation

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + F(q) = 0, \tag{1}$$

where the dependent variable $q(x,t)$ represents the wave profile. Also, k is a constant and m is a positive integer with $m \geq 1$. In fact, if $m = 1$, equation (1) reduces to the regular KGE. In this paper, the following five forms of the function $F(q)$ will be considered.

$$F(q) = aq^m - bq^{2m}, \tag{2}$$

$$F(q) = aq^m - bq^{3m}, \tag{3}$$

$$F(q) = aq^m - bq^n, \tag{4}$$

$$F(q) = aq^m - bq^n + cq^{2n-m}, \tag{5}$$

$$F(q) = aq^m - bq^{n-m} + cq^{n+m}. \tag{6}$$

These five cases will be respectively labeled as Forms I-V. In all of these five forms, a, b and c are real valued constants.

The KGE, given by (1), has at least two conserved quantities. They are the momentum (P) and the energy (E) that are respectively given by

$$P = -\int_{-\infty}^{\infty} \left(q^m\right)_t \left(q^m\right)_x \mathrm{d}x \tag{7}$$

and

$$E = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\left[\left(q^m\right)_t\right]^2 + \frac{1}{2}\left[\left(q^m\right)_x\right]^2 + f(q) \right\} \mathrm{d}x, \tag{8}$$

where $f(q)$ is the anti-derivative of $F(q)$ that is given by

$$f(q) = \int^{q} F(s)\mathrm{d}s. \tag{9}$$

2.1. Perturbation Terms

The perturbed generalized KGE that is going to be studied in this paper is given by

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + F(q) = \varepsilon R, \tag{10}$$

where ε is a perturbation parameter and R represents the perturbation terms. In presence of perturbation terms, the conserved quantities do not stay conserved. Instead they vary adiabatically. Thus, the adiabatic variation of the momentum (P) and energy (E) are given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \varepsilon\int_{-\infty}^{\infty} \left(q^m\right)_x R\mathrm{d}x \tag{11}$$

and

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \varepsilon\int_{-\infty}^{\infty} \left(q^m\right)_t R\mathrm{d}x. \tag{12}$$

The perturbatiion terms that are going to be considered in this paper are given by

$$R = \alpha q^m + \beta\left(q^m\right)_t + \gamma\left(q^m\right)_x + \delta\left(q^m\right)_{xt} + \lambda\left(q^m\right)_{tt} + \sigma\left(q^m\right)_{xxt} + \nu\left(q^m\right)_{xxxx}. \tag{13}$$

These perturbation terms arise in the study of long Josephson junctions. Here, in (13), α is the loss term that apperas in field theory, while β accounts for dissipative losses in Josephson junction theory due to tunneling of normal electrons across the dielectric barrier, while σ accounts for losses due to a current along the barrier. Also, the perturbation term due to γ is generated by a small inhomogeneous part of the local inductance while λ the capacity inhomogeneity. The higher order spatial dispersion term is given by ν. Here $\beta < 0$ while $\sigma > 0$.

Thus, the perturbed KGE that is going to be studied is given by

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + F(q) = \varepsilon\left\{\alpha q^m + \beta\left(q^m\right)_t + \gamma\left(q^m\right)_x + \delta\left(q^m\right)_{xt} + \lambda\left(q^m\right)_{tt} + \sigma\left(q^m\right)_{xxt} + \nu\left(q^m\right)_{xxxx}\right\}.$$

3. ADIABATIC DYNAMICS

In this section, the adiabatic dynamics of the soliton velocity will be obtained for all the five forms of nonlinearity. The soliton perturbation theory will be therefore applied to each of the five forms of nonlinearity. This study will be conducted in the following five subsections.

3.1. Form-I

In this case by virtue of equations (1) and (2), the gKGE is given by

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + aq^m - bq^{2m} = 0. \tag{14}$$

The 1-soliton solution of (15) is given by

$$q(x,t) = \frac{A}{\cosh^{\frac{2}{m}}[B(x - vt)]}, \tag{15}$$

where the amplitude A and the width B are respectively given by

$$A = \left(\frac{3a}{2b}\right)^{\frac{1}{m}} \tag{16}$$

and

$$B = \sqrt{\frac{a}{4\left(k^2 - v^2\right)}}. \tag{17}$$

For (15), the momentum (P) is given by

$$P = \frac{8\sqrt{a}A^{2m}v}{15\sqrt{k^2 - v^2}}. \tag{18}$$

The perturbed gKGE is given by

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + aq^m - bq^{2m} = \varepsilon R, \tag{19}$$

where R is in (13). The adiabatic variation of the soliton velocity, in presence of these perturbation terms is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{15\left(k^2 - v^2\right)^{\frac{3}{2}}}{8k^2\sqrt{a}A^{2m}}\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{15\varepsilon\left(k^2 - v^2\right)^{\frac{3}{2}}}{8k^2\sqrt{a}A^{2m}}\int_{-\infty}^{\infty}\left(q^m\right)_x R\mathrm{d}x. \tag{20}$$

Thus, for the perturbation terms given by (13), equation (21) reduces to

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\varepsilon\beta}{k^2}\left\{v^3 - \frac{\gamma}{\beta}v^2 + \left(\frac{5\sigma a}{7\beta} - k^2\right)v + \frac{\gamma\kappa^2}{\beta}\right\}. \tag{21}$$

Now, separating variables, (22) gives

$$t = \frac{k^2}{\varepsilon\beta}\int\frac{\mathrm{d}v}{v^3 - \frac{\gamma}{\beta}v^2 + \left(\frac{5\sigma a}{7\beta} - k^2\right)v + \frac{\gamma\kappa^2}{\beta}}. \tag{22}$$

Equation (23) will now be studied based on the structure of the roots of the cubic in v that is located in the denominator of the right hand side of (23). This is:

$$v^3 - \frac{\gamma}{\beta}v^2 + \left(\frac{5\sigma a}{7\beta} - k^2\right)v + \frac{\gamma\kappa^2}{\beta}. \tag{23}$$

Therefore there are four possible cases that can arise in this situation. They are as follows:

Case 1. Suppose the cubic given by (24) has three real distinct roots, v_1, v_2 and v_3. In this case (23) reduces to

$$\frac{\varepsilon\beta}{k^2}t = \int \frac{\mathrm{d}v}{(v - v_1)(v - v_2)(v - v_3)}, \tag{24}$$

where

$$v_1 + v_2 + v_3 = \frac{\gamma}{\beta}, \tag{25}$$

$$v_1 v_2 + v_2 v_3 + v_3 v_1 = \frac{7\sigma a}{5\beta} - k^2, \tag{26}$$

$$v_1 v_2 v_3 = -\frac{\gamma k^2}{\beta}. \tag{27}$$

Thus, (23) integrates to

$$t = \frac{k^2}{\varepsilon\beta}\frac{1}{(v_1 - v_2)(v_2 - v_3)(v_3 - v_1)}\left[(v_2 - v_3)\ln\left|\frac{u - v_1}{v - v_1}\right| + (v_3 - v_1)\ln\left|\frac{u - v_2}{v - v_2}\right| + (v_1 - v_2)\ln\left|\frac{u - v_3}{v - v_3}\right|\right],$$

where $v(t = 0) = u$ represents the initial velocity of the soliton. This is an implicit solution and it shows that the velocity will exponentially decay depending on the sign of β.

Case 2. Suppose the cubic given by (24) has one real root of multiplicity two. In that case, equation (23) can be restructured as

$$\frac{\varepsilon\beta}{k^2}t = \int \frac{\mathrm{d}v}{(v - v_1)^2(v - v_2)}, \tag{28}$$

where v_1 is the real root of multiplicity two and v_2 is the real root of multiplicity one. This leads to

$$t = \frac{k^2}{\varepsilon\beta(v_1 - v_2)}\left[\frac{1}{v_1 - v_2}\ln\left|\frac{(u - v_1)(v - v_2)}{(u - v_2)(v - v_1)}\right| + \frac{(v - u)(v + u - 2v_1)}{(u - v_1)^2(v - v_1)^2}\right]. \tag{29}$$

In this case the solution is implicit too.

Case 3. In this case, the assumption is that there is one real root of multiplicity three. Hence (23) modifies to

$$\frac{\varepsilon\beta}{k^2}t = \int \frac{\mathrm{d}v}{(v - v_1)^3}, \tag{30}$$

where v_1 is the real root of multiplicity two. Therefore this gives

$$v = v_1 + \frac{k(u - v_1)}{\sqrt{k^2 - 2\varepsilon\beta t(u - v_1)^2}}. \tag{31}$$

This is the case where the solution is explicit in v. In this case,

$$\lim_{t\to\infty} v(t) = v_1. \tag{32}$$

which shows that the limiting value of the soliton velocity is v_1 for large time t.

Case 4. For this case, the assumption is that the cubic in v has one real root and two imaginary roots. Thus, (23) reduces to

$$\frac{\varepsilon\beta}{k^2}t = \int \frac{\mathrm{d}v}{(v-v_1)\left(v^2+c^2\right)}, \tag{33}$$

where v_1 is the real root and c^2 is the product of the imaginary roots. This form integrates to

$$t = \frac{k^2\left(v_1^2+c^2\right)}{\varepsilon\beta}\left[\ln\left|\frac{u-v_1}{v-v_1}\right| - \frac{1}{2}\ln\left(\frac{u^2+c^2}{v^2+c^2}\right) - \frac{v_1}{c}\tan^{-1}\left\{\frac{c(u-v)}{uv+c^2}\right\}\right]. \tag{34}$$

Thus the adiabatic behavior of the soliton velocity will be one of the foor types depending on the values of the perturbation parameters.

3.2. Form-II

In this case, equations (1) and (3) together gives

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + aq^m - bq^{3m} = 0. \tag{35}$$

The 1-soliton solution to (37) is given by

$$q(x,t) = \frac{A}{\cosh^{\frac{1}{m}}[B(x-vt)]}, \tag{36}$$

where the amplitude and the width are respectively given by

$$A = \left(\frac{2a}{b}\right)^{\frac{1}{2m}} \tag{37}$$

and

$$B = \sqrt{\frac{a}{\left(k^2-v^2\right)}}. \tag{38}$$

For this form, the momentum is given by

$$P = \frac{2A^{2m}v\sqrt{a}}{3\left(k^2-v^2\right)^{\frac{1}{2}}}. \tag{39}$$

In this case, the perturbed gKGE is therefore given by

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + aq^m - bq^{3m} = \varepsilon R \tag{40}$$

so tht the adiabtic variation of the soliton velocity is governed by

$$\frac{dv}{dt}=\frac{3\left(k^2-v^2\right)^{\frac{3}{2}}}{2\sqrt{a}A^{2m}k^2}\frac{dP}{dt}=\frac{3\left(k^2-v^2\right)^{\frac{3}{2}}}{2\sqrt{a}A^{2m}k^2}\int_{-\infty}^{\infty}\left(q^m\right)_x R dx \tag{41}$$

For the perturbation terms given by (13), equation (43) reduces to

$$t=\frac{k^2}{\varepsilon\beta}\int\frac{dv}{v^3-\frac{\gamma}{\beta}v^2+\left(\frac{57\sigma a}{5\beta}-k^2\right)v+\frac{\gamma\kappa^2}{\beta}} \tag{42}$$

This leads to the same four cases of integration as in Form-I. The only difference is the coefficient of the v term in the cubic in v.

3.3. Form-III

Here, equations (1) and (4) together imply

$$\left(q^m\right)_{tt}-k^2\left(q^m\right)_{xx}+aq^m-bq^n=0. \tag{43}$$

This is the generalized form of the nonlinear KGE. The special case $m=1$ reduces to the first type of nonlinear KGE that was studied along with its perturbation terms [10]. In particular the case $m=1$ with $n=3$ is called the Φ^6 model that appears in solid state Physics, Condensed Matter Physics as well as Quantum Field Theory [6]. Also one restriction is that $m\neq n$. The 1-soliton solution to (45) is given by

$$q(x,t)=\frac{A}{\cosh^{\frac{2}{n-m}}[B(x-vt)]}, \tag{44}$$

where the amplitude and the inverse width are respectively given by

$$A=\left[\frac{a(n+m)}{2bm}\right]^{\frac{1}{n-m}} \tag{45}$$

and

$$B=\sqrt{\frac{a(n-m)^2}{4m^2\left(k^2-v^2\right)}}. \tag{46}$$

The momentum of the soliton is given by

$$P=\frac{2m\sqrt{a}A^{2m}v}{(n-m)^2\left(k^2-v^2\right)^{\frac{1}{2}}}\frac{\Gamma\left(\frac{2m}{n-m}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2m}{n-m}+\frac{1}{2}\right)}. \tag{47}$$

When the perturbation terms are turned on, (45) changes to

$$\left(q^m\right)_{tt}-k^2\left(q^m\right)_{xx}+aq^m-bq^n=\varepsilon R, \tag{48}$$

so that the adiabtic variation of the soliton velocity is governed by

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{(n-m)^2\left(k^2-v^2\right)^{\frac{3}{2}}}{2m\sqrt{a}A^{2m}k^2}\frac{\mathrm{d}P}{\mathrm{d}t}\frac{\Gamma\left(\frac{2m}{n-m}+\frac{1}{2}\right)}{\Gamma\left(\frac{2m}{n-m}\right)\Gamma\left(\frac{1}{2}\right)}=\frac{\varepsilon(n-m)^2\left(k^2-v^2\right)^{\frac{3}{2}}}{2m\sqrt{a}A^{2m}k^2}\frac{\Gamma\left(\frac{2m}{n-m}+\frac{1}{2}\right)}{\Gamma\left(\frac{2m}{n-m}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-\infty}^{\infty}\left(q^m\right)_x R\mathrm{d}x. \quad (49)$$

For the perturbation terms given by (13), equation (51) integrates to

$$t=\frac{k^2}{\varepsilon\beta}\int\frac{dv}{v^3-\frac{\gamma}{\beta}v^2+\left\{\frac{4\sigma am(m+2n)(n-m)}{\beta(3n+m)}-k^2\right\}v+\frac{\gamma\kappa^2}{\beta}}, \quad (50)$$

which leads to the same analysis and conclusions as in the previous two forms of nonlinearity.

3.4. Form-IV

In this case, equations (1) and (5) together implies

$$\left(q^m\right)_{tt}-k^2\left(q^m\right)_{xx}+aq^m-bq^n+cq^{2n-m}=0 \quad (51)$$

In this case, the 1-soliton solution is given by [10]

$$q(x,t)=\frac{A}{\left(D+\cosh[B(x-vt)]\right)^{\frac{1}{n-m}}}, \quad (52)$$

where the amplitude (A) and the inverse width (B) are respectively given by

$$A=\left[\frac{am(n+m)D}{b}\right]^{\frac{1}{n-m}}, \quad (53)$$

$$B=(n-m)\sqrt{\frac{a}{\left(k^2-v^2\right)}}, \quad (54)$$

while the constant D is

$$D=\frac{b\sqrt{n}}{\sqrt{nb^2-acm(m+n)^2}}. \quad (55)$$

The momentum of the soliton is given by

$$P=\frac{m^2A^{2m}v\sqrt{a}}{2^{\frac{2m}{n-m}}(n-m)\sqrt{v^2-k^2}}F\left(\frac{2n}{n-m},\frac{2m}{n-m},\frac{2m}{n-m}+\frac{3}{2};\frac{1-D}{2}\right)B\left(\frac{2m}{n-m},\frac{3}{2}\right), \quad (56)$$

where in (58), $F(u,v;w;z)$ is the Gauss' hypergeometric function defined as

$$F(u,v;w;z)=\frac{\Gamma(w)}{\Gamma(u)\Gamma(v)}\sum_{n=0}^{\infty}\frac{\Gamma(u+n)\Gamma(v+n)}{\Gamma(w+n)}\frac{z^n}{n!} \quad (57)$$

and $B(u,v)$ is the beta function and $\Gamma(u)$ is the Euler's gamma function. It is clear from (58) that it is necessary to have

$$m\neq n \quad (58)$$

for the existence of solitons. With the perturbation terms are turned on, (53) changes to

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + aq^m - bq^n + cq^{2n-m} = \varepsilon R, \tag{59}$$

so that the adiabtic variation of the soliton velocity is governed by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{2^{\frac{2m}{n-m}}\left(k^2-v^2\right)^{\frac{3}{2}}(n-m)}{m^2A^{2m}k^2\sqrt{a}M_1}\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{2^{\frac{2m}{n-m}}\left(k^2-v^2\right)^{\frac{3}{2}}(n-m)}{m^2A^{2m}k^2\sqrt{a}M_1}\int_{-\infty}^{\infty}\left(q^m\right)_x R\mathrm{d}x. \tag{60}$$

This leads to the adiabatic variation of the soliton velocity being given by

$$t = \frac{k^2}{\varepsilon\beta}\int\frac{\mathrm{d}v}{v^3 - \frac{\gamma}{\beta}v^2 + \left(J-k^2\right)v + \frac{\gamma\kappa^2}{\beta}} \tag{61}$$

where

$$J = \frac{\sigma a}{\beta}\left\{(n-m)^2 + \frac{n^2M_2}{M_1} + \frac{(n-m)M_3}{4M_1} - \frac{2n(n-m)M_4}{M_1} - \frac{n(n-m)M_5}{2M_1}\right\}, \tag{62}$$

with

$$M_1 = F\left(\frac{2n}{n-m}, \frac{2m}{n-m}, \frac{2m}{n-m}+\frac{3}{2}; \frac{1-D}{2}\right)B\left(\frac{2m}{n-m}, \frac{3}{2}\right), \tag{63}$$

$$M_2 = F\left(\frac{4n-2m}{n-m}, \frac{2m}{n-m}, \frac{2m}{n-m}+\frac{5}{2}; \frac{1-D}{2}\right)B\left(\frac{2m}{n-m}, \frac{3}{2}\right), \tag{64}$$

$$M_3 = F\left(\frac{2n}{n-m}, \frac{2n}{n-m}, \frac{2m}{n-m}+\frac{5}{2}; \frac{1-D}{2}\right)B\left(\frac{2n}{n-m}, \frac{5}{2}\right), \tag{65}$$

$$M_4 = F\left(\frac{3n-m}{n-m}, \frac{2m}{n-m}, \frac{2m}{n-m}+\frac{5}{2}; \frac{1-D}{2}\right)B\left(\frac{2m}{n-m}, \frac{5}{2}\right), \tag{66}$$

and

$$M_5 = F\left(\frac{3n-m}{n-m}, \frac{n+m}{n-m}, \frac{2m}{n-m}+\frac{5}{2}; \frac{1-D}{2}\right)B\left(\frac{n+m}{n-m}, \frac{3}{2}\right). \tag{67}$$

Thus, (63) leads to the same adiabatic dynamics of the soliton velocity through those same four cases.

3.5. Form-V

In this case, equations (1) and (6) together imply that the generalized KGE is

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + aq^m - bq^{m-n} + cq^{n+m} = 0. \tag{68}$$

The topological 1-soliton solution to (70) is given by

$$q(x,t) = A\tanh^{\frac{2}{n}}[B(x-vt)], \tag{69}$$

where the free parameters A and B are given by

$$A = \left[\frac{4mb}{(2m-n)a}\right]^{\frac{1}{n}} \tag{70}$$

and

$$B = \sqrt{\frac{an^2}{8m^2\left(v^2 - k^2\right)}}. \tag{71}$$

In addition, the constraint relation between the nonlinear coefficients a, b, c and the exponents m and n given by

$$4m^2\left(a^2 + 4bc\right) = n^2 a^2 \tag{72}$$

must hold in order for the solitons to exist. The momentum of the soliton is given by

$$P = \frac{8m^2 v A^{2m} B}{(4m-n)(4m+n)}. \tag{73}$$

Equations (72) and (75) imply that

$$n \neq \pm 4m. \tag{74}$$

With the perturbation terms, equation (70) modifies to

$$\left(q^m\right)_{tt} - k^2\left(q^m\right)_{xx} + aq^m - bq^{m-n} + cq^{n+m} = \varepsilon R. \tag{75}$$

Thus the adiabatic variation of the soliton velocity in this case is given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\varepsilon\beta}{k^2}\left[v^3 - \frac{\gamma}{\beta}v^2 + \left\{\frac{8\sigma an^2\left(3n^2 - 6mn + 2m^2\right)}{\beta\left(16m^2 - 9n^2\right)} - k^2\right\}v + \frac{\gamma\kappa^2}{\beta}\right]. \tag{76}$$

With the perturbation terms given by (13), the law of adiabatic variation of the soliton velocity takes the form

$$t = \frac{k^2}{\varepsilon\beta}\int \frac{\mathrm{d}v}{v^3 - \frac{\gamma}{\beta}v^2 + \left\{\frac{8\sigma an^2\left(3n^2 - 6mn + 2m^2\right)}{\beta\left(16m^2 - 9n^2\right)} - k^2\right\}v + \frac{\gamma\kappa^2}{\beta}}, \tag{77}$$

which again leads to the four cases as observed in the first form of nonlinearity. It needs to be noted that equations (78) and (79) imply that

$$3n \neq \pm 4m \tag{78}$$

for the soliton perturbation theory to be valid, in this case.

4. CONCLUSIONS

This paper studied the soliton perturbation theory of the gKGE in presence of fully nonlinear perturbation terms. The key observation is that the structure of the adiabatic dynamics of the soliton velocity stays the same as that of the regular KGE with its perturbation terms that was studied in 2009 [9]. This situation is similar to the topological solitons that are observed in the case of sine-Gordon equation and its generalization to full nonlinearity [6, 8]. This is a very important observation for the case of KGE that is being made for the first time in this paper. In future, the quasi-stationary soliton solution of the gKGE will be obtained by the method of multiple scale perturbation theory. Those results will be reported in future.

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