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UNSTEADY POISEUILLE FLOW OF SECOND GRADE FLUID IN A TUBE OF ELLIPTICAL CROSS SECTION

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The unsteady flow of an incompressible second grade fluid in an infinitely long tube of elliptical cross-section is considered under constant pressure gradient. The resulting governing equation is a time dependent PDE which is solved for exact solution using separation of variables method.

Key words: Second-grade fluid, Elliptic cross-section, Exact solution, Velocity profile, Separation of variables.

1. INTRODUCTION

Many industrial materials including clay coatings, drilling muds, suspensions, certain oils and greases, polymer melts, elastomers and many emulsions have been categorized as non-Newtonian fluids. Newtonian fluids can be described by a single model but it is difficult to suggest a single model which exhibits all properties of non-Newtonian fluids. The classification of non-Newtonian fluids has remained much confused. However, non-Newtonian fluids may be classified as: (i) fluids for which the shear stress depends on the shear rate (ii) fluids for which the relation between the shear stress and shear rate depends on time; (iii) fluids which possess both elastic and viscous properties called visco-elastic fluids or elastico-viscous fluids[1]. To recommend a single constitutive equation for use in the cases described in (i), (ii) and (iii) it does not seem possible because of great diversity in the physical structure of non-Newtonian fluids. Therefore, many constitutive equations for non-Newtonian fluids have been proposed. Most of them are empirical or semi-empirical.

Although many constitutive equations have been suggested, many questions are still unsolved. Some of the continuum models do not give satisfactory results in accordance with the available experimental data. Therefore, in many practical applications, empirical or semi-empirical equations have been used. The constitutive equation of a second grade fluid is a linear relation between the stress and the first Rivlin-Ericksen tensor, the square of the first Rivlin-Ericksen tensor and the second Rivlin-Ericksen tensor [5]. The constitutive equation has three coefficients. There are some restrictions on these coefficients due to the Clausius-Duhem inequality and due to assumption that the Helmholtz free energy is minimum in equilibrium. A comprehensive discussion on the restrictions for these coefficients has been given by Dunn and Fosdick [6], and Dunn and Rajagopal [7]. The equation of motion of incompressible second grade fluids is of higher order than the Navier-Stokes equation. The Navier-Stokes equation is a second order partial differential equation, but the equation of motion of a second grade fluid is a third-order partial differential equation. A marked difference between the case of the Navier-Stokes theory and that for fluids of second grade is that ignoring the non-linearity in Navier-Stokes does not lower the order of the equation, however, ignoring the higher order non-linearities in the case of the second grade fluids, reduces the order of the equation. Exact solutions are very important for many reasons. They provide a standard for checking the accuracies of many approximate methods such as numerical and empirical. Although computer techniques make the complete numerical integration of the non-linear equations feasible, the accuracy of the results can be established by a comparison with an exact solution. Many attempts to collect the exact solution of the non-linear equations for unsteady flow of second grade fluid have been done by different researcher for different geometries. The only comprehensive review is that due to [8,9]. However, there are many new exact solutions which have been published in journals or in review articles. Two recent reviews, one for unsteady flows [10-12] and the other for steady flows of non-Newtonian fluids have been published by [13].

An exact solution is defined as a solution of the non-linear governing equations and the continuity equation. An exact solution may be in a closed form, in a series form or in a form expressed by a numerical method. Most of the exact solutions for unsteady flows are in series forms. They may be slowly convergent or rapidly convergent. However,it is possible to replace a rapidly convergent series with a slowly convergent one. Then this provides very important facilities. Steady or unsteady flows, one can be obtained as a result of several effects. This can be some kind of motion of the boundaries, application of a body force, wall that applies a tangential stress on the fluid or application of a pressure gradient. One or two of these effects can be applied together to the fluid. If the fluid is initially at rest, the motion of the fluid may eventually become steady or remain unsteady. This fact depends on the boundary condition and the kind of effects exerted on the fluid to set it in motion. In this paper, the flow is obtained as a result of sudden application of a pressure gradient, therefore this is time-dependent problem. The examination of this flows is not only done to characterize it but also to follow its development in time.

2. GOVERNING EQUATIONS

The basic equations governing the flow of an incompressible second grade fluid in the absence of body forces and thermal effects are

$$\mathrm{div}\mathbf{V} = 0, \tag{1}$$

$$\rho\dot{\mathbf{V}} = -\nabla p + \mathrm{div}\tau, \tag{2}$$

where ρ is the constant density, $\mathbf{V}$ is the velocity vector, p is the pressure, τ is the stress tensor, $\dot{\mathbf{V}}$ denotes the material derivative. Assuming that the flow is unsteady and two dimensional, we seek the velocity profile of the form

$$\mathbf{V} = (u(y,z,t),0,0). \tag{3}$$

The stress tensor τ defining a second grade fluid is given by [7–9]

$$\boldsymbol{\tau} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$

where μ is the coefficient of viscosity and α_1, and α_2 are material constants associated with the non-linear terms. The Rivlin-Erickson tensor, A_n, are defined as:

$$A_0 = I\text{-the identity tensor, and}$$

$$A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}(\nabla V) + (\nabla V)^t A_{n-1,} \quad n \geq 1.$$

For unsteady two dimensional flow of a second grade fluid, equation (2) in components form yield:
x-component:

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \alpha_1\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right); \tag{4}$$

y-component:

$$0 = -\frac{\partial p}{\partial y} + (2\alpha_1 + \alpha_2)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\right); \tag{5}$$

z-component:

$$0 = -\frac{\partial p}{\partial z} + (2\alpha_1 + \alpha_2)\frac{\partial}{\partial y}(\frac{\partial u}{\partial u}\frac{\partial u}{\partial z}) + \frac{\partial}{\partial z}(\frac{\partial u}{\partial z})^2 = 0. \tag{6}$$

The Clausius-Duhem inequality and the condition that the Helmholtz free energy is minimum in equilibrium provide the following restrictions [1, 2]:

$$\mu \ge 0, \quad \alpha_1 \ge 0 \text{ and } \quad (2\alpha_1 + \alpha_2) = 0. \tag{7}$$

A comprehensive discussion on the restrictions for μ , α_1 and α_2 can be found in the work by Dunn and Rajagopal [1]. The sign of the material moduli α_1 and α_2 is the subject of much controversy [2–4]. Making use of equation (7) in equations (5) and (6) we get

$$\frac{\partial p}{\partial y} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} = 0$$

showing that $p = p(x)$, Therefore equations(4), (5) and (6) reduces to single equation, i.e.,

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\ (\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + \beta\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}). \tag{8}$$

where $\beta = \frac{\alpha_1}{\rho}$.

3. PROBLEM FORMULATION

Consider the flow of an incompressible, isothermal second grade fluid in an infinitely long tube, under constant pressure gradient $\frac{\partial p}{\partial x}$ and negligible gravity. The tube has an elliptical cross-section with semi-axes *a* and *b* (Fig. 1). The flow is considered to be unsteady, and two dimensional. Accordingly the flow velocity u has one non-vanishing component u_x, which depends on the coordinates y and z given in equation(8). Boundary conditions require that the flow velocity vanishes at the wall of the tube, i.e. on the ellipse

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

and that the gradient of the velocity vanishes at the centre of the tube, $y = z = 0$.

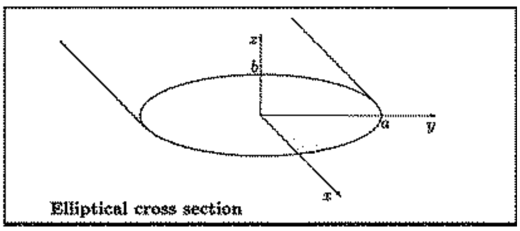

Fig. 1 – Two-dimensional Poiseuille flow in a tube of elliptical cross-section.

4. SOLUTION OF THE PROBLEM

Erdogan has presented the unsteady flows of an incompressible viscous fluid in rectangular and circular cross-sections. In this paper we have solved unsteady two dimensional flow problem exactly using separation of variables [15]. We have converted the unsteady problem given in quation (8) into steady and transient problems using following transformation

$$u(y,z,t) = f(y,z) + g(y,z,t). \tag{9}$$

Steady problem is given by

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{10}$$

we solve the steady problem by assuming $f(y,z)$ of the following form [14]

$$f(y,z) = k\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right) \tag{11}$$

using equation (11) in equation (10), we find that

$$f(y,z) = -\frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{a^2b^2}{a^2+b^2}\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right). \tag{12}$$

The unsteady part is given by

$$\frac{\partial g}{\partial t} = \nu\left(\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \beta\frac{\partial}{\partial t}\left(\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) \tag{13}$$

subject to following boundary and initial conditions

$$g(a,z,t) = 0,\ g(y,b,t) = 0,\ g(y,z,0) = f(y,z),\ \frac{\partial g}{\partial y}(0,z,t) = 0,\ \frac{\partial g}{\partial z}(y,0,t) = 0 \tag{14}$$

we will solve the above IBVP using separation of variables method and assuming following in equations (13) and (14)

$$g(y,z,t) = Y(y)Z(z)T(t). \tag{3}$$

The resulting system of differential equations is

$$\begin{aligned} &Y'' + J^2 Y = 0, && Y'(0) = Y(a) = 0,\\ &Z'' + L^2 Z = 0, && Z'(0) = Z(b) = 0,\\ &T' + \left(\frac{k^2}{1+\beta k^2}\right)T = 0, && T(0) = -f(y,z). \end{aligned} \tag{16}$$

The solutions obtained for differential equations in (16) are

$$\begin{aligned} Y_m &= B_m \cos\frac{(2m+1)\pi y}{2a},\ m = 0,1,2,....,\\ Z_n &= D_n \cos\frac{(2n+1)\pi z}{2b},\ n = 0,1,2,....,\\ T_{mn} &= \exp\left[-\frac{\left(\frac{(2m+1)\pi}{2a}\right)^2 + \left(\frac{(2n+1)\pi}{2b}\right)^2}{1+\beta\left(\left(\frac{(2m+1)\pi}{2b}\right)^2 + \left(\frac{(2n+1)\pi}{2b}\right)^2\right)}t\right]. \end{aligned} \tag{17}$$

The solution of the unsteady problem is given by

$$g(y,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{mn}\cos\frac{(2m+1)\pi y}{2a}\cos\frac{(2n+1)\pi z}{2b}\exp\left[-\frac{\left(\frac{(2m+1)\pi}{2a}\right)^2 + \left(\frac{(2n+1)\pi}{2b}\right)^2}{1+\beta\left(\left(\frac{(2m+1)\pi}{2b}\right)^2 + \left(\frac{(2n+1)\pi}{2b}\right)^2\right)}t\right], \tag{18}$$

where

$$D_{mn} = \frac{4}{ab}\int_0^a\int_0^b - f(y,z)\cos\frac{(2m+1)\pi y}{2a}\cos\frac{(2n+1)\pi z}{2b}\ \mathrm{d}y\mathrm{d}z \tag{19}$$

or

$$D_{mn} = -\frac{8}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{a^2b^2}{a^2+b^2}\frac{(-1)^{m+n}}{(2m+1)(2n+1)\pi^2}\left(-1+\frac{8}{(2m+1)^2\pi^2}+\frac{8}{(2n+1)^2\pi^2}\right) \tag{20}$$

and the complete velocity distribution is given by

$$u(y,z,t) = -\frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{a^2b^2}{a^2+b^2}(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{mn}\cos\frac{(2m+1)\pi y}{2a}\cos\frac{(2n+1)\pi z}{2b}\cdot$$
$$\cdot\exp\left[-\frac{(\frac{(2m+1)\pi}{2a})^2+(\frac{(2n+1)\pi}{2b})^2}{1+\beta((\frac{(2m+1)\pi}{2a})^2+(\frac{(2n+1)\pi}{2b})^2)}t\right], \tag{21}$$

where D_{mn} is given by equation 20.

5. CONCLUSION

In this paper, a problem is studied in order to show the effect of the applied pressure gradient in a channel of elliptical cross-section on unsteady flow of a fluid of second grade. Exact solution is obtained using separation of variables.

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