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A NEW PROOF FOR THE COMPLETE POSITIVITY OF THE BOOLEAN PRODUCT OF COMPLETELY POSITIVE MAPS BETWEEN C*-ALGEBRAS

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We give a new proof concerning the complete positivity of the Boolean product of contractive, completely positive maps between C*-algebras (see [9]). This is inspired by a technique due to M. Bozejko, M. Leinert and R. Speicher from [6] and uses an extension of a W.L.Paschke and E. Stormer's criterion for the positivity of a matrix over a C*-algebra[13] [18].

Key words: Complete positivity, Stinespring dilation, GNS representation, Universal free product (*-,C*-)algebra, Boolean product of linear maps.

1. INTRODUCTION

The Boolean product of linear functionals on algebras is defined on the associated universal free product algebra without unit, and on involutive algebras it preserves the positivity (see, e.g., [15]).

This product originates in M. Bozejko's investigations on positive definite functions on the free group(see, e.g., [5]) and corresponds to the Boolean independence and partial cumulants going back to W. von Waldenfels' work on the pressure broadening of spectral lines (see, e.g., [20]).

This Boolean product and the involved independence are fundamental in the so-called Boolean quantum probability theory and related topics (see, e.g., [17, 2, 1, 3, 8, 10]). This theory is one of the three noncommutative probability theories (the other being R.L.Hudson's Boson or Fermion probability theory and D.V. Voiculescu's free probability theory) issued from an associative product which does not depend on the order of its factors and fulfills a universal rule for mixed moments (according to R. Speicher's answer[16] to M. Schürmann's conjecture [15] on the universal products of *-algebraic probability spaces).

In [9], we considered the Boolean product for linear maps between algebras and showed, by a direct proof, it preserves the complete positivity in C*-algebraic setting.

In this Note, we give a new proof of the same fact, inspired from that in [6] concerning the positivity of the conditionally free product of positive linear functionals on involutive algebras.

2. COMPLETE POSITIVITY, UNIVERSAL FREE PRODUCT (*-)ALGEBRAS, AND BOOLEAN PRODUCT OF LINEAR MAPS

Let A be a (complex) *-algebra (i.e., a complex algebra endowed with a conjugate linear involution *, which is an anti-isomorphism). We denote by A_+ the set of positive elements in A; i.e. of finite sums $\sum a_i * a_i$, with $a_i \in A$. For any positive integer n, let $M_n(A)$ be the *-algebra of $n \times n$ matrices $\left[a_{ij} \right]$ with entries from *A*.

In particular, if A is a C*-algebra, A_+ determines an order structure on the real linear subspace of self-adjoint elements in A, and $M_n(A)$ becomes a C*-algebra.

The next criterion of positivity for a matrix over a C*-algebra is due to W.L. Paschke and E. Stormer (see Prop. 6.1 in [13], and Th.2.2 in [18]).

PROPOSITION 2.1. *Let B be a C*-algebra and* $[b_{ij}]_{i,j=\overline{1,n}} \in M_n(B)$ **.** *Then*

$$[b_{ij}]_{i,j=\overline{1,n}} \in M_n(B)_+ \text{ if and only if } \sum_{i,j=1}^{n} b_i^* b_{ij} b_j \in B_+, \text{for all } b ,..., b_n \in B.$$

Let B be another *-algebra and $Q : A \longrightarrow B$ be a map. We say Q is positive if $Q(A_+) \subset B_+$.

For any positive integer n, let $Q_n : M_n(A) \longrightarrow M_n(B)$ be the inflation map given by $Q_n([a_{ij}]) = [Q(a_{ij})]$, for $\left[a_{ij}\right] \in M_n(A)$. Then Q is called *n*-positive if the map Q_n induced by Q is positive. The map Q is completely positive if it is *n*-positive, for all positive integer n.

The famous Stinespring dilation theorem characterizes the unital completely positive maps defined on unital C*-algebras as dilations in the sense of Gelfand-Naimark (see, e.g., [14]); and extends the GNS theorem concerning the (positive functionals, i.e.,) states on this kind of *-algebras.

THEOREM 2.2. *Let A be a unital C*-algebra, L(H) be the bounded linear operators on a Hilbert space H, and* $Q : A \longrightarrow L(H)$ *be a unital completely positive map. Then there exists a unique (up to a unitary equivalence) Stinespring dilation* (K, π) *of* Q*, where* $K \supset H$ *is a Hilbert space, and* $\pi : A \longrightarrow L(K)$ *is a unital *-representation, such that* $Q(a) = P_H^K \pi(a)|H$ *, for* $a \in A$*; and* $\overline{sp}\, \pi(A)H = K$.

The following fact was established by M.-D. Choi and E.G. Effros via a C. Lance's non-unital version of Stinespring dilation theorem (see Lemma 3.9 in [7]).

LEMMA 2.3. *Let A and B be C*-algebras, and* $Q : A \longrightarrow B$ *be a contractive, completely positive map. Let* $\widetilde{A}$ *and* $\widetilde{B}$ *be the unitizations of A and B , and* $\widetilde{Q} : \widetilde{A} \longrightarrow \widetilde{B}$ *be the unitization of* Q*. Then* $\widetilde{Q}$ *is completely positive*.

The universal free product (*-, C*-)algebra is the coproduct in the category of (complex) (*-, C*-) algebras, non necessary unital [4, 6, 9, 10, 11, 15, 19].

We denote by $*_0 A_i$, the non-unital universal (or *full*) free product C*-algebra corresponding to a family $(A_i)_{i \in I}$ of C*-algebras.

As linear space, a realization of the universal free product associated to a family of (*-)algebras $(A_i)_{i \in I}$ is

$$A = \oplus_{n \ge 1} \oplus_{i_1 \ne \dots \ne i_n} A_{i_1} \otimes \ldots \otimes A_{i_n} .$$

By natural operations, A is organized as a (*-)algebra.

In particular, if A_i are C*-algebras, A satisfies the Combes axiom, i.e. for every $a \in A$, there exists a scalar $\lambda(a) > 0$ with $x^* a * ax \le \lambda(a) x^* x$, for all $x \in A$

After separation and completion of the corresponding universal free product *-algebra A in its enveloping C*-seminorm

$$\|a\| = \sup\{\|\pi(a)\|; \pi \text{ *-representation of } A \text{ asboundedoperators on a Hilbert space}\}$$

one can realize the universal (or full) free product $*_0 A_i$ in the category of C*-algebras.

Let B and A_i be (complex) algebras, and $Q_i : A_i \longrightarrow B$ be linear maps; $i \in I$.

In [9], we considered the Boolean product $Q = \bullet Q_i$ as the unique linear map defined on the universal free product A of the algebras A_i, $i \in I$, such that

$$Q(a_1 ... a_n) = Q_{i_1}(a_1) ... Q_{i_n}(a_n) ,$$

for all $n \ge 1$, with $i_1 \ne ... \ne i_n$, and $a_k \in A_{i_k}$, if $k = 1,...,n$; with respect to the natural embeddings of A_i into A arising from the free product construction; and we directly proved the following result.

THEOREM 2.4. *Let A_i and B be C*-algebras, and $Q_i : A_i \longrightarrow B$ be contractive,completely positive maps; $i \in I$. Then the Boolean product $Q = \bullet Q_i$ defined on the *-algebraic free product of the algebras A_i, $i \in I$, is contractive, with respect to the enveloping C*-seminorm, and completely positive.*

Therefore, Q extends to a unique map $Q : *_0 A_i \longrightarrow B$ which is contractive and completely positive.

3. THE NEW PROOF OF THE THEOREM 2.4

To this aim, we establish the lemma below; its proof is inspired from that of Th.2.2 in [6] concerning the positivity of the conditionally free product φ, of unital positive linear functionals on unital involutive algebras; where one shows that $\varphi(a^* a) \geq |\varphi(a)|^2$, for all a in the unital *-algebraic free product .

LEMMA 3.1. *Let A_i and B be C*-algebras, and $Q_i : A_i \longrightarrow B$ be contractive, completely positive maps; $i \in I$.Let A be the *-algebraic free product of $(A_i)_{i\in I}$ and $Q : A \longrightarrow B$* be *the Boolean product of $(Q_i)_{i\in I}$. Then*

$$[Q(a_i^* a_j)]_{i,j=\overline{1,n}} \geq [Q(a_i)^* Q(a_j)]_{i,j=\overline{1,n}}$$

in $M_n(B)$, for all $n \geq 1$, and all $a_1,...,a_n \in A$. *In particular, Q is completely positive.*

Proof. Denote by $W=\{a_1...a_n; n \geq 1, a_k \in A_{i_k}, i_1 \neq ... \neq i_n\}$ the set of reduced words in A. For $w = a_1...a_n \in W$, call *n* the length of *w* and a_1 *the first letter of w.* If $x = \sum_k w^{(k)} \in A$ call the length of *x* the maximal length in this representation of *x*.

Denote by $\widetilde{A}_i$ and $\widetilde{B}$ the unitizations of A_i and *B* (by **C**, the field of complex numbers), and by $\tilde{Q}_i : \tilde{A}_i \longrightarrow \tilde{B}$ the unitization of Q_i defined by $\widetilde{Q}_i(a \oplus \lambda 1) = Q_i(a) \oplus \lambda 1$ ($a \in A_i, \lambda \in \mathbf{C}$). The complete positivity and the contractivity of Q_i imply the complete positivity of $\tilde{Q}_i$, due to Choi-Effros' Lemma 2.3.

The first step in proving Lemma 3.1 is the next Lemma 3.2.

LEMMA 3.2. *The Boolean product $Q = \bullet Q_i$ is a Schwarz map, i.e.,*

$$Q(a^* a) \geq Q(a)^* Q(a), \textit{ for all } a \in A.$$

In particular, Q is a positive map.

Proof. By the very definition of the Boolean product, it suffices to prove the asserted inequality for every random variable $x(i)$ in A represented as $\sum_k w^{(k)}$ with $w^{(k)} \in W$ having the first letter in a same A_i ; if $i \in I$.

Suppose that such a word $x(i)$ has p terms of length one**;** else, the argument is similar**.**

Therefore, $x(i) = \sum_{k=1}^{p} a^{(k)} + \sum_{k=p+1}^{N} a^{(k)} y^{(k)}$, with $a^{(k)} \in A_i$; and $y^{(k)} \in W$**,** but the first letter of $y^{(k)}$ does not belong to A_i .

Let $b_k = 1 \in \widetilde{B}$**,** for $k = \overline{1,p}$; and $b_k = Q(y^{(k)})$, for $k = \overline{p+1,N}$**.**

Note that

$$Q(x(i)^* x(i))) - Q(x(i))^* Q(x(i)) = \sum_{k,l=1}^{N} b_k^* b_{kl} b_l \ ,$$

where

$$b_{kl} := Q_i(a^{(k)} * a^{(l)}) - Q_i(a^{(k)}) * Q_i(a^{(l)}) .$$

Then, it is enough to observe that the $(N+1)$-positivity of $\tilde{Q}_i$ implies the positivity of $[b_{kl}]_{k,l}$ in $M_N(B)$; because $\sum_{k,l=1}^{N} b_k^* b_{kl} b_l \in B_+$; due to the Paschke- Stormer criterion for the positivity of a matrix in $M_N(B)$ (i.e, the above Prop. 2.1). □

COROLLARY 3.3. *The Boolean product of contractive linear functionals defined on C*-algebras preserves the positivity.*

In view of the previous Lemma 3.2, suppose in the sequel that $n \geq 2$.

The very definition of the Boolean product assures us it suffices to prove the inequality asserted in Lemma 3.1 only for all words $x_s(i)$ in A, $s = 1,...,n$, represented as $\sum_k w^{(k)}$ with $w^{(k)} \in W$ having the first letter in a same A_i ; if $i \in I$; such that these representations contain the same number of terms, for all $s = 1,...,n$.

Suppose that every such $x_s(i)$ with $i = \overline{1,p}$ has the length 1; every such $x_s(i)$ with $i = \overline{p+1,n}$ has the length greater or equal to 2, and p_s terms of length 1; else, the argument is similar.

Therefore, $x_s(i) = \sum_{k=1}^{N} a_s^{(k)}$ for $s = \overline{1,p}$; and $x_s(i) = \sum_{k=1}^{p_s} a_s^{(k)} + \sum_{k=p_s+1}^{N} a_s^{(k)} y_s^k$, for $s = \overline{p+1,n}$; with some $a_s^{(k)} \in A_i$; and $y_s^{(k)} \in W$, but the first letter of $y_s^{(k)}$ does not belong to A_i .

Denote $b_s^{(k)} = 1 \in \tilde{B}$, for all $k = \overline{1,N}$, if $s = \overline{1,p}$; but $b_s^{(k)} = 1 \in \tilde{B}$, for $k = \overline{1,p_s}$, and $b_s^{(k)} = Q(y_s^{(k)})$, for $k = \overline{p_s+1,N}$, if $s = \overline{p+1,n}$.

Then, note that we may express

$$[Q(x_s(i) * x_t(i))]_{s,t} - [Q(x_s(i)) * Q(x_t(i))]_{s,t} = [\sum_{k,l=1}^{N} b_s^{(k)} * b_{s,t}(k,l) b_t^{(l)}]_{s,t=\overline{1,n}} \text{, in } M_n(B),$$

where

$$b_{s,t}(k,l) := Q_i(a_s^{(k)} * a_t^{(l)}) - Q_i(a_s^{(k)}) * Q_i(a_t^{(l)}) \text{ for } k,l = \overline{1,N} \text{, and } s,t = \overline{1,n} .$$

In the same way as before, we may deduce that $[b_{s,t}(k,l)]_{s,t=\overline{1,n}}^{k,l=\overline{1,N}}$ belongs to $M_{nN}(B)_+$, by the $(nN+1)$-positivity of $\tilde{Q}_i$. Thus, the positivity of $\left[\sum_{k,l=1}^{N} b_s^{(k)} * b_{s,t}(k,l) b_t^{(l)}\right]_{s,t=\overline{1,n}}$ in $M_n(B)$ is a consequence of one inference in the following extension of the mentioned Paschke-Stormer positivity criterion in Prop 2.1 to *block-matrices* over a C*-algebra.

LEMMA 3.4. *Let n, N be positive integers*, **B** *be an arbitrary C*-algebra*; $T_{ij} \in M_N(B)$, for $i,j = 1,2,...,n$; and $b_1, b_2, ..., b_n \in B^{\oplus N}$. *For* $i,j = 1,2,...,n$, *define* $S_{ij} := T_{ij}$, if $i \geq j$, *and* $S_{ij} := S_{ji}{}^*$, if $i < j$; *and, also* $s_{ij} := b_i{}^* S_{ij} b_j$, if $i \geq j$, *and* $s_{ij} := s_{ji}{}^*$, if $i < j$.

Then $[S_{ij}]_{i,j} \in M_{nN}(B)_+ \iff [s_{ij}]_{i,j} \in M_n(B)_+$.

Proof. By the GNS representation, we may assume that $B \subseteq L(H)$, with a Hilbert space H. Thus, the data in the Lemma 3.4 appear as bounded operators $b_1, b_2, ..., b_n \in L(H, H^{\oplus N})$, $S \in L(H^{\oplus nN})$, and $s \in L(H^{\oplus n})$.

Notice this identity (involving the operators s, S and the corresponding scalar products) holds:

$< sh, h >=< Sbh, bh >$ for all $h := h_1 \oplus \cdots \oplus h_n \in H^{\oplus n}$; denoting here $bh := b_1 h_1 \oplus \cdots \oplus b_n h_n \in H^{\oplus nN}$.

Consequently, one inference in Lemma 3.4 became trivial.

Modulo a decomposition of H into a direct sum of orthogonal subspaces, it is enough to suppose for the rest, that B has a cyclic vector $\xi \in H$.

Let $f := f_1 \oplus \cdots \oplus f_n \in H^{\oplus nN}$. Therefore, we find a sequence $b_m := b_m^1 \oplus \cdots \oplus b_m^n \in B^{\oplus nN}$, $m \geq 1$, such that $b_m \xi := b_m^1 \xi \oplus \cdots \oplus b_m^n \xi$ converges to f in $H^{\oplus nN}$.

The conclusion results via the identity before and the continuity of both S and the scalar product on $H^{\oplus nN}$.

These facts finish the proof of Lemma 3.1.

It was remarked by F. Boca in [4] that the classical Stinespring dilation Theorem 2.2 is still true for unital completely positive maps on unital *-algebras verifying the Combes axiom.

Let $\widetilde{A}$ be the unitization of A , and $\widetilde{Q} : \widetilde{A} \longrightarrow \widetilde{B}$ be the unitization of Q . Remark that the inequality in Lemma 3.1 is equivalent to the same kind of inequality corresponding to $\widetilde{Q}$.

Therefore, the Theorem 2.4 follows from Lemma 3.1 via a Stinespring construction. □

In the same way, one can prove that the amalgamated Boolean (or, moreover, conditionally free) product of linear maps preserves the complete positivity in C*-algebraic setting (for these and other extensions see, e.g., [10, 11, 12]).

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