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MODULE DERIVATIONS INTO ITERATED DUALS OF BANACH ALGEBRAS

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In this paper, we define the n-weak module amenability for a Banach algebra A which is a Banach module over another Banach algebra Θ with compatible actions, and find the relation between n-weak module amenability of A and n-weak amenability A/J, where J is the closed ideal of A generated by elements of the form $(a\cdot\alpha)b-a(\alpha\cdot b)$ for $a\in\mathsf{A}$ and $\alpha\in\Theta$. As a consequence we prove that $\ell^1(S)$ is ($2n+1$)-weakly module amenable as an $\ell^1(E)$-module, for each $n\in\mathsf{N}$, where S is an inverse semigroup with the set of idempotents E.

Key words: Banach module, Module derivation, n-weak module amenability, Inverse semigroup.

1. INTRODUCTION

The concept of n-weak amenability was introduced by Dales, Ghahramani and Grønbæk in [7]. A Banach algebra A is *n-weakly amenable* if every bounded derivation from A into $\mathsf{A}^{(n)}$, nth dual space of A is inner. A 1-weakly amenable Banach algebra is called weakly amenable. Johnson in [10] showed that the group algebra $L^1(G)$ is amenable if and only if G is amenable, and so for any amenable group G, $L^1(G)$ is n-weakly amenable for each natural number n. Also he showed in [11] that for any locally compact group G, the group algebra $L^1(G)$ is weakly amenable. Later in [12] he proved that $L^1(G)$ is n-weakly amenable for each $n\in\mathsf{N}$ whenever G is a free group. The same is true for an arbitrary locally compact group G when n is odd [7, Theorem 4.1]. Mewomo in [13] investigated the n-weak amenability of semigroup algebras and showed that for a Rees matrix semigroup S, $\ell^1(S)$ is n-weakly amenable when n is odd. He obtained a similar result for a regular semigroup S with finitely many idempotents [13, Theorem 4.5].

The notion of weak module amenability of a Banach algebra A which is a Banach module over another Banach algebra Θ with compatible actions is defined in [4] and studied in [2]. The main result of [4] is that $\ell^1(S)$ is weakly module amenable, as an $\ell^1(E)$-module, when S is commutative. The definition of weak module amenability is modified in [2] and the above result is proved for an arbitrary inverse semigroup (with trivial left action). One should note that the ideal J in [2] is defined to be the closed ideal of A generated by elements of the form $(a\cdot\alpha)b-a(\alpha\cdot b)$, for $a\in\mathsf{A}$ and $\alpha\in\Theta$, whereas in [4] it was defined as the closed ideal of A generated by elements of the form $\alpha\cdot(ab)-(ab)\cdot\alpha$, for $a\in\mathsf{A}$ and $\alpha\in\Theta$. These two ideals happen to be the same for the algebra $\ell^1(S)$ with the corresponding actions of $\ell^1(E)$ in [2] and [4], but the definition in [2] (adapted here) has the advantage that J is also a Banach A-submodule of A.

In this paper, we define the n-weak module amenability for Banach algebras and investigate this concept for inverse semigroup algebra $\ell^1(S)$. The motivation of writing this paper is Theorem 3.15 (main result) which show that the inverse semigroup algebra $\ell^1(S)$ is always n-weakly module amenable as an $\ell^1(E)$-module when n is odd.

2. NOTATIONAS AND PRELIMINARY RESULTS

Throughout this paper, A and Θ are Banach algebras such that A is a Banach Θ-bimodule with compatible actions, that is $\alpha\cdot(ab)=(\alpha\cdot a)b,(ab)\cdot\alpha=a(b\cdot\alpha)$ for all $a,b\in\mathsf{A}$, and $\alpha\in\Theta$. Let X be a Banach A-bimodule and a Banach Θ-bimodule with compatible actions, that is

$$\alpha\cdot(a\cdot x)=(\alpha\cdot a)\cdot x, a\cdot(\alpha\cdot x)=(a\cdot\alpha)\cdot x,(\alpha\cdot x)\cdot a=\alpha\cdot(x\cdot a)\quad(a\in\mathsf{A},\alpha\in\Theta,x\in X)$$

and similarly for the right or two-sided actions. Then we say that X is a Banach A-Θ-module. Moreover, if $\alpha\cdot x=x\cdot\alpha$ for all $\alpha\in\Theta, x\in X$ then X is called a *commutative* A-Θ-module. Note that when A acts on itself by algebra multiplication, it is not in general a Banach A-Θ-module, as we have not assumed the compatibility condition $a\cdot(\alpha\cdot b)=(a\cdot\alpha)\cdot b$ for all $a,b\in\mathsf{A}$ and $\alpha\in\Theta$. If A is a commutative Θ-module and acts on itself by multiplication from both sides, then it is also a Banach A-Θ-module.

Let X' be the conjugate (dual) space of X consisting of all bounded linear functionals on X. We set $X''=(X')'$ and regard X as a subspace of X'' by the canonical embedding. If X is a Banach A-Θ-module, then so are X' and X'' by the usual actions. For $n\in\mathsf{Z}^+$, we define $X^{(n)}$ inductively by $X^{(n)}=(X^{(n-1)})'$, then $X^{(n)}$ is a Banach A-Θ-module. In particular, if A is a Banach Θ-module with compatible actions, then so are the iterated dual spaces $\mathsf{A}^{(n)}$. If moreover A is a commutative Θ-module, then $\mathsf{A}^{(n)}$ is a commutative A-Θ-module.

Let X and Y be A-Θ-modules, then a left A-Θ-module homomorphism from X to Y is a norm-continuous map $T:X\to Y$ with $T(x\pm y)=T(x)\pm T(y)$ and $T(\alpha\cdot x)=\alpha\cdot T(x), T(a\cdot x)=a\cdot T(x),$ for all $x,y\in X$, $a\in\mathsf{A}$ and $\alpha\in\Theta$. A right or two-sided A-Θ-module homomorphism is defined similarly. For $n\in\mathsf{Z}^+$, the canonical embedding $X^{(n)}\to X^{(n+2)};x\mapsto\hat{x}$ and the projection $P:X^{(n+2)}\to X^{(n)};\Phi\mapsto\overline{\Phi}$ are A-Θ-module homomorphisms, where $\overline{\Phi}$ is the restriction of Φ to $X^{(n-1)}$.

Let E be a linear space and let F be a family of w^*-continuous functionals on E. The weak topology generated the family F is denoted by $\sigma(E,\mathsf{F})$, in particular we denote $\sigma(\mathsf{A}'',\mathsf{A}')$ by σ. Let ∇ and $\lozenge$ be the first and second Arens products on the second dual space A'', respectively. Then

$$F\nabla G=\lim_j\lim_k a_jb_k \quad\text{in } (\mathsf{A}'',\sigma),\qquad F\lozenge G=\lim_k\lim_j a_jb_k\quad\text{in } (\mathsf{A}'',\sigma),$$

where $F=\lim_j\hat{a}_j$ and $G=\lim_k\hat{b}_k$ in (A'',σ). When these two products coincide on A'', we say that A is *Arens regular* (for more details refer to [6]). All over this paper, A'' and Θ'' are Banach algebras with the first Arens product.

Let X be a Banach A-Θ-module. We define actions A'' and Θ'' on X'' such that X'' is a Banach A''-Θ''-module with the compatible actions. For $\Phi\in X''$ and $\lambda\in X'$, define $\Phi\cdot\lambda\in\Theta'$ by $\langle\Phi\cdot\lambda,\alpha\rangle=\langle\Phi,\lambda\cdot\alpha\rangle$, where $\alpha\in\Theta$. For $\Im\in\Theta''$ and $\Phi\in X''$, define $\Im\cdot\Phi,\Phi\cdot\Im\in X''$ by $\langle\Im\cdot\Phi,\lambda\rangle=\langle\Im,\Phi\cdot\lambda\rangle$ and $\langle\Phi\cdot\Im,\lambda\rangle=\langle\Phi,\Im\cdot\lambda\rangle;(\lambda\in X')$. Finally, for $\Im\in\Theta'',\lambda\in X'$ and $x\in X$, define $\lambda\cdot x\in\Theta'$ and $\Im\cdot\lambda\in X'$ by

$$\langle\lambda\cdot x,\alpha\rangle=\langle\lambda,x\cdot\alpha\rangle,\quad\langle\Im\cdot\lambda,x\rangle=\langle\Im,\lambda\cdot x\rangle,\quad(\lambda\in X',\alpha\in\Theta).$$

If in the above actions we replace Θ by A, we obtain the actions of A'' on X''. Similarly, A'' is Θ''-bimodule. Let $\Im\in\Theta''$, $G\in\mathsf{A}''$, and $\Phi\in X''$. Take the nets $(\alpha_j)\subset\Theta$, $(a_k)\subset\mathsf{A}$, and $(x_l)\subset X$ such that $\hat{\alpha}_j\xrightarrow{w^*}\Im$, $\hat{a}_k\xrightarrow{w^*}G$, and $\hat{x}_l\xrightarrow{w^*}\Phi$. It follows from the above discussion that module actions Θ'' on A'' are

$$\Im\cdot G=\lim_j\lim_k\alpha_j\cdot a_k,\quad G\cdot\Im=\lim_k\lim_j a_k\cdot\alpha_j\quad\text{in } (\mathsf{A}'',\sigma),$$

and X'' with the respect to the module actions

$$\Im\cdot\Phi=\lim_j\lim_l \alpha_j\cdot x_l,\quad \Phi\cdot\Im=\lim_l\lim_j x_l\cdot\alpha_j \quad \text{in } (X'',\sigma(X'',X')),$$

is a Θ''-bimodule. Also, we have

$$G\cdot\Phi=\lim_k\lim_l a_k\cdot x_l,\quad \Phi\cdot G=\lim_l\lim_k x_l\cdot a_k \quad \text{in } (X'',\sigma(X'',X')).$$

Take $G,H\in \mathsf{A}''$, $\Im\in\Theta''$. Then $\Im\cdot(G\nabla H)=(\Im\cdot G)\nabla H$ and $(G\nabla H)\cdot\Im=G\nabla(H\cdot\Im)$. Therefore A'' is a Banach Θ''-bimodule with compatible actions. Also, for each $\Im\in\Theta''$, $G\in\mathsf{A}''$, and $\Phi\in X''$, we have $G\cdot(\Im\cdot\Phi)=(G\cdot\Im)\cdot\Phi, \Im\cdot(G\cdot\Phi)=(\Im\cdot G)\cdot\Phi$ and $\Im\cdot(\Phi\cdot G)=(\Im\cdot\Phi)\cdot G$. Hence, X'' is a Banach A''-Θ''-module with the left compatible actions and similarly for the right or two-sided actions. Therefore we have the following result:

PROPOSITION 2.1. *If X is a Banach A-Θ-module with compatible actions, then X'' is a Banach A''-Θ''-module.*

A bounded map $D:\mathsf{A}\to X$ is called a Θ-*module derivation* if $D(\alpha\cdot a)=\alpha\cdot D(a), D(a\cdot\alpha)=D(a)\cdot\alpha$, and $D(a\pm b)=D(a)\pm D(b), D(ab)=D(a)\cdot b+a\cdot D(b)$, for all $\alpha\in\Theta$ and $a,b\in\mathsf{A}$. If X is a commutative A-Θ-module, then each $x\in X$ defines a module derivation $D_x(a)=a\cdot x-x\cdot a$ for all $a\in\Theta$. These are called the *inner* module derivations. The Banach algebra A is called *module amenable* (as an Θ-module) if for any commutative Banach A-Θ-module X, each Θ-module derivation $D:\mathsf{A}\to X'$ is inner [1]. We use the notations $Z_\Theta(\mathsf{A},X')$ for the set of all Θ-module derivations $D:\mathsf{A}\to X'$, $B_\Theta(\mathsf{A},X')$ for those which are inner, and $H_\Theta(\mathsf{A},X')$ for the quotient group $Z_\Theta(\mathsf{A},X')/B_\Theta(\mathsf{A},X')$. Therefore A is Θ-module amenable if and only if $H_\Theta(\mathsf{A},X')=\{0\}$, for each commutative Banach A-Θ-module X.

Consider the module projective tensor product $\mathsf{A}\hat{\otimes}_\Theta\mathsf{A}$ which is isomorphic to the quotient space $(\mathsf{A}\hat{\otimes}_\Theta\mathsf{A})/I$, where I be the closed ideal of the projective tensor product $\mathsf{A}\hat{\otimes}\mathsf{A}$ generated by elements of the form $a\cdot\alpha\otimes b-a\otimes\alpha\cdot b$ for $\alpha\in\Theta, a,b\in\mathsf{A}$. Also consider the closed ideal J of A generated by elements of the form $(a\cdot\alpha)b-a(\alpha\cdot b)$ for $\alpha\in\Theta,a,b\in\mathsf{A}$. Then I and J are A-submodules and Θ-submodules of $\mathsf{A}\hat{\otimes}\mathsf{A}$ and A, respectively, and the quotients $\mathsf{A}\hat{\otimes}_\Theta\mathsf{A}$ and A/J are A-modules and Θ-modules. Also, A/J is an Banach A-Θ-modules with the compatible actions when A acts on A/J canonically.

3. N-WEAK MODULE AMENABILITY OF BANACH ALGEBRAS

Let X be a normed space, let $\mathsf{Y}\le\mathsf{X}$ and $\mathsf{Z}\le\mathsf{X}'$ be subspaces of X and X', respectively. The annihilators $\mathsf{Y}^\perp$ of Y and ${}^\perp\mathsf{Z}$ of $\mathsf{Z}\le\mathsf{X}'$ are defined by $\mathsf{Y}^\perp=\{f\in\mathsf{X}':\langle f,x\rangle=0 \quad (x\in\mathsf{Y})\}$ and ${}^\perp\mathsf{Z}=\{x\in\mathsf{X}:\langle f,x\rangle=0 \quad (f\in\mathsf{Z})\}$. Then $\mathsf{Y}^\perp$ and ${}^\perp\mathsf{Z}$ are closed subspaces of X' and X, respectively. Also ${}^\perp(\mathsf{Y}^\perp)$ is the norm closure Y in X, and $({}^\perp\mathsf{Z})^\perp$ is the $\sigma(\mathsf{X}',\mathsf{X})$- closure of Z in X'.

Let A and Θ be as in the previous section and X be a Banach A-Θ-module. Let I and J be the corresponding closed ideals of $\mathsf{A}\hat{\otimes}\mathsf{A}$ and A, respectively. For $Y\le\mathsf{A}^{(n)}$ and $n\ge 0$, define $Y^{(n\perp)}$ by induction: $Y^{(0\perp)}=Y\le\mathsf{A}$, $Y^{(1\perp)}=Y^\perp\le\mathsf{A}'$, and $Y^{(n\perp)}=(Y^{((n-2)\perp)})^{\perp\perp}\le(\mathsf{A}^{(n-2)})''=\mathsf{A}^{(n)}$. It is well-known that $(\mathsf{A}/J)^{(2n)}=\mathsf{A}^{(2n)}/J^{(2n\perp)}$ and $(\mathsf{A}/J)^{(2n-1)}=J^{((2n-1)\perp)}$.

In the following lemma, we show that J is indeed equal to the closed subspace generated by elements of the form $(a\cdot\alpha)b-a(\alpha\cdot b)$.

LEMMA 3.1. *With the above notation,* $J=\overline{span}\{(a\cdot\alpha)b-a(\alpha\cdot b):\alpha\in\Theta,a,b\in\mathsf{A}\}$.

Proof. Let $\mathrm{J}=\{(a\cdot\alpha)b-a(\alpha\cdot b):\alpha\in\Theta,a,b\in\mathsf{A}\}$, then J is the closed ideal generated by J. But $\overline{span}(\mathrm{J})$ is a closed ideal of A containing the generators of J, hence $J\subseteq\overline{span}(\mathrm{J})$. The reverse inclusion is trivial.

Definition 3.2. Let $\mathcal{A}$ be a Banach algebra, $n \in \mathbb{N}$. Then $\mathcal{A}$ is called n- weakly module amenable (as an Θ-module) if $(\mathcal{A}/J)^{(n)}$ is a commutative $\mathcal{A}$-Θ-module, and each module derivation $D: \mathcal{A} \to (\mathcal{A}/J)^{(n)}$ is inner; that is, $D(a) = a \cdot y - y \cdot a =: D_y(a)$ for some $y \in (\mathcal{A}/J)^{(n)}$ and each $a \in A$. A Also $\mathcal{A}$ is called permanently weakly module amenable if $\mathcal{A}$ is n-weakly module amenable for each $n \in \mathbb{N}$.

When $\Theta = \mathbb{C}$ (the set of complex numbers), we have $J = \{0\}$ and $\mathcal{A}^{(n)}$ is automatically a commutative $\mathcal{A}$-$\mathbb{C}$-module. Also derivations and module derivations are the same, hence n- weak module amenability is the same as n-weak amenability for $\mathcal{A}$. In general, it follows from the above definition that every module amenable Banach algebra is permanently weakly module amenable. Also we have the following simple result.

PROPOSITION 3.3. *Let $\mathcal{A}$ be a Banach algebra, $n \in \mathbb{N}$. Suppose that $\mathcal{A}$ is $(n+2)$-weakly module amenable. Then $\mathcal{A}$ is n-weakly module amenable.*

LEMMA 3.4. *For each odd $n \geq 1$, if $Y \leq \mathcal{A}^{(n)}$ is a commutative $\mathcal{A}$-Θ-module, then $Y \leq J^{(n\perp)}$.*

Proof. We argue by induction on odd n's. For $n = 1$, from the fact that Y is a commutative $\mathcal{A}$-submodule of $\mathcal{A}^{(n)}$,

$$\begin{aligned} 0 &= \langle y \cdot \alpha - \alpha \cdot y, ab \rangle = \langle y \cdot \alpha, ab \rangle - \langle \alpha \cdot y, ab \rangle = \langle b \cdot (y \cdot \alpha), a \rangle - \langle (\alpha \cdot y) \cdot a, b \rangle \\ &= \langle (b \cdot y) \cdot \alpha, a \rangle - \langle \alpha \cdot (y \cdot a), b \rangle = \langle \alpha \cdot (b \cdot y), a \rangle - \langle (y \cdot a).\alpha, b \rangle = \langle b \cdot y, a \cdot \alpha \rangle - \langle y \cdot a, \alpha \cdot b \rangle \\ &= \langle y, (a \cdot \alpha) b \rangle - \langle y, a(\alpha \cdot b) \rangle = \langle y, (a \cdot \alpha) b - a(\alpha \cdot b) \rangle, \end{aligned}$$

for each $\alpha \in \Theta, a, b \in \mathcal{A}, y \in Y$. It follows from the above lemma and the linearity and continuity of y as a functional on $\mathcal{A}$ that $Y \leq J^{\perp}$. Next if $n \geq 3$ is odd and $Y \leq \mathcal{A}^{(n)}$ is a commutative $\mathcal{A}$-Θ-module, then $\langle y, \alpha \cdot a \rangle = \langle y \cdot \alpha, a \rangle = 0,$ for each $\alpha \in \Theta, a \in^{\perp} Y, y \in Y,$ hence $\Theta \cdot^{\perp} Y \subseteq^{\perp} Y,$ and the same for the right action. Thus $^{\perp}Y \leq A^{(n-1)}$ is an $\mathcal{A}$-Θ-module. On the other hand, $\langle y, b \cdot \alpha - \alpha \cdot b \rangle = \langle \alpha \cdot y - y \cdot \alpha, b \rangle = 0,$ for each $\alpha \in \Theta, y \in Y, b \in \mathcal{A}^{(n-3)},$ we have $b \cdot \alpha - \alpha \cdot b \in^{\perp} Y.$ Hence for each $\alpha \in \Theta, z \in^{\perp\perp} Y,$ and $b \in \mathcal{A}^{(n-3)}$ $\langle z \cdot \alpha - \alpha \cdot z, b \rangle = \langle z, \alpha \cdot b - b \cdot \alpha \rangle = 0.$ Hence $z \cdot \alpha - \alpha \cdot z = 0,$ thus $^{\perp\perp}Y \leq \mathcal{A}^{(n-2)}$ is a commutative $\mathcal{A}$-Θ-module, and by the induction hypothesis, $^{\perp\perp}Y \leq J^{((n-2)\perp)}.$ Therefore $Y \leq (^{\perp\perp}Y)^{\perp\perp} \leq (J^{((n-2)\perp)})^{\perp\perp} = J^{(n\perp)}.$ •

PROPOSITION 3.5. *Let $\mathcal{A}$ be a Banach algebra. If $(\mathcal{A}/J)^{(n)}$ is a commutative $\mathcal{A}$-Θ-module and both $\mathcal{A}$ and Θ have bounded approximate identities then n-weak amenability of $\mathcal{A}$ implies its n-weak module amenability, for each odd n.*

Proof. Let $D: \mathcal{A} \to (\mathcal{A}/J)^{(n)}$ be a bounded Θ-module derivation and $\{\alpha_j\}$ be a bounded approximate identity for Θ. By Cohen factorization theorem, for each $a \in \mathcal{A}$ there are $\beta, \gamma \in \Theta$, $b \in \mathcal{A}$, and $\Phi \in \mathcal{A}^{(n)}$ such that $a = \beta \cdot b$, and $D(a) = \gamma \cdot \Phi$. Then for each $a \in \mathcal{A}$ and $\mu \in \mathbb{C}$ we have

$$\begin{aligned} D(\mu a) &= D(\mu(\beta \cdot b)) = \lim_j D(\mu(\alpha_j \beta) \cdot b) = \lim_j D(\mu \alpha_j \cdot a) \\ &= \lim_j \mu \alpha_j \cdot D(a) = \lim_j \mu \alpha_j \cdot (\gamma \cdot \Phi) = \mu(\gamma \cdot \Phi) = \mu D(a). \end{aligned}$$

Therefore D is linear. Since n is odd, $(\mathcal{A}/J)^{(n)} = J^{(n\perp)}$ could be considered as a submodule of $\mathcal{A}^{(n)}$, and $D: \mathcal{A} \to \mathcal{A}^{(n)}$ is a bounded derivation. By assumption, there is $y \in \mathcal{A}^{(n)}$ such that $D = D_y$. We claim that $y \in J^{(n\perp)}$. Since D is a module derivation, $(\alpha \cdot y - y \cdot \alpha) \cdot a = \alpha \cdot D(a) - D(\alpha \cdot a) = 0,$ for each $\alpha \in \Theta, a \in \mathcal{A}$. Since $\mathcal{A}$ has a bounded approximate identity, by Cohen factorization theorem, $\alpha \cdot y - y \cdot \alpha = x \cdot b$, for some $b \in \mathcal{A}$ and $x \in \mathcal{A}'$. By the above equality, $x \cdot ba = (x \cdot b) \cdot a = 0$, for each $a \in \mathcal{A}$. Again, since $\mathcal{A}$ has a bounded approximate identity, the above equality implies that $x.b = 0$, that is $\alpha \cdot y = y \cdot \alpha$, for each $\alpha \in \Theta$. An argument similar to the first display in the proof of Lemma 3.4 shows that $y \in J^{\perp} \leq J^{(n\perp)}$, proving the claim. Therefore D is inner as a module derivation into $(\mathcal{A}/J)^{(n)}$.

As we will see later in the last section, there is a Banach algebra which is n-weakly module amenable for $n \in \mathsf{N}$, but it is not weakly amenable, so the converse of the above Proposition is false.

PROPOSITION 3.6. *Let X be a Banach A-Θ-module and $D: \mathsf{A} \to X$ be a Θ-module derivation. Then the second transpose $D'': \mathsf{A}'' \to X''$ of D is a Θ''-module derivation.*

Proof. Obviously, D'' is bounded. For each $\Im \in \Theta''$ and $G \in \mathsf{A}''$, we can obtain $D''(G \cdot \Im) = D''(G) \cdot \Im$ and $D''(\Im \cdot G) = \Im \cdot D''(G)$. Now it follows from the proof of [7, Proposition 1.17] that $D''(F \nabla G) = D''(F) \cdot G + F \cdot D''(G)$, for all $F, G \in \mathsf{A}''$. This completes the proof.

We modify the definition of module Arens regularity [15] to avoid complications such as those discussed in [16]. Here we do not require the maps $R_\lambda : \mathsf{A} \to \mathsf{A}$ to be Θ-module homomorphism. The results of [15] after correction [16] remain valid with this new definition.

Definition 3.7. A is called module Arens regular (as an Θ-module) if Θ-module homomorphisms $R_\lambda : \mathsf{A} \to \mathsf{A}; a \mapsto a \cdot \lambda$ are weakly compact for any $\lambda \in J^\perp$.

Recall that $\sigma(\mathsf{A}'', J^\perp)$ is the weak topology generated the family $J^\perp$ of w^*-continuous functionals on A''. For $n \in \mathsf{N}$, we consider the Arens products ∇ and $\lozenge$ on Banach algebra $\mathsf{B} = \mathsf{A}^{(2n)}$. On the other hand, B is module Arens regular if and only if the map $\mathsf{B}'' \to \mathsf{B}''; G \mapsto F \nabla G$ is $\sigma(\mathsf{B}'', J_{\mathsf{B}}^\perp)$-continuous for any $F \in \mathsf{B}'' = \mathsf{A}^{(2n+2)}$ (see [15, Theorem 2.3]), where J_{B} is the corresponding ideal of B. In the following results we consider the first Arens product for $\mathsf{A}^{(2n)}$ for all $n \in \mathsf{N}$.

LEMMA 3.8. *Let Y be a Banach A-Θ-module. Then the map $P: Y^{(4)} \to Y''$ is a left A''-Θ''-module. Also P is a A''-Θ''-module map when the map $T: \mathsf{A}'' \to Y''; F \mapsto \Phi \cdot F$ is w^*-w^*-continuous for each fixed $\Phi \in Y''$. In particular, if A is module Arens regular then the map $P: \mathsf{A}^{(4)} \to \mathsf{A}''$ is a A''-Θ''-module homomorphism.*

Proof. We can show that $P(F \cdot \Upsilon) = F \cdot P(\Upsilon)$ and $P(\Im \cdot \Upsilon) = \Im \cdot P(\Upsilon)$, for all $\Upsilon \in X^{(4)}$, $F \in \mathsf{A}''$ and $\Im \in \Theta''$. Since T is w^*-w^*-continuous, we conclude that $P(\Upsilon \cdot \Im) = P(\Upsilon) \cdot \Im$ and $P(\Upsilon \cdot F) = P(\Upsilon) \cdot F$. When A is module Arens regular, then the map $G \mapsto F \nabla G$ is $\sigma(\mathsf{A}'', J^\perp)$-continuous, for any $F \in \mathsf{A}''$ (see [15, Theorem 2.3]) and the above argument applies to $P: \mathsf{A}^{(4)} \to \mathsf{A}''$.

PROPOSITION 3.9. *Let A be a Banach Θ-module, and $Y = (\mathsf{A}/J)^{(2n)}$, and $D: \mathsf{A} \to Y$ be a Θ-module derivation.* If $\mathsf{A}^{(2n-2)}$ *is module Arens regular, then there is a Θ''-module derivation $\widetilde{D}: \mathsf{A}^{(2n)} \to Y$ such that $\widetilde{D}(\widetilde{a}) = D(a)$ for all $a \in \mathsf{A}$, where $\widetilde{a}$ is the canonical image a in $\mathsf{A}^{(2n)}$.*

Proof. It follows from Proposition 3.6 that $D'': \mathsf{A}'' \to Y'' = (\mathsf{A}''/J^{\perp\perp})^{(2n+2)}$ is a Θ''-module derivation. Since $\mathsf{B} = \mathsf{A}^{(2n-2)}$ is module Arens regular, the projection $P: \mathsf{B}^{(4)} \to \mathsf{B}''$ is a B''-Θ''-module homomorphism by Lemma 3.8. Let $F \in \mathsf{A}''$ and $\Phi \in \mathsf{B}^{(4)}$, we have $P(F \cdot \Phi) = F \cdot P(\Phi)$, therefore P is an A''-Θ''-module homomorphism. For $n = 1$, we get $\widetilde{D} = P \circ D''$ by Lemma 3.8. Now the result follows by a straightforward induction argument.

COROLLARY 3.10. *Let A be a Banach Θ-module and $Y = (\mathsf{A}/J)''$ be a commutative A-Θ-module. If A is module Arens regular and $H_{\Theta''}(\mathsf{A}'', Y) = \{0\}$, then A is 2-weakly module amenable.*

Proof. Assume that $D: \mathsf{A} \to Y$ is a A-module derivation. By Proposition 3.9, there exists $\widetilde{D} \in Z_{\Theta''}(\mathsf{A}, Y)$ such that $D(\hat{a}) = D(a)$ for all $a \in \mathsf{A}$. By assumption, there exists $F \in Y$ such that $\widetilde{D}(G) = F \nabla G - G \nabla F, (G \in \mathsf{A}'')$. Hence, for each $a \in \mathsf{A}$, $D(a) = \widetilde{D}(a) = F \cdot a - a \cdot F$. Therefore, A is 2-weakly module amenable.

COROLLARY 3.11. *Let A be a Banach commutative Θ-module. If for $n \in \mathsf{N} \cup \{0\}$, $\mathsf{A}^{(2n)}$ is module Arens regular and $H_\Theta(\mathsf{A}^{(2n+2)}, \mathsf{A}^{(2n+2)}) = \{0\}$, then A is $(2n)$-weakly module amenable for all $n \in \mathsf{N}$.*

Proof. By the hypothesis and Corollary 3.10 and A and A'' are 2-weakly module amenable. Let $m \in \mathsf{N}$ and let A be $(2m)$-weakly module amenable. We show that A is $(2m+2)$-weakly module amenable. Since A'' satisfies the same properties as A, we may suppose A'' is $(2m)$-weakly module amenable. Put $\mathsf{B} = \mathsf{A}^{(2m)}$ and assume that $D : \mathsf{A} \to \mathsf{B}''$ is a Θ-module derivation. Hence $D'' : \mathsf{A}'' \to \mathsf{B}^{(4)}$ is a Θ''-module derivation by Proposition 3.6. Also by Lemma 3.8 the projection $P : \mathsf{B}^{(4)} \to \mathsf{B}''$ is a B''-Θ''-module morphism. Since $\widetilde{D} = P \circ D'' \in Z_{\Theta''}(\mathsf{A}'', \mathsf{B}'') = Z_{\Theta''}(\mathsf{A}'', (\mathsf{A}'')^{(2m)})$, there exists $\Psi \in \mathsf{B}''$ such that $\widetilde{D}(F) = F \cdot \Psi - \Psi \cdot F \quad (F \in \mathsf{A}'')$, and so $D : \mathsf{A} \to \mathsf{B}''$ is inner. •

PROPOSITION 3.12. Let A be a Banach Θ-module and $Y = (\mathsf{A}/J)^{(n)}$ be a commutative A-Θ-module. Then $H_{\Theta}(\mathsf{A}, Y) = \{0\} \Leftrightarrow H_{\Theta}(\mathsf{A}/J, Y) = \{0\}$.

Proof. Suppose that $H_{\Theta}(\mathsf{A}/J, Y) = \{0\}$. The Θ-commutativity of Y and the compatibility of actions of A and Θ on n-th predual of Y show that $J \cdot Y = Y \cdot J = 0$. Hence, the following module actions are well-defined

$$(a+J)\cdot y := a \cdot y, \quad y \cdot (a+J) := y \cdot a \quad (y \in Y, a \in \mathsf{A}),$$

therefore Y is a Banach A/J-module. Assume that $D : \mathsf{A} \to Y$ is a module derivation. Define $\widetilde{D} : \mathsf{A}/J \to Y$ via $\widetilde{D}(a+J) = D(a)$. It is easy to check that $D((a\cdot\alpha)b - a(\alpha\cdot b)) = 0$ for all $\alpha \in \mathsf{A}$ and $a, b \in \mathsf{A}$. On the other hand, Since J is a closed ideal, the restriction of D to J is zero. Therefore $\widetilde{D}$ is well-defined. By hypothesis, $H_{\Theta}(\mathsf{A}, Y) = \{0\}$. Conversely, if $D : \mathsf{A}/J \to Y$ is a module derivation, then the derivation $: \widetilde{D} : \mathsf{A} \to Y$ defined by $\widetilde{D}(a) = D(a+J)$ is inner. •

We say that Θ acts trivially on A from left (right) if for each $\alpha \in \Theta$ and $a \in \mathsf{A}$, $\alpha \cdot a = f(\alpha)a$ (resp. $a \cdot \alpha = f(\alpha)a$), where f is a continuous linear functional on Θ.

LEMMA 3.13. *If Θ acts on A trivially from the left or right and A/J has a right bounded approximate identity, then for each $\alpha \in \Theta$ and $a \in \mathsf{A}$ we have $f(\alpha)a - a\cdot\alpha \in J$.*

Proof. We prove the result for the left trivial action. Assume that $(e_j + J)$ is a right bounded approximate identity for A/J. For each $\alpha \in \Theta$ and $a \in \mathsf{A}$, we have $\lim_j (a\cdot\alpha + J)(e_j + J) = a\cdot\alpha + J$, and so

$$\left\| (a\cdot\alpha)e_j - a\cdot\alpha + J \right\| \to 0 \tag{3.1}$$

as $j \to \infty$. Since for each j, $(a\cdot\alpha)e_j - f(\alpha)ae_j \in J$, we have

$$\left\| (a\cdot\alpha)e_j - f(\alpha)ae_j + J \right\| = 0. \tag{3.2}$$

Again, from the defnition of bounded approximate identity, $\lim_j f(\alpha)(a+J)(e_j+J) = f(\alpha)a + J.$ Hence

$$\left\| f(\alpha)ae_j - f(\alpha)a + J \right\| \to 0 \tag{3.3}$$

as $j \to \infty$. Using (3.1), (3.2), (3.3), and triangle inequality we obtain the desired result.

The above Lemma shows that when Θ acts on A trivially from left or right, then the actions of Θ on A/J from both sides are trivial, that is $\alpha\cdot(a+J) = (a+J)\cdot\alpha = f(\alpha)a + J$ for all $a \in \mathsf{A}$ and $\alpha \in \Theta$. Thus the actions of A on $(\mathsf{A}/J)^{(n)}$ are trivial, for all $n \geq 1$. In particular, $(\mathsf{A}/J)^{(n)}$ is a commutative A-Θ-module, for each $n \in \mathsf{N}$.

Recall that a left Banach A-module X is called a left essential A-module if the linear span of $\mathsf{A}\cdot X = \{a\cdot x : a \in \mathsf{A}, x \in X\}$ is dense in X. Right essential A-modules and (two-sided) essential A-bimodules are defined similarly.

THEOREM 3.14. *Let $n \geq 1$ be odd.* Let A/J *has a left or right identity and Θ acts trivially on A from left. If A is n-weakly module amenable, then A/J is n-weakly amenable. The converse is true if A is a right essential Θ-module.*

Proof. Let $D:\mathsf{A}/J \to (\mathsf{A}/J)^{(n)}$ be a derivation and let $\widetilde{D}:\mathsf{A} \to (\mathsf{A}/J)^{(n)}$ be defined by $\widetilde{D}(a) = D(a+J)$, for $a \in \mathsf{A}$. It follows from Lemma 3.13 that $\widetilde{D}(\alpha \cdot a) = \alpha \cdot \widetilde{D}(a)$, and $\widetilde{D}(a \cdot \alpha) = \widetilde{D}(a) \cdot \alpha$, for each $\alpha \in \Theta$ and $a \in \mathsf{A}$. Also $\widetilde{D}(a \pm b) = \widetilde{D}(a) \pm \widetilde{D}(b)$ and $\widetilde{D}(ab) = \widetilde{D}(a)\cdot b + a \cdot \widetilde{D}(b)$, for all $a,b \in \mathsf{A}$. Thus $\widetilde{D}$ is a module derivation. Hence, there exists $\Phi \in (\mathsf{A}/J)^{(n)}$ such that $\widetilde{D}(a) = a\cdot\Phi - \Phi\cdot a$ .Therefore $D(a+J) = (a+J)\cdot\Phi - \Phi\cdot(a+J)$, and so D is inner. For the converse, since A/J is a commutative A-module, it is enough show that an arbitrary derivation $D:\mathsf{A} \to (\mathsf{A}/J)^{(n)}$ is a module derivation. Define $\widetilde{D}:\mathsf{A}/J \to (\mathsf{A}/J)^{(n)}$ via $\widetilde{D}(a+J) = D(a)$. By the proof of Proposition 3.12, that is well-defined. Now, we show that $\widetilde{D}$ is C-linear. If A is an essential right Θ-module, then every Θ-module derivation is also a derivation, and so it is linear. If $a \in \mathsf{A}$, then there is a sequence $(b_n) \subseteq \mathsf{A}\cdot\Theta$ such that $\lim_n b_n = a$. Assume that $b_n = \sum_{m=1}^{P_n} a_{n,m}\cdot\alpha_{n,m}$ for some finite sequences $(a_{n,m})_{m=1}^{m=P_n} \subseteq \mathsf{A}$ and $(\alpha_{n,m})_{m=1}^{m=P_n} \subseteq \Theta$. Let $\lambda \in \mathsf{C}$. Then

$$D(\lambda b_n) = D(\lambda \sum_{m=1}^{P_n} a_{n,m}\cdot\alpha_{n,m}) = \sum_{m=1}^{P_n} D(a_{n,m}\cdot(\lambda\alpha_{n,m})) = \sum_{m=1}^{P_n} D(a_{n,m})\cdot(\lambda\alpha_{n,m}) = \sum_{m=1}^{P_n} \lambda D(a_{n,m}\cdot\alpha_{n,m}) = \lambda D(b_n)$$

and so, by the continuity of D, $D(\lambda a) = \lambda D(a)$ This shows that $\widetilde{D}$ is linear, and so it is inner. Therefore D is an inner module derivation.

An inverse semigroup is a discrete semigroup S such that for each $s \in S$, there is a unique element $s' \in S$ with $ss's = s$ and $s'ss' = s'$. Elements of the form ss' are called idempotents of S and denoted by E. Suppose that S is an inverse semigroup with the set idempotents E, endowed with the partial order $e \leq d \Leftrightarrow ed = e (e,d \in E)$. It is easy to show that E is a (commutative) subsemigroup of S. In particular $\ell^1(E)$ could be regard as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions [1]. Here we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s, \quad \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \tag{3.4}$$

for all $s \in S$ and $e \in E$. In this case, the ideal J (see section 2) is the closed linear span of $\{\delta_{set} - \delta_{st} \quad s,t \in S, e \in E\}$. We consider an equivalence relation on S as follows: $s \approx t \Leftrightarrow \delta_s - \delta_t \in J (s,t \in S)$. For an inverse semigroup S, the quotient $S/\approx$ is a discrete group (see [3] and [14]). As in [15, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(S/\approx)$. With the notations of previous section, $\ell^1(S)/J$ is a commutative $\ell^1(E)$-bimodule with the following actions:

$$\delta_e \cdot (\delta_s + J) = \delta_s + J, \quad (\delta_s + J)\cdot\delta_e = \delta_{se} + J \quad (s \in S, e \in E).$$

It is proved in [2, Corollary 3.5] that if S is an inverse semigroup with an upward directed set of idempotents E (that is condition D_1 of [8]) then $\ell^1(S)$ is weakly module amenable as an $\ell^1(E)$-module. In the following theorem, we remove this condition and extend this result for $(2n+1)$-weak module amenability.

THEOREM 3.15. *Let S be an inverse semigroup with the set of idempotents E. Then, for each $n \in \mathsf{N}$, $\ell^1(S)$ is $(2n+1)$-weakly module amenable as an $\ell^1(E)$-module with trivial left action.*

Proof. With the actions $\ell^1(E)$ on $\ell^1(S)$ considered in (3.4), $\mathsf{A} = \ell^1(S)$ is always a right essential $\ell^1(E)$-module. If $f \in \ell^1(S)$, we have $f = \sum_{s\in S} f(s)\delta_s = \sum_{s\in S} f(s)\delta_s * \delta_{s^*s} = \sum_{s\in S} f(s)\delta_s \cdot \delta_{s^*s}$, that belongs to the closed linear span of $\ell^1(S)\cdot\ell^1(E) = \{\delta_s \cdot \delta_e : e \in E, s \in S\}$. Since $S/\approx$ is a discrete group, the group algebra $\ell^1(S/\approx)$ has an identity. Now, the result follows from [7, Theorem 4.1] and Theorem 3.14 with $\mathsf{A}=\ell^1(S)$ and $\Theta=\ell^1(E)$.

Let S be an inverse semigroup with the set of idempotents E. If $C^*(S)$ is the enveloping C^*-algebra of $\ell^1(S)$ (see [9]), then by continuity the action of $\ell^1(E)$ on $\ell^1(S)$ extends to an action of $C^*(E)$ on $C^*(S)$ and we have the following result:

THEOREM 3.16. *Let S be an inverse semigroup with the set of idempotents E, then for each $n\in\mathbb{Z}^+$, $C^*(S)$ is n-weakly module amenable as a $C^*(E)$-module with trivial left action.*

Proof. Since, the C^*-algebra $C^*(E)$ has a bounded approximate identity, by Cohen factorization theorem $C^*(S)$ is a right essential $C^*(E)$-module. Thus, it follows from the proof of Theorem 3.14 that each $C^*(E)$-module derivation on $C^*(S)$ is a $\mathbb{C}$-linear derivation. Since, the C^*-algebra $C^*(S)$ is n-weakly amenable [7, Theorem 2.1], it is also n-weakly module amenable as a $C^*(E)$-module.

Example 3.17. (i) Consider the inverse semigroup $S=\bigcup_{n\in\mathbb{N}} I_n$, where I_n is the semigroup of all partial one-one maps on $\{1,2,\dots,n\}$ [5, Remark 2.7(5)]. Then $\ell^1(S)$ is not weakly amenable, but it has a bounded approximate identity (since $E=E_S$ is obviously upward directed, and so satisfies the Duncan-Namioka condition D_1 [8]). It is easy to see that $S/\approx$ is the group $G=\bigcup_{n\in\mathbb{N}} S_n$ of finite permutations of $\mathbb{N}$. Now $\ell^1(S/\approx)$ is ($2n+1$)-weakly amenable, for each $n\in\mathbb{Z}^+$ [7]. It follows from Theorem 3.14 that $\ell^1(S)$ is ($2n+1$)-weakly module amenable as an $\ell^1(E)$-module, for each $n\in\mathbb{Z}^+$.

(ii) Let S be the bicyclic inverse semigroup generated by a and b, that is $\mathsf{S}=\{a^m b^n : m,n\geq 0\}$ for which $(a^m b^n)^* = a^n b^m$. The set of idempotents of S is $E_{\mathsf{S}}=\{a^n b^n : n=0,1,...\}$ with the order $a^n b^n \leq a^m b^m \Leftrightarrow m\leq n.$ It is shown in [3] that $\ell^1(\mathsf{S})$ is $\ell^1(E_{\mathsf{S}})$-module amenable, and so it is n-weakly module amenable for each $n\in\mathbb{N}$, but is not even weakly amenable [5, Theorem 2.8].

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