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TERNARY HOMOMORPHISMS BETWEEN UNITAL TERNARY C^*-ALGEBRAS

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Let A,B be two unital ternary C^*-algebras. We prove that every almost unital almost linear mapping $h:A\to B$ which satisfies $h([3^nu3^nvy]_A)=[h(3^nu)h(3^nv)h(y)]_B$ for all $u,v\in U(A)$, all $y\in A$, and all $n=0,1,2,...$, is a ternary homomorphism. Also, for a unital ternary C^*-algebra A of real rank zero, every almost unital almost linear continuous mapping $h:A\to B$ is a ternary homomorphism when $h([3^nu3^nvy]_A)=[h(3^nu)h(3^nv)h(y)]_B$ holds for all $u,v\in I_1(A_{sa})$, all $y\in A$, and all $n=0,1,2,...$. Furthermore, we investigate the Hyers-Ulam-Rassias stability of ternary homomorphisms between unital ternary C^*-algebras.

Key words: Ternary homomorphism, Ternary C^*-algebra.

1. INTRODUCTION

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists such as Cayley [5] who introduced the notions of cubic matrix, which in turn ([1,7,23,24,33,35]) was generalized by Kapranov at al. [22].

Following the terminology of Ref. [8], a nonempty set G with a ternary operation $[.,.,.]:G^3\to G$ is called a ternary groupoid and is denoted by $(G,[.,.,.])$. The ternary groupoid $(G,[.,.,.])$. is called commutative if $[x_1,x_2,x_3]=[x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}]$ for all $x_1,x_2,x_3\in G$ and all permutations σ of $\{1,2,3\}$. If a binary operation $\circ$ is defined on G such that $[x,y,z]=(x\circ y)\circ z$ for all $x,y,z\in G$, then we say that $[.,.,.]$ is derived from $\circ$.

We say that $(G,[.,.,.])$ is a ternary semigroup if the operation $[.,.,.]$ is associative, i.e., if $[[x,y,z],u,v]=[x,[y,z,u],v]=[x,y,[z,u,v]]$ holds for all $x,y,z,u,v\in G$ (see Ref. [2, 3, 13]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary produc $(X,Y,Z)\mapsto(X,Y,Z)$ of A^3 into A, which is C-linear in the outer variables, conjugate C-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$ and satisfies $\|[x,y,z]\|\le\|x\|.\|y\|.\|z\|$ and $\|[x,x,x]\|=\|x\|^3$. If a C^*-ternary algebra $(A,[.,.,.])$ has an identity, i.e., an element $e\in A$ such that $x=[x,e,e]=[e,e,x]$ for all $x\in A$, then it is routine to verify that A, endowed with $xoy:=[x,e,y]$ and $x^*:=[e,x,e]$, is a unital C^*-algebra. Conversely, if (A,o) is a unital C^*-algebra, then $[x,y,z]:=xo\,y^*oz$ makes A into a C^*-ternary algebra. A C-linear mapping $H:A\to B$ is called a C^*-ternary algebra homomorphism if

$$H([x,y,z])=[Hx),H(y),H(z)],$$

for all $x, y, z \in A$. Ternary structures and their generalization the so-called n-ary structures, raise certain hops in view of their applications in physics [2, 10, 13, 23, 36].

The study of stability problems originated from a famous talk given by S. M. Ulam [34] in 1940: "under what condition does there exist a homomorphism near an approximate homomorphism?" In the next year 1941, D. H. Hyers [15] answered affirmatively the question of Ulam. This stability phenomenon is called the Hyers-Ulam stability of the additive functional equation $g(x+y) = g(x) + g(y)$. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [32].

The stability phenomenon that was introduced and proved by Th. M. Rassias is called Hyers-Ulam-Rassias stability. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [6,9,11,12,14–18,20,27–31].

Throughout this paper, let A be a unital ternary C^*-algebra with unit e, and B a unital ternary Banach algebra with unit element e_B. Let $U(A)$ be the set of unitary elements in A, $A_{sa} := \{x \in A \,|\, x = x^*\}$, and $I_1(A_{sa}) = \{v \in A_{sa} \,|\, \|v\| = 1, v \in Inv(A)\}$. In this paper, we prove that every almost unital almost linear mapping $h : A \to B$ is a homomorphism when $h([3^n u 3^n vy]_A) = [h(3^n u) h(3^n v) h(y)]_B$ for all $u, v \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \ldots$. Also, for a unital ternary C^*-algebra A of real rank zero, every almost unital almost linear continuous mapping $h : A \to B$ is a ternary homomorphism when $h([3^n u 3^n vy]_A) = [h(3^n u) h(3^n v) h(y)]_B$ holds for all $u, v \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \ldots$. Furthermore, we investigate the Hyers-Ulam-Rassias stability of ternary *-homomorphisms between unital ternary C^*-algebras. Note that a unital ternary C^*-algebra is of real rank zero, if the set of invertible self-adjoint elements is dense in the set of self-adjoint elements [4]. We denote the algebraic center of A by $Z(A)$.

2. TERNARY HOMOMORPHISMS ON UNITAL TERNARY C^*-ALGEBRAS

Following the same approach as in [26], we obtain the next theorem.

Theorem 2.1. *Let $f : A \to B$ be a mapping such that $f(0) = 0$ and that*

$$f([3^n u 3^n vy]_A) = [f(3^n u) f(3^n) v f(y)]_B, \tag{2.1}$$

for all $u, v \in U(A)$, all $y \in A$, and all $n = 0,1,2,\ldots$. Assume as well that there exists a function $\phi : (A - \{0\})^2 \to [0, \infty)$ such that $\tilde{\phi}(x, y) = \sum_{n=0}^{\infty} 3^{-n} \phi(3^n x, 3^n y) < \infty$ for all $x, y \in A - \{0\}$ and that

$$\left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) \right\| \le \phi(x, y) \tag{2.2}$$

for all $\mu \in T$ and all $x, y \in A$. If $\lim_n \frac{f(3^n e)}{3^n} \in I_1(B_{sa}) \cap Z(B)$, then the mapping $f : A \to B$ is a ternary homomorphism.

Proof. Set $\mu = 1$ in (2.2), it follows from Theorem 1 of [19] that there exists a unique additive mapping $h : A \to B$ such that

$$\|f(x) - h(x)\| \le \frac{1}{3}\left(\tilde{\phi}(x, -x) + \tilde{\phi}(-x, 3x)\right) \tag{2.3}$$

for all $x \in A - \{0\}$. This mapping is given by $h(x) = \lim_n \frac{f(3^n x)}{3^n}$ for all $x \in A$. By the same reasoning as in the proof of Theorem 1 of [26], h is C-linear. It follows from (2.1) that

$$h([uvy]_A) = \lim_n \frac{f([3^n u 3^n vy]_A)}{9^n} = \lim_n \frac{[f(3^n u) f(3^n v) f(y)]_B}{9^n} = [h(u)h(v)f(y)]_B, \tag{2.4}$$

for all $u, v \in U(A)$, all $y \in A$.

Since h is additive, then by (2.4), we have $3^n h([uvy]_A) = h([uv(3^n y)]_A = [h(u)h(v)f(3^n y)]_B$ for all $u, v \in U(A)$ and all $y \in A$.

Hence,

$$h([uvy]_A) = \lim_n [h(u)h(v)\frac{f(3^n y)}{3^n}]_B = [h(u)h(v)h(y)]_B \tag{2.5}$$

for all $u, v \in U(A)$ and all $y \in A$. By the assumption, we have $h(e) = \lim_n \frac{f(3^n e)}{3^n} \in U(B)$ hence, it follows by (2.4) and (2.5) that $[h(e)h(e)h(y)]_B = h([eey]_A) = [h(e)h(e)f(y)]_B$ for all $y \in A$. We denote the unit element of B by e_B. Since $h(e)$ belongs to $I_1(B_{sa})$, then

$$\begin{aligned}
h(y) &= [e_B e_B h(y)]_B = [[h(e)^{-1} e_B h(e)]_B e_B h(y)]_B = [h(e)^{-1}[e_B h(e) e_B]_B h(y)]_B = \\
&= [h(e)^{-1}[e_B e_B h(e)]_B h(y)]_B = [h(e)^{-1} e_B [e_B h(e) h(y)]_B]_B = \\
&= [h(e)^{-1}[e_B e_B h(e)]_B h(y)]_B = [h(e)^{-1} e_B [e_B h(e) h(y)]_B]_B = \\
&= [h(e)^{-1} e_B [[h(e)^{-1} e_B h(e)]_B h(e) h(y)]_B]_B = [h(e)^{-1} e_B [h(e)^{-1} e_B [h(e)h(e)h(y)]_B]_B]_B = \\
&= [h(e)^{-1} e_B [h(e)^{-1} e_B [h(e)h(e)f(y)]_B]_B]_B = [h(e)^{-1} e_B [[h(e)^{-1} e_B h(e)]_B h(e) f(y)]_B]_B = \\
&= [h(e)^{-1}[e_B e_B h(e)]_B h(y)]_B = [h(e)^{-1} e_B [e_B h(e) f(y)]_B]_B = [h(e)^{-1}[e_B h(e) e_B]_B f(y)]_B = \\
&= [[h(e)^{-1} e_B h(e)]_B e_B f(y)]_B = [e_B e_B f(y)]_B = \\
&= f(y), \quad \text{for all } y \in A.
\end{aligned}$$

We have to show that f is a ternary homomorphism. For every $a, b \in A$, we define $a \Diamond b := [aeb]_A$. Then $\Diamond : A \times A \to A$ is a binary product for which $(A, \Diamond)$ may be considered as a (binary) C^*-algebra. Also, we have $a \in U(A, [\ ]_A)$ if and only if $a \in U((A, \Diamond))$ for all $a \in A$. Now, let $a, b \in A$. By Theorem 4.1.7 of [21], a, b are finite linear combinations of unitary elements, i.e., $a = \sum_{i=1}^n c_i u_i, b = \sum_{j=1}^m d_j v_j (c_i, d_j \in C, u_i, v_j \in U(A))$, it follows from (2.5) that

$$\begin{aligned}
f([aby]_A) &= h([aby]_A) = h([(\sum_{i=1}^n c_i u_i)(\sum_{j=1}^m \mathrm{d}_j v_j) y])_A = \\
&= h\left([\sum_{i=1}^n \sum_{j=1}^m c_i \mathrm{d}_j u_i v_j y]_A\right) = h\left(\sum_{i=1}^n \sum_{j=1}^m c_i \mathrm{d}_j [u_i v_j y]_A\right) = \\
&= \sum_{i=1}^n \sum_{j=1}^m c_i \mathrm{d}_j h([u_i v_j]_A) = \sum_{i=1}^n \sum_{j=1}^m c_i \mathrm{d}_j [h(u_i) h(v_j) h(y)]_B = \\
&= [\sum_{i=1}^n \sum_{j=1}^m c_i \mathrm{d}_j h(u_i) h(v_j) h(y)]_B = [h(\sum_{i=1}^n c_i u_i) h(\sum_{j=1}^m \mathrm{d}_j v_j) h(y)]_B = \\
&= [h(a)h(b)h(y)]_B, \quad \text{for all } y \in A.
\end{aligned}$$

This completes the proof of theorem.

Corollary 2.2. *Let* $p \in (0,1), \theta \in [0,\infty)$ *be real numbers. Let* $f : A \to B$ *be a mapping such that* $f(0) = 0$ *and that*

$$f([3^n u 3^n v y]_A) = [f(3^n u) f(3^n) v f(y)]_B$$

for all $u, v \in U(A)$, *all* $y \in A$, *and all* $n = 0,1,2,...$. *Suppose that*

$$\left\| 2f(\frac{\mu x + \mu y}{2}) - \mu f(x) - \mu f(y) \right\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $\mu \in T$ *and all* $x, y \in A$. *If* $\lim_n \frac{f(3^n e)}{3^n} \in I_1(B_{sa})$, *then the mapping* $f : A \to B$ *is a ternary homomorphism*.

Proof. Set $\phi(x,y) := \left(\|x\|^p + \|y\|^p\right)$ all $x, y \in A$. Then by Theorem 2.1 we get the desired result.

Theorem 2.3. *Let* A *be a ternary* C^**-algebra of real rank zero. Let* $f : A \to B$ *be a continuous mapping such that* $f(0) = 0$ *and that*

$$f([3^n u 3^n v y]_A) = [f(3^n u) f(3^n) v f(y)]_B \tag{2.6}$$

for all $u, v \in I_1(A_{sa})$ *all* $y \in A$, *and all* $n = 0,1,2,...$. *Suppose that there exists a function* $\phi : \left(A - \{0\}\right)^2 \to [0,\infty)$ *satisfying* (2.2) *and* $\tilde{\phi}(x,y) < \infty$ *for all* $x, y \in A - \{0\}$. If $\lim_n \frac{f(3^n e)}{3^n} \in I_1(B_{sa})$, *then the mapping* $f : A \to B$ *is a ternary homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique *C*-linear mapping $h : A \to B$ satisfying (2.3). It follows from (2.6) that

$$h([uvy]_A) = \lim_n \frac{f([3^n u 3^n v y]_A)}{9^n} = \lim_n \frac{[f(3^n u) f(3^n v) f(y)]_B}{9^n} = [h(u)h(v)f(y)]_B \tag{2.7}$$

for all $u, v \in I_1(A_{sa})$, and all $y \in A$. By additivity of h and (2.7), we obtain that

$3^n h([uvy]_A) = h([uv(3^n y)]_A = [h(u)h(v)f(3^n y)]_B$, for all $u, v \in I_1(A_{sa})$ and all $y \in A$.

Hence,

$$h([uvy]_A) = \lim_n [h(u)h(v)\frac{f(3^n y)}{3^n}]_B = [h(u)h(v)h(y)]_B \text{, for all } u, v \in I_1(A_{sa})) \text{ and all } y \in A. \tag{2.8}$$

By the assumption, we have

$$h(e) = \lim_n \frac{f(3^n e)}{3^n} \in U(B).$$

Similar to the proof of Theorem 2.1, it follows from (2.7) and (2.8) that $h = f$ on A. So h is continuous. On the other hand A is real rank zero. One can easily show that $I_1(A_{sa})$ is dense in $\{x \in A_{sa} : \|x\| = 1\}$. Let $u, v \in \{x \in A_{sa} : \|x\| = 1\}$ There are $\{t_n\}, \{z_n\}$ in $I_1(A_{sa})$ such that $\lim_n t_n = u$, $\lim_n z_n = v$. Since h is continuous, it follows from (2.8) that

$$h([uvy]_A) = h(\lim_n (t_n z_n y)) = \lim_n h([(t_n z_n y)]_A) = \lim_n [h(t_n)h(z_n)h(y)]_B = [h(u)h(v)h(y)]_B, \tag{2.9}$$

for all $y \in A$. Now, let $a, b \in A$. Then we have $a = a_1 + \mathrm{i}a_2, b = b_1 + \mathrm{i}b_2$, where $a_1 := \frac{a + a^*}{2}$, $b_1 := \frac{b + b^*}{2}$ and $a_2 := \frac{a - a^*}{2i}$, $b_2 := \frac{b - b^*}{2i}$ are self-adjoint. First consider $a_2 = b_2 = 0$, $a_1, b_1 \neq 0$. Since h is C-linear, it follows from (2.9) that

$$f([aby]_A) = h\left([aby]_A\right) = h\left([a_1b_1y]_A\right) = h\left(\|a_1\|\|b_1\|\left[\frac{a_1}{\|a_1\|}\frac{b_1}{\|b_1\|}y\right]_A\right) =$$

$$= \|a_1\|\|b_1\| h\left(\left[\frac{a_1}{\|a_1\|}\frac{b_1}{\|b_1\|}y\right]_A\right) = \|a_1\|\|b_1\|\left[h\left(\frac{a_1}{\|a_1\|}\right)h\left(\frac{b_1}{\|b_1\|}\right)h(y)\right]_B =$$

$$= \left[h\left(\|a_1\|\frac{a_1}{\|a_1\|}\right)h\left(\|b_1\|\frac{b_1}{\|b_1\|}\right)h(y)\right]_B = \left[h(a_1)h(b_1)h(y)\right]_B = \left[f(a)f(b)f(y)\right]_B, \text{ for all } y \in A.$$

Now, consider $a_1 = b_1 = 0$, $a_2, b_2 \neq 0$. Since h is C-linear, it follows from (2.9) that

$$f\left([aby]_A\right) = h\left([aby]_A\right) = h\left([ia_2ib_2y]_A\right) = -h\left(\|a_2\|\|b_2\|\left[\frac{a_2}{\|a_2\|}\frac{b_2}{\|b_2\|}y\right]_A\right) =$$

$$= -\|a_2\|\|b_2\| h\left(\left[\frac{a_2}{\|a_2\|}\frac{b_2}{\|b_2\|}y\right]_A\right) = -\|a_2\|\|b_2\|\left[h\left(\frac{a_2}{\|a_2\|}\right)h\left(\frac{b_2}{\|b_2\|}\right)h(y)\right]_B =$$

$$= \left[h\left(i\|a_2\|\frac{a_2}{\|a_2\|}\right)h\left(i\|b_2\|\frac{b_2}{\|b_2\|}\right)h(y)\right]_B = \left[h(ia_2)h(ib_2)h(y)\right]_B =$$

$$= \left[f(a)f(b)f(y)\right]_B, \quad \text{for all } y \in A.$$

Suppose $a_2 = b_1 = 0$, $a_1, b_2 \neq 0$. Then by (2.9), we have

$$f\left([aby]\right)_A = h\left([aby]\right)_A = h\left(\left[a_1(ib_2)y\right]_A\right) = h\left(i\|a_1\|\|b_2\|\left[\frac{a_1}{\|a_1\|}\frac{b_2}{\|b_2\|}y\right]_A\right) =$$

$$= i\|a_1\|\|b_2\| h\left(\left[\frac{a_1}{\|a_1\|}\frac{b_2}{\|b_2\|}y\right]_A\right) = i\|a_1\|\|b_2\|\left[h\left(\frac{a_1}{\|a_1\|}\right)h\left(\frac{b_2}{\|b_2\|}\right)h(y)\right]_B =$$

$$= \left[h\left(\|a_1\|\frac{a_1}{\|a_1\|}\right)h\left(i\|b_2\|\frac{b_2}{\|b_2\|}\right)h(y)\right]_B = \left[h(a_1)h(ib_2)h(y)\right]_B =$$

$$= \left[f(a)f(b)f(y)\right]_B, \quad \text{for all } y \in A.$$

Similarly we can show that

$$f\left([aby]_A\right) = \left[f(a)f(b)f(y)\right]_B,$$

for all $y \in A$ if $a_1 = b_2 = 0$, $a_2, b_1 \neq 0$. In the case that $b_2 = 0, a_1, a_2, b_1 \neq 0$, we have

$$f\left([aby]_A\right) = h\left([aby]_A\right) = h\left([(a_1 + ia_2)b_1y]_A\right) = h\left([a_1b_1y]_A\right) + ih\left([a_2b_1y]_A\right) =$$

$$= h\left(\|a_1\|\|b_1\|\left[\frac{a_1}{\|a_1\|}\frac{b_1}{\|b_1\|}y\right]_A\right) + ih\left(\|a_2\|\|b_1\|\left[\frac{a_2}{\|a_2\|}\frac{b_1}{\|b_1\|}y\right]_A\right) =$$

$$=\|a_1\|\|b_1\|h\left(\left[\frac{a_1}{\|a_1\|}\frac{b_1}{\|b_1\|}y\right]_A\right)+i\|a_2\|\|b_1\|h\left(\left[\frac{a_2}{\|a_2\|}\frac{b_1}{\|b_1\|}y\right]_A\right)=$$

$$=\|a_1\|\|b_1\|\left[h\left(\frac{a_1}{\|a_1\|}\right)h\left(\frac{b_1}{\|b_1\|}\right)h(y)\right]_B+i\|a_2\|\|b_1\|\left[h\left(\frac{a_2}{\|a_2\|}\right)h\left(\frac{b_1}{\|b_1\|}\right)h(y)\right]_B=$$

$$=\left[h\left(\|a_1\|\frac{a_1}{\|a_1\|}\right)h\left(\|b_1\|\frac{b_1}{\|b_1\|}\right)h(y)\right]_B+i\left[h\left(\|a_2\|\frac{a_2}{\|a_2\|}\right)h\left(\|b_1\|\frac{b_1}{\|b_1\|}\right)h(y)\right]_B=$$

$$=[h(a_1)h(b_1)h(y)]_B+i[h(a_2)h(b_1)h(y)]_B=[h(a_1+ia_2h(b_1)h(y))]_B=$$

$$=[f(a)f(b)f(y)]_B,\qquad \text{for all } y\in A.$$

By a same reasoning above, we can show that

$$f([aby]_A)=[f(a)f(b)f(y)]_B$$

for all $y\in A$ if $a_2=0,\ a_1,b_1,b_2\neq 0$. Now consider $b_1=0,\ a_1,a_2,b_2\neq 0$. Then by (2.9), we have

$$f([aby]_A)=h([aby]_A)=h([(a_1+ia_2)(ib_2)y]_A)=h([ia_1b_2y]_A)-h([a_2b_2y]_A)=$$

$$=ih\left(\|a_1\|\|b_2\|\left[\frac{a_1}{\|a_1\|}\frac{b_2}{\|b_2\|}y\right]_A\right)-h\left(\|a_2\|\|b_2\|\left[\frac{a_2}{\|a_2\|}\frac{b_2}{\|b_2\|}y\right]_A\right)=$$

$$=i\|a_1\|\|b_2\|h\left(\left[\frac{a_1}{\|a_1\|}\frac{b_2}{\|b_2\|}y\right]_A\right)-\|a_2\|\|b_2\|h\left(\left[\frac{a_2}{\|a_2\|}\frac{b_2}{\|b_2\|}y\right]_A\right)=$$

$$=i\|a_1\|\|b_2\|\left[h\left(\frac{a_1}{\|a_1\|}\right)h\left(\frac{b_2}{\|b_2\|}\right)h(y)\right]_B-\|a_2\|\|b_2\|\left[h\left(\frac{a_2}{\|a_2\|}\right)h\left(\frac{b_2}{\|b_2\|}\right)h(y)\right]_B= \tag{2.10}$$

$$=\left[h\left(\|a_1\|\frac{a_1}{\|a_1\|}\right)ih\left(\|b_2\|\frac{b_2}{\|b_2\|}\right)h(y)\right]_B+\left[ih\left(\|a_2\|\frac{a_2}{\|a_2\|}\right)ih\left(\|b_2\|\frac{b_2}{\|b_2\|}\right)h(y)\right]_B=$$

$$=[h(a_1)ih(b_2)h(y)]_B+[ih(a_2)ih(b_2)h(y)]_B=[h(a_1+ia_2)h(ib_2)h(y)]_B=$$

$$=[f(a)f(b)f(y)]_B,\qquad \text{for all } y\in A.$$

Also, by a same reasoning, we can see that

$$f([aby]_A)=[f(a)f(b)f(y)]_B,\qquad \text{for all } y\in A \text{ if } a_1=0,a_2,b_1,b_2\neq 0.$$

Finally consider that $a_1,a_2,b_1,b_2\neq 0$. Then by (2.9), we have

$$f([aby]_A)=h([aby]_A)=h([(a_1+ia_2)(b_1+ib_2)y]_A)=$$

$$=h([a_1b_1y]_A)+h([ia_1b_2y]_A)+h([ia_2b_1y]_A)-h([ia_2b_2y]_A)=$$

$$=h\left(\|a_1\|\|b_1\|\left[\frac{a_1}{\|a_1\|}\frac{b_1}{\|b_1\|}y\right]_A\right)+ih\left(\|a_1\|\|b_2\|\left[\frac{a_1}{\|a_1\|}\frac{b_2}{\|b_2\|}y\right]_A\right)+ih\left(\|a_2\|\|b_2\|\left[\frac{a_2}{\|a_2\|}\frac{b_1}{\|b_1\|}y\right]_A\right)-h\left(\|a_2\|\|b_2\|\left[\frac{a_2}{\|a_2\|}\frac{b_2}{\|b_2\|}y\right]_A\right)=$$

$$=\|a_1\|\|b_1\|h\left(\left[\frac{a_1}{\|a_1\|}\frac{b_1}{\|b_1\|}y\right]_A\right)+i\|a_1\|\|b_2\|h\left(\left[\frac{a_1}{\|a_1\|}\frac{b_2}{\|b_2\|}y\right]_A\right)+$$

$$+i\|a_2\|\|b_1\|h\left(\left[\frac{a_2}{\|a_2\|}\frac{b_1}{\|b_1\|}y\right]_A\right)-\|a_2\|\|b_2\|h\left(\left[\frac{a_2}{\|a_2\|}\frac{b_2}{\|b_2\|}y\right]_A\right)=$$

$$=\|a_1\|\|b_1\|\left[h\left(\frac{a_1}{\|a_1\|}\right)h\left(\frac{b_1}{\|b_1\|}\right)h(y)\right]_B+i\|a_1\|\|b_2\|\left[h\left(\frac{a_1}{\|a_1\|}\right)h\left(\frac{b_2}{\|b_2\|}\right)h(y)\right]_B+$$

$$+i\|a_2\|\|b_1\|\left[h\left(\frac{a_2}{\|a_2\|}\right)h\left(\frac{b_1}{\|b_1\|}\right)h(y)\right]_B-\|a_2\|\|b_2\|\left[h\left(\frac{a_2}{\|a_2\|}\right)h\left(\frac{b_2}{\|b_2\|}\right)h(y)\right]_B=$$

$$=\left[h\left(\|a_1\|\frac{a_1}{\|a_1\|}\right)h\left(\|b_1\|\frac{b_1}{\|b_1\|}\right)h(y)\right]_B+\left[h\left(\|a_1\|\frac{a_1}{\|a_1\|}\right)ih\left(\|b_2\|\frac{b_2}{\|b_2\|}\right)h(y)\right]_B+$$

$$+\left[ih\left(\|a_2\|\frac{a_2}{\|a_2\|}\right)h\left(\|b_1\|\frac{b_1}{\|b_1\|}\right)h(y)\right]_B+\left[ih\left(\|a_2\|\frac{a_2}{\|a_2\|}\right)ih\left(\|b_2\|\frac{b_2}{\|b_2\|}\right)h(y)\right]_B=$$

$$=\left[h(a_1)h(b_1)h(y)\right]_B+\left[h(a_1)ih(b_2)h(y)\right]_B+\left[ih(a_2)h(b_1)h(y)\right]_B+\left[ih(a_2)ih(b_2)h(y)\right]_B=$$

$$=\left[h(a_1+ia_2)h(b_1+ib_2)h(y)\right]_B=\left[f(a)f(b)f(y)\right]_B,\quad \text{for all } y\in A.$$

Hence, $f([aby]_A)=[f(a)f(b)f(y)]_B$ for all $a,b,y\in A$ and f is ternary homomorphism.

Corollary 2.4. *Let A be a ternary C^*-algebra of real rank zero. Let $p\in(0,1),\theta\in[0,\infty)$ be real numbers. Let $f:A\to B$ be a mapping such that $f(0)=0$ and that*

$$f([3^n u 3^n v y]_A)=[f(3^n u)f(3^n)vf(y)]_B \tag{2.11}$$

for all $u,v\in I_1(A_{sa})$, all $y\in A$, and all $n=0,1,2,...$. Suppose that

$$\left\|2f\left(\frac{\mu x+\mu y}{2}\right)-\mu f(x)-\mu f(y)\right\|\le\theta\left(\|x\|^p+\|y\|^p\right)$$

for all $\mu\in T$ and all $x,y\in A$. If $\lim\limits_n\dfrac{f(3^n e)}{3^n}\in U(B)$, then the mapping $f:A\to B$ is a ternary homomorphism.

Proof. Set $\phi(x,y):=\left(\|x\|^p+\|y\|^p\right)$ for all $x,y\in A$. Then by Theorem 2.3 we get the desired result.

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