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IDEAL AMENABILITY AND APPROXIMATE IDEAL AMENABILITY OF MATRIX BANACH ALGEBRAS

F. HABIBIAN[1], H. SAMEA[2], A. REJALI[3]

[1]Department of Mathematics, University of Semnan, Semnan, Iran
[2]Department of Mathematics, University of Bu-Ali Sina, Hamedan, Iran
[3]Department of Mathematics, University of Isfahan, Isfahan, Iran
Email: fhabibian@semnan.ac.ir

This work was intended as an attempt to introduce and investigate the approximate ideal amenability of Banach algebras. We show that the approximate ideal amenability and approximate weak amenability of matrix Banach algebra $\mathfrak{E}_p(I)$ ($1\le p<\infty$) are equivalent. As a consequence, we prove that the convolution Banach algebra $\mathbf{L}^2(G)$ is approximately ideally amenable and, further, is ideally amenable if and only if G is finite or abelian.

Key words: Amenability, Banach algebra, Group algebra, Ideal amenability.

INTRODUCTION

The Banach algebras $\mathfrak{E}_p(I)$ $p\in[1,\infty)\cup\{0\}$, were introduced and extensively studied in [5, Section 28]. Recently amenability of these Banach algebras was studied in [7].

The aim of the present paper is to investigate the ideal amenability, and according to the work of F. Ghahramani and R. J. Loy [4], approximate ideal amenability of Banach algebras $\mathfrak{E}_p(I)$ $(1\le p<\infty)$.

The organization of this paper is as follows. The preliminaries and notations are given in section one. In Section two, among the other results, we prove that the matrix Banach algebras $\mathfrak{E}_p(I)$ $(1\le p<\infty)$ are approximately ideally amenable. Finally, section three is devoted to find some results for ideal amenability and approximate ideal amenability of certain Banach algebras on compact groups.

1. PRELIMINARIES

For a Banach algebra $\mathfrak{A}$, an $\mathfrak{A}$– *bimodule* will always refer to a *Banach* $\mathfrak{A}$*–bimodule* X, that is a Banach space which is algebraically an $\mathfrak{A}$– bimodule, and for which there is a constant $C_{\mathfrak{A},X}>0$ such that

$$(a\in\mathfrak{A}, x\in X).\qquad \|a.x\|,\|x.a\|\le C_{\mathfrak{A},X}\|a\|\|x\|$$

The Banach space X^* with the dual module multiplications defined by

$$(f.a)(x)=f(a.x) \text{ and } (a.f)(x)=f(x.a)\quad (a\in\mathfrak{A}, f\in X^*, x\in X),$$

is a Banach $\mathfrak{A}$– bimodule which is called the dual Banach $\mathfrak{A}$– bimodule.

A Banach algebra $\mathfrak{A}$ is always a Banach $\mathfrak{A}$ – bimodule with the product of $\mathfrak{A}$. An approximate identity for $\mathfrak{A}$ is a net $(e_\alpha)_\alpha$ in $\mathfrak{A}$ such tht for each $a\in\mathfrak{A}$, $\lim_\alpha e_\alpha.a=\lim_\alpha a.e_\alpha=a$. A derivation from $\mathfrak{A}$ into an $\mathfrak{A}$ – bimodule X is a bounded linear map *D*, such that $D(ab)=D(a).b+a.D(b)$, for all $a,b\in\mathfrak{A}$.

if $x\in X$, then $ad_x:\mathfrak{A}\to X$ defined by $ad_x(a)=a.x-x.a$ $(a\in\mathfrak{A})$, is a derivation. Such derivations are called inner. Denote by $Z^1(\mathfrak{A},X)$ the space of all continuous derivations from $\mathfrak{A}$ into X and by $N^1(\mathfrak{A},X)$ the space of all inner derivations from $\mathfrak{A}$ into X. Then $N^1(\mathfrak{A},X)$ is a subspace of $Z^1(\mathfrak{A},X)$. The quotient space $H^1(\mathfrak{A},X)=Z^1(\mathfrak{A},X)/N^1(\mathfrak{A},X)$ is called the first cohomology group with coefficients in X.

All the amenability theories are related to the question of whether $H^1(\mathfrak{A},X)=\{0\}$ for certain X. A Banach algebra $\mathfrak{A}$ is said to be contractible or super amenable if $H^1(\mathfrak{A},X)=\{0\}$ for all Banach $\mathfrak{A}$ – bimodules X, amenable if $H^1(\mathfrak{A},X^*)=\{0\}$ for all Banach $\mathfrak{A}$ – bimodules X, weakly amenable if $H^1(\mathfrak{A},\mathfrak{A}^*)=\{0\}$, and ideally amenable if $H^1(\mathfrak{A},I^*)=\{0\}$ for every closed ideal I of $\mathfrak{A}$. For more information we refer the reader to [6,1,2].

A derivation $D:\mathfrak{A}\to X$ is approximately inner, if there exists a net $(\xi_\alpha)\subset X$ such that for every $a\in\mathfrak{A}$, $D(a)=\lim_\alpha ad_{\xi_\alpha}(a)$, the limit being in norm. A Banach algebra $\mathfrak{A}$ is called approximately weakly amenable if each $D\in Z^1(\mathfrak{A},\mathfrak{A}^*)$ is approximately inner [4].

Let H be a finite dimensional Hilbert space of dimension n, and let B(H) be the space of all linear operators on H. Clearly we can identify B(H) with $\mathbb{M}_n(\mathbb{C})$ (the space of all $n\times n$-matrices on $\mathbb{C}$). For $A\in\mathbb{M}_n(\mathbb{C})$, let $A^*\in\mathbb{M}_n(\mathbb{C})$ by $(A^*)_{ij}=\bar{A}_{ij}\,(1\le i,j\le n)$, and let |A| denote the unique positive-definite square root of AA^*. A is called unitary, if $A^*A=AA^*=I$, where I is the $n\times n$-identity matrix. For $E\in B(H)$, Let $(\lambda_1,\dots,\lambda_n)$ be the sequence of eignvalues of operator |E|, written in any order.

Define $\|E\|_{\varphi_\infty}=\max\{|\lambda_i|:1\le i\le n\}$, and $\|E\|_{\varphi_p}=(\sum_{i=1}^n|\lambda_i|^p)^{\frac{1}{p}}$ $(1\le p\le\infty)$. For more details see Definition [5, D.37] and Theorem [5, D.40].

Let I be an arbitrary index set. For each $i\in I$, let H_i be a finite dimensional Hilbert space of dimension d_i, and let $a_i\ge 1$. The *-algebra $\prod_{i\in I}B(H_i)$ will denoted by $\mathfrak{E}(I)$; scalar multiplication, addition, multiplication and the adjoint of an element are defined coordinatewise. Let $E=(E_i)$ be an element of $\mathfrak{E}(I)$. For $1\le p\le\infty$, define

$$\|E\|_p=\left(\sum_{i\in I}a_i\,\|E_i\|_{\varphi_p}^p\right)^{\frac{1}{p}},$$

and

$$\|E\|_\infty=\sup\{\|E\|_{\varphi_\infty}:i\in I\}.$$

Let $\mathfrak{E}_p(I)$ be the set of all $E\in\mathfrak{E}(I)$ for which $\|E\|_p<\infty$, and $\mathfrak{E}_0(I)$ be the set of all $E\in\mathfrak{E}(I)$ such that $\{i\in I:\|E_i\|_{\varphi_\infty}\ge e\}$ is finite for all $e>0$. By [5, Theorems 28.25, 28.27, and 28.32(v)], both $(\mathfrak{E}_p(I),\|.\|_p)$ $(1\le p<\infty)$, and $(\mathfrak{E}_p(I),\|.\|_\infty)$ are Banach algebras.

Let G be a compact group with dual $\hat{G}$ (the set of all irreducible representations of G). For each $\pi\in\hat{G}$, let H_π be the representation space of π. The algebras $\mathfrak{E}_p(\hat{G})$ for $p\in[1,\infty)$, are defined as above with each a_π equal to the dimension d_π of $\pi\in\hat{G}$ [5, Definition 28.34].

For a locally compact group G and a function f: $G\to\mathbb{C}$, $\check{f}$ is defined by $\check{f}(x)=f(x^{-1})$ $(x\in G)$. Let A(G) (or with the notation $\mathfrak{R}$ (G) defined in [5,35,16]) consist of all functions h in $C_0(G)$ which can be written in at least one way as $\sum_{n=1}^\infty f_n*\check{g}_n$, where f_n, $g_n\in L^2(G)$ and $\sum_{n=1}^\infty\|f_n\|_2\|g_n\|_2<\infty$. For $h\in A(G)$, define

$$\|h\|_{A(G)} = \inf\left\{\sum_{n=1}^{\infty}\|f_n\|_2\|g_n\|_2 : h=\sum_{n=1}^{\infty} f_n * \check{g}_n\right\}.$$

Recall that (A (G), $\|.\|_{A(G)}$) is a compact group, A (G) under convolution product and the norm $\|.\|_{A(G)}$ defines a Banach algebra which is isometrically algebra isomorphic with $\mathfrak{E}_1(\hat{G})$ [5, Theorem 34.32].

2. IDEAL AND APPROXIMATE IDEAL AMENABILITY OF CERTAIN MATRIX ALGEBRAS

In this section, we follow F. Ghahramani and R. J. Loy [4] to define the concept of approximate ideal amenability.

Definition 2.1. A Banach algebra $\mathfrak{A}$ is called approximately ideally amenable if, for any closed ideal I of $\mathfrak{A}$, every derivation D: $\mathfrak{A} \to I^*$ is approximately inner.

Remark 2.2. It is clear that each ideally amenable Banach algebra $\mathfrak{A}$ is approximately ideally amenable. Furthermore, for a commutative Banach algebra $\mathfrak{A}$, ideal amenability and approximate ideal amenability are equivalent.

Throughout the paper for $A \in \mathbb{M}_{d_i}(\mathbb{C})$ we define A^i as an element of $\mathfrak{E}(I)$ by

$$(A^i)_j = \begin{cases} A \text{ for } j=i \\ 0 \text{ otherwise.} \end{cases}$$

Let $\mathfrak{T}$ be a closed ideal of $\mathfrak{E}_p(I)$. Define

$$\mathfrak{T}_i = \{ A \in \mathbb{M}_{d_i}(\mathbb{C}) : \exists \tilde{A} \in \mathfrak{T} \text{ s.t. } \tilde{A}_i = A\}.$$

The reminder of this section will be devoted to investigate approximate ideal amenability of $\mathfrak{E}_p(I)$ for $1 \le p < \infty$.

Lemma 2.3. *Let $\mathfrak{T}$ be a closed ideal of $\mathfrak{E}_p(I)$, then $\mathfrak{T}_i = 0$ or $\mathfrak{T}_i = \mathbb{M}_{d_i}(\mathbb{C})$.*

Proof. Let $B \in \mathbb{M}_{d_i}(\mathbb{C})$. Clearly $B^i \in \mathfrak{E}_p(I)$. For $A \in \mathfrak{T}_i$, there exists $\tilde{A} \in \mathfrak{T}$ such that $\tilde{A}_i = A$ and since $\mathfrak{T}$ is an ideal of $\mathfrak{E}_p(I)$, then $\tilde{A}B^i \in \mathfrak{T}$. But $(\tilde{A}B^i)_i$=AB, hence $AB \in \mathfrak{T}_i$. This implies that $\mathfrak{T}_i$ is an ideal of $\mathbb{M}_{d_i}(\mathbb{C})$. We know that $\mathbb{M}_{d_i}(\mathbb{C})$ is simple, therefore, $\mathfrak{T}_i = 0$ or $\mathfrak{T}_i = \mathbb{M}_{d_i}(\mathbb{C})$.

Corollary 2.4. *Each ideal of $\mathfrak{E}_p(I)$ ($1 \le p < \infty$) is of the form $\mathfrak{E}_p(I')$ where $I' = \{i \in I : \mathfrak{T}_i = \mathbb{M}_{d_i}(\mathbb{C})\}$.*

Notation. Throughout the rest of the paper for $1 \le p < \infty$, let q denote the exponent conjugate to *p*, that is $\frac{1}{p}+\frac{1}{q}=1$. For $p = 1$, let $q = 0$ (no ∞).

Remark 2.5. By [5, Theorem 28.32], $\mathfrak{E}_p(I)$ is a Banach $\mathfrak{E}_p(I)$–bimodule with the product of $\mathfrak{E}(I)$. Also by [5, Theorem 28.31], the mapping T: $\mathfrak{E}_p(I) \to \mathfrak{E}_q(I)^*$ given by

$$< B, T(A)> = \sum_{i\in I} a_i \operatorname{tr}(B_i A_i) \quad (A \in \mathfrak{E}_p(I), B \in \mathfrak{E}_p(I)),$$

is an isometric Banach space isomorphism. Then $\mathfrak{E}_p(I)$ can be identified with the dual Banach $\mathfrak{E}_q(I)$-bimodule $\mathfrak{E}_q(I)^*$ with the product of $\mathfrak{E}(I)$.

According to the above remark and corollary 2.4, we have:

Corollary 2.6. *Let* $\mathfrak{T}$ *be a closed ideal of* $\mathfrak{E}_p(I)$ $(1\le p<\infty)$. *Then* $\mathfrak{T}^*$ *is a Banach* $\mathfrak{E}_q(I)$*-bimodule with the product of* $\mathfrak{E}(I)$ *isomorphic with* $\widetilde{\mathfrak{E}_q(I')}\subseteq\mathfrak{E}_q(I)$ *where* $\widetilde{\mathfrak{E}_q(I')}=\{A\in\mathfrak{E}_q(I): A_i=0\ (i\notin I')\}$.

We are thus led to the following main result.

Theorem 2.7. *For each* $1\le p<\infty$, *the Banach* $\mathfrak{E}_p(I)$ *is approximately ideally amenable*.

Proof. Let I be a closed ideal of $\mathfrak{E}_p(I)$ and D: $\mathfrak{E}_p(I)\to\mathfrak{T}^*$ be a continuous derivation. By corollary 2.6, $\mathfrak{T}^*=\widetilde{\mathfrak{E}_q(I')}$ and we can consider D: $\mathfrak{E}_p(I)\to\widetilde{\mathfrak{E}_q(I')}$. Fix $i\in I$ and let I_i be the identity element of $\mathbb{M}_{d_i}(\mathbb{C})$. It is easily seen that I^i is a central idempotent in $\mathfrak{E}_p(I)$ and hence by [1, Proposition 1.8.2(ii)], $D(I^i)=0$. From this we have

$$D(A\,I^i)=D(A)\,I^i+AD\,(I^i)=D(A)\,I^i,$$

and thus $D\,(\mathbb{M}_{d_i}(\mathbb{C}))\subseteq\mathbb{M}_{d_i}(\mathbb{C})$. Now define $D_i:\mathbb{M}_{d_i}(\mathbb{C})\to\mathbb{M}_{d_i}(\mathbb{C})$ by $D_i(A)=(D(A^i))_i$ for each $A\in\mathbb{M}_{d_i}(\mathbb{C})$ Of course D_i is a well–defined continuous derivation. By super amenability of $\mathbb{M}_{d_i}(\mathbb{C})$ [1, theorem 1.9.21], there exists $E\in\mathbb{M}_{d_i}(\mathbb{C})$ such that $D_i=ad_{E_i}$. For each finite set F of I define I_F by

$$(I_F)_i=\begin{cases} I_i & \text{for } i\in F\\ 0 & \text{otherwise.}\end{cases}$$

If $\mathcal{F}=\{F\subseteq I: F \text{ is finite}\}$, then $\mathcal{F}$ is a directed set with

$$F_1\le F_2 \text{ if and only if } F_1\subseteq F_2.$$

Recall that $(I_F)_F$ is an approximate identity for $\mathfrak{E}_p(I)$ [7, Theorem 4.3]. Now for each finite set F of I' we define

$$(E_F)_i=\begin{cases} E_i & \text{for } i\in F\\ 0 & \text{otherwise.}\end{cases}$$

Let $A\in\mathfrak{E}_p(I)$. For $i\in F$,

$$D(A\,I_F)_i=(D(A_i^i))_i=D_i(A_i)=A_iE_i-E_iA_i=(AE_F-E_FA)_i\,,$$

and for $i\notin F$,

$$D(A\,I_F)_i=0=(AE_F-E_FA)_i\,,$$

this clearly forces

$$D(A\,I_F)_i=AE_F-E_FA\,.$$

Then

$$D(A)=D(\lim_F AI_F)=\lim_F D(AI_F)=\lim_F(AE_F-E_FA)\,.$$

But $(E_F)\subset\mathfrak{E}_q(I')$, and accordingly $\mathfrak{E}_p(I)$ is approximately ideally amenable.

Example 2.8. Let I be an arbitrary set. For $1\le p<\infty$, $\ell^p(I)$ with pointwise multiplication is equal to $\mathfrak{E}_p(I)$, whenever for each $i\in I$ we take $H_i=\mathbb{C}$ and $a_i=1$. Therefore by the above theorem ($\ell^p(I)$, .) is an approximately ideally amenable Banach algebra.

3. APPLICATIONS TO COMPACT GROUPS

Let G be a compact group, and λ be the normalized Haar measure on G, then the function space $L^2(G,\lambda) = L^2(G)$ is a Banach algebra under convolution product. For more information see [5, Theorem 28.46]. In this section we present some applications of theorem 2.7, to a number of interesting convolution Banach algebras on compact groups.

Corollary 3.1. *Let G be a compact group. Then the Banach algebra* $L^2(G,*)$ *is approximately ideally amenable.*

Proof. Note that by [5, Peter–Weyl theorem 28.43], the Banach algebra $L^2(G)$ is isometrically algebra isomorphic with $\mathfrak{E}_2(\hat{G})$. Now by theorem 2.7, $\mathfrak{E}_2(\hat{G})$ and accordingly $L^2(G)$ is approximately ideally amenable.

Corollary 3.2. *Let G be a compact group. Then the Banach algebra* (A(G),*) *is approximately ideally amenable.*

Proof. As noted in the introduction, A(G) is isometrically algebra isomorphic with $\mathfrak{E}_1(\hat{G})$ and by theorem 2.7, the rest is evident.

Theorem 3.3. *Approximate ideal amenability (resp. ideal amenability) and approximate weak amenability (resp. weak amenability)* of $\mathfrak{E}_p(I)$ $(1 \le p < \infty)$ *are equivalent.*

Proof. Suppose that $\mathfrak{E}_p(I)$, is approximately weakly amenable. If $\mathfrak{T}$ is a closed ideal of $\mathfrak{E}_p(I)$, so by corollary 2.6, $\mathfrak{T}^* = \widetilde{\mathfrak{E}_q(I')}$. Let D: $\mathfrak{E}_p(I) \to \widetilde{\mathfrak{E}_q(I')} \subseteq \mathfrak{E}_q(I)$ be a continuous derivation. Define $\tilde{D}: \mathfrak{E}_p(I) \to \mathfrak{E}_q(I)$ by $\tilde{D}(A) = D(A)$ $(A \in \mathfrak{E}_p(I))$.

Since $\mathfrak{E}_p(I)$ is approximately weakly amenable, then there exists $(E_\alpha) \subseteq \mathfrak{E}_q(I)$ such that $\tilde{D}(A) = \lim_\alpha (AE_\alpha - E_\alpha A)$ for all $A \in \mathfrak{E}_p(I)$. Define

$$(\tilde{E}_\alpha)_i = \begin{cases} (E_\alpha)_i & \text{for } i \in I' \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\tilde{E}_\alpha \in \widetilde{\mathfrak{E}_q(I')} = \mathfrak{T}^*$. Now if $i \in I'$

$$(A\tilde{E}_\alpha - \tilde{E}_\alpha A)_i = A_i(\tilde{E}_\alpha)_i - (\tilde{E}_\alpha)_i A_i = A_i(E_\alpha)_i - (E_\alpha)_i A_i = (AE_\alpha - E_\alpha A)_i .$$

Further, if $i \notin I'$,

$$(A\tilde{E}_\alpha - \tilde{E}_\alpha A)_i = 0 ,$$

and hence $\lim_\alpha (A\tilde{E}_\alpha - \tilde{E}_\alpha A)_i = 0$. From $D(\mathfrak{E}_p(I)) \subseteq \widetilde{\mathfrak{E}_q(I')}$ we have $(D(A)_i) = 0$ $(i \notin I')$. Then for $i \notin I'$,

$$0 = (D(A)_i) = \left(\lim_\alpha (AE_\alpha - E_\alpha A)_i\right) = \lim_\alpha (AE_\alpha - E_\alpha A)_i$$

and for $i \notin I'$,

$$\lim_\alpha (A\tilde{E}_\alpha - \tilde{E}_\alpha A)_i = \lim_\alpha (AE_\alpha - E_\alpha A)_i .$$

From this D is approximately inner and $\mathfrak{E}_p(I)$) is approximately ideally amenable.

Theorem 3.4. *Let G be a compact group. The following assertions are equivalent:*

(i) (L^2(G),*) *is ideally amenable*.
(ii) (L^2(G),*) *is weakly amenable.*
(iii) G *is finite or abelian.*

Proof. Since $L^2(\mathrm{G}) = \mathfrak{C}_2(\hat{\mathrm{G}})$, so by theorem 3.3, (i) and (ii) are equivalent. Let G be a finite group, then $(L^2(\mathrm{G}),*) = (\ell^2(\mathrm{G}),*) = (\ell^1(\mathrm{G}),*)$ But $(\ell^1(\mathrm{G}),*)$ is weakly amenable and so (iii) implies (ii).

Let G be an abelian group. By corollary 3.1, L^2(G) is approximately ideally amenable and hence each derivation D: $L^2(\mathrm{G}) \to L^2(\mathrm{G})^*$ is zero. This means that (iii) implies (ii).

Let G be an infinite and non–abelian group. Then there exist $x, y \in$ G such that $xy \neq yx$. The mapping $\mathrm{D}_x : L^2(\mathrm{G}) \to L^2(\mathrm{G})^*$ defined by

$$\mathrm{D}_x(f) = \delta_x * f - f * \delta_x \quad (f \in L^2(\mathrm{G})),$$

is a non-inner derivation. To see this we refer the reader to [3, Remark 3.2]. Therefore, (G) is not weakly amenable. This proves that (ii) implies (iii).

Corollary 3.5. *Let G be a non-abelian infinite group. Then* (L^2(G),∗) *is an approximately ideally amenable Banach algebra which is not ideally amenable.*

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