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POSSIBILISTIC OPTIMIZATION WITH APPLICATION TO PORTFOLIO SELECTION

Costin-Ciprian POPESCU, Cristinca FULGA

Department of Mathematics, Academy of Economic Studies, Bucharest, Romania
E-mail: ciprian.popescu@csie.ase.ro

This paper proposes a type of fuzzy numbers called *superior-LR-piecewise* (*s-LR*), whose theoretical properties are investigated. We also propose a chance constraint-type possibilistic programming model; the coefficients of the model are *s-LR* fuzzy numbers and a Value at Risk (*VaR*)-type *s-LR* parameter which controls the chances (in possibilistic terms) of obtaining unsatisfactory value of the objective function. The results obtained have a high degree of generality, but an immediate application may refer to the portfolio selection problem for which both modelling and solving aspects are discussed.

Key words: Fuzzy number; Possibilistic mean value; Linear programming; Portfolio selection.

1. INTRODUCTION

The theory of fuzzy sets, introduced by Zadeh [17], has been applied in various fields of science. One of them is Financial Mathematics, and, more precisely, the portfolio selection issue [6, 8, 10, 16]. One of the main theoretical tool used is the fuzzy mathematical programming (FMP) [9, 15]. In some situations, the fuzzy approach is used as an alternative to classical probabilistic approach [2, 8, 15], in other more complex (multistage) situations, both methods are used in successive steps [5]. In this work, our goal is to create a new framework for the portfolio optimization problem. We first propose a special type of fuzzy numbers called *superior-LR-piecewise* (*s-LR*) whose theoretical properties are investigated. By defining this new fuzzy concept, we better capture the two major aspects in modelling the securities rate of returns in portfolio optimization: the statistical data information and the experts' knowledge. We also propose a chance constraint-type possibilistic programming model in which the coefficients are *s-LR* fuzzy numbers. We use a Value at Risk (VaR)-type *s-LR* parameter which controls the chances (in possibilistic terms) of obtaining unsatisfactory total return. Since our approach provides a model that can be reduced to solving several linear programming models, the benefit is not only theoretical but also computational.

2. PRELIMINARY THEORETICAL RESULTS

Definition 1. *Consider a subset* F_1 *of the whole set of fuzzy numbers denoted by* F *. A fuzzy number* $\widetilde{a}$ *is considered to belong to the set* F_1 *if the following requirements are met: there exist the sets* $A, B \subset \mathbb{R}$, $A = \left\{ l_i \,\middle|\, i = \overline{1,4}, l_i < l_{i+1} \right\}$, $B = \left\{ r_j \,\middle|\, j = \overline{1,4}, r_j < r_{j+1} \right\}$, $l_4 < r_5$, *and* $\widetilde{a}$ *is characterized by the membership function* $a : \mathbb{R} \to [0,1]$*, with* $a(x) = 0$ *for* $x \in (-\infty, l_1) \cup (r_8, \infty)$ *and*

$$a(x)=\begin{cases}0.5(x-l_1)(l_2-l_1)^{-1}, & x\in[l_1,l_2]\\ 0.25(x-l_k)(l_{k+1}-l_k)^{-1}+4^{-1}k, & x\in[l_k,l_{k+1}], \quad k=\overline{2,3}\\ 1, & x\in[l_4,r_4]\\ 0.25(x-r_{k+1})(r_k-r_{k+1})^{-1}-4^{-1}k+2, & x\in[r_k,r_{k+1}], \quad k=\overline{5,6}\\ 0.5(x-r_8)(r_7-r_8)^{-1}, & x\in[r_7,r_8]\ .\end{cases} \tag{1}$$

We also note the number $\tilde{a}=((l_1,...,l_4),(r_5,...,r_8))$. As in [3] and [7], the fuzzy number $\tilde{a}\in F_1$ can also be written in the parametrized form $\tilde{a}\equiv\{(\underline{a}(t),\overline{a}(t),t)|t\in[0,1]\}$, where $\underline{a}(t),\overline{a}(t):[0,1]\to\mathbb{R}$ and

$$\underline{a}(t)=\begin{cases}2t(l_2-l_1)+l_1, t\in[0,0.5]\\ 4t(l_3-l_2)+(3l_2-2l_3), t\in[0.5,0.75],\\ 4t(l_4-l_3)+(4l_3-3l_4), t\in[0.75,1],\end{cases} \quad \overline{a}(t)=\begin{cases}2t(r_7-r_8)+r_8, t\in[0,0.5]\\ 4t(r_6-r_7)+(3r_7-2r_6), t\in[0.5,0.75]\\ 4t(r_5-r_6)+(4r_6-3r_5), t\in[0.75,1].\end{cases} \tag{2}$$

The addition and the scalar multiplication (the scalars are real numbers) can be expressed as

$$\tilde{a}+\tilde{b}\equiv\{(\underline{a}(t)+\underline{b}(t),\overline{a}(t)+\overline{b}(t),t)|t\in[0,1]\}, \tag{3}$$

$$k\tilde{a}\equiv\{(k\underline{a}(t),k\overline{a}(t),t)|t\in[0,1]\}, \text{ if } k\in\mathbb{R},k>0, \tag{4}$$

$$k\tilde{a}\equiv\{(-k\overline{a}(t),-k\underline{a}(t),t)|t\in[0,1]\}, \text{ if } k\in\mathbb{R},k<0. \tag{5}$$

Using previous lines, the following result emerges.

PROPOSITION 1. *If* $\tilde{a},\tilde{b}\in F_1$, $\tilde{a}=((l_{a1},...,l_{a4}),(r_{a5},...,r_{a8}))$, $\tilde{b}=((l_{b1},...,l_{b4}),(r_{b5},...,r_{b8}))$, *and* $u,v\in\mathbb{R},u>0,v<0$*, then*

$$\tilde{a}+\tilde{b}=((l_{a1}+l_{b1},...,l_{a4}+l_{b4}),(r_{a5}+r_{b5},...,r_{a8}+r_{b8})), \tag{6}$$

$$u\tilde{a}=((ul_{a1},...,ul_{a4}),(ur_{a5},...,ur_{a8})),\ v\tilde{a}=((vr_{a8},...,vr_{a5}),(vl_{a4},...,vl_{a1})). \tag{7}$$

Example 1. Let $\tilde{a}=((1,4,7,8),(10,15,16,20))$ and $\tilde{b}=((1,2,5,6),(9,11,14,22))$. The graphical representations of $\tilde{a}+\tilde{b}$ (Fig. 1) and $2\tilde{a}$ (Fig. 2) are below.

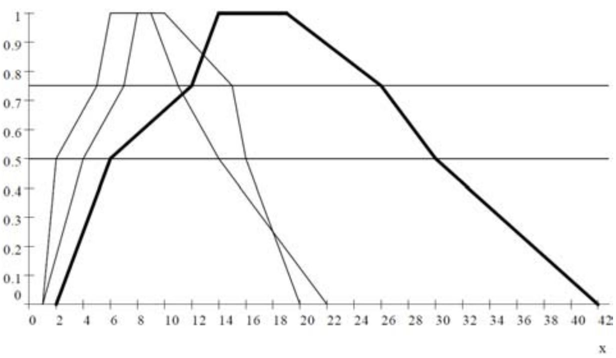

Fig. 1 – The sum of two numbers from F_1.

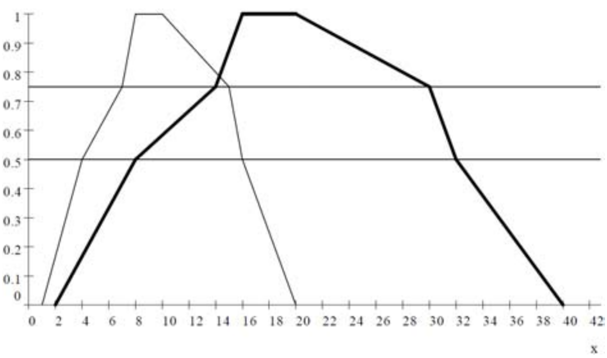

Fig. 2– Scalar multiplication.

Carlsson and Fullér [1] defined the crisp possibilistic mean value of a fuzzy number $\tilde{a}$ as

$$E(\tilde{a})=\left(2\int_0^1 t\mathrm{d}t\right)^{-1}\left[\int_0^1 t\underline{a}(t)\,\mathrm{d}t+\int_0^1 t\overline{a}(t)\,\mathrm{d}t\right]=\int_0^1 t\left[\underline{a}(t)+\overline{a}(t)\right]\mathrm{d}t\,. \tag{8}$$

PROPOSITION 2. *Consider* $\tilde{a}=((l_1,\dots,l_4),(r_5,\dots,r_8))\in F_1$. *The following statement holds;*

$$E(\tilde{a})=96^{-1}\left[4(l_1+r_8)+15(l_2+r_7)+18(l_3+r_6)+11(l_4+r_5)\right]. \tag{9}$$

Proof: From relation (8), we have

$$E(\tilde{a})=J_1+J_2\,, \tag{10}$$

where $J_1=\int_0^1 t\underline{a}(t)\,\mathrm{d}t$ and $J_2=\int_0^1 t\overline{a}(t)\,\mathrm{d}t$.

On the other hand, we have

$$J_1=\int_0^{0.5} t\left[2t(l_2-l_1)+l_1\right]\mathrm{d}t+\int_{0.5}^{0.75} t\left[4t(l_3-l_2)+(3l_2-2l_3)\right]\mathrm{d}t+\int_{0.75}^{1} t\left[4t(l_4-l_3)+(4l_3-3l_4)\right]\mathrm{d}t=$$

$$=96^{-1}(4l_1+15l_2+18l_3+11l_4).$$

J_2 is calculated similarly and using (10), the final result is obtained.

In Dubois and Prade [4] and Liu [11] the possibility of $\tilde{a}\le\tilde{b}$ (for $\tilde{a},\tilde{b}\in F$) and the possibility of $\tilde{a}\le k$ (for $\tilde{a}\in F$, $k\in\mathbb{R}$) are defined as

$$\mathrm{Pos}(\tilde{a}\le\tilde{b})=\sup\left\{\min(a(x),b(y))\,\middle|\,x,y\in\mathbb{R},x\le y\right\}, \tag{11}$$

$$\mathrm{Pos}(\tilde{a}\le k)=\sup\left\{a(x)\,\middle|\,x\in\mathbb{R},x\le k\right\}. \tag{12}$$

PROPOSITION 3. *If* $\tilde{a}=((l_1,\dots,l_4),(r_5,\dots,r_8))\in F_1$ *then*

$$\mathrm{Pos}(\tilde{a}\le 0)=\begin{cases}0, & l_1\ge 0\\ -0.5l_1(l_2-l_1)^{-1}, & l_1\le 0\le l_2\\ -0.25l_i(l_{i+1}-l_i)^{-1}+4^{-1}i, & l_i\le 0\le l_{i+1}, \quad \text{for } i=\overline{2,3}\\ 1, & l_4\le 0\,.\end{cases} \tag{13}$$

THEOREM 1. *Consider* p *in* (0, 1) *and a fuzzy number* $\tilde{a}=((l_1,\dots,l_4),(r_5,\dots,r_8))\in F_1$. *Then* $\mathrm{Pos}(\tilde{a}\le 0)\le p$ *if and only if*

(i) $(1-2p)l_1+2pl_2\ge 0$, *when* $l_1\le 0\le l_2$;

(ii) $(3-4p)l_2+(-2+4p)l_3\ge 0$, *when* $l_2\le 0\le l_3$;

(iii) $(4-4p)l_3+(-3+4p)l_4\ge 0$, *when* $l_3\le 0\le l_4$.

Proof: (i) Assume $\mathrm{Pos}(\tilde{a}\le 0)\le p$. Taking into account (13), we obtain $-0.5l_1(l_2-l_1)^{-1}\le p$. Since $l_2-l_1>0$, we have $2p(l_2-l_1)+l_1\ge 0$, which leads to $(1-2p)l_1+2pl_2\ge 0$. Reciprocally, if $(1-2p)l_1+2pl_2\ge 0$, the conclusion is obtained by reasoning in reverse order. Analogously, we obtain (ii) and (iii).

THEOREM 2. *Let* $\tilde{a} = ((l_1,...,l_4),(r_5,...,r_8)) \in F_1$ *be a fuzzy number and* $p \in (0,1)$ *a real number. Then the following assertions hold.*

(i) *If* $l_1 \geq 0$ *then* $\mathrm{Pos}(\tilde{a} \leq 0) \leq p$ *;*

(ii) *If* $l_1 < 0$ *then* $\mathrm{Pos}(\tilde{a} \leq 0) \leq p$ *if and only if* $\exists i \in \{1,2,3\}$ *such that* $l_i l_{i+1} \leq 0 \leq (1-h_i) l_i + h_i l_{i+1}$, *where* $h_i = p(-i^2 + 5i - 2) + (0.5i^2 - 3.5i + 3)$.

Proof: (i) From (13) we get $l_1 \geq 0 \Rightarrow \mathrm{Pos}(\tilde{a} \leq 0) = 0 \leq p$.

(ii) We suppose that $\mathrm{Pos}(\tilde{a} \leq 0) \leq p$. If $l_4 \leq 0$ then $\mathrm{Pos}(\tilde{a} \leq 0) = 1 > p$. Thus, $l_4 > 0$. We have the following possible situations: $l_1 < 0 \leq l_2$, $l_2 \leq 0 < l_3$, or $l_3 \leq 0 < l_4$; $l_1 < 0 \leq l_2$, results in $l_1 l_2 \leq 0$ and $\mathrm{Pos}(\tilde{a} \leq 0) = -0.5 l_1 (l_2 - l_1)^{-1} \leq p$. Hence $0 \leq (1-2p) l_1 + 2p l_2$, or $0 \leq (1-h_1) l_1 + h_1 l_2$. Reciprocally, for $i = 1$, we have $l_1 l_2 \leq 0 \leq (1-h_1) l_1 + h_1 l_2$ and $l_1 < 0 \leq l_2$. On the other hand, from the right side of the previous double inequality, we deduce $\mathrm{Pos}(\tilde{a} \leq 0) \leq p$. The other cases ($i = 2$ and $i = 3$) are similar.

3. THE POSSIBILISTIC PROGRAMMING MODEL

Consider the possibilistic optimization model

$$\max_{\mathbf{x} \in \Lambda} \left[E\left(\sum_{j=1}^{n} x_j \tilde{a}_j \right) \right], \tag{14}$$

where

$$\Lambda = \left\{ \mathbf{x} \in \mathbf{R}^n \middle| \mathrm{Pos}\left(\sum_{j=1}^{n} x_j \tilde{a}_j \leq \tilde{b} \right) \leq p, \sum_{j=1}^{n} x_j = 1, \alpha_j \leq x_j \leq \beta_j, \forall j = \overline{1,n} \right\}, \tag{15}$$

and α_j, β_j are real numbers which belong to the closed interval $[0,1]$, $\tilde{a}_j$ are basically any shape fuzzy numbers and $\mathbf{x} = (x_1, x_2, ..., x_n)^{\mathrm{T}} \in \mathbb{R}^n$ represents the decision vector. The model (14) cannot be solved in this general form. In literature, triangular and trapezoidal numbers are preferred because they are convenient to use. The problem is that this choice leads to limited results in true reflection of the phenomena studied. By modelling the uncertain parameters in the model as *s-LR* fuzzy numbers, we better capture the two major aspects: the statistical data information and the experts' knowledge. Therefore, in (14) we consider $\tilde{a}_j$ and *s-LR* fuzzy numbers, i.e., of the form $\tilde{a}_j = ((l_{j1},...,l_{j4}),(r_{j5},...,r_{j8})) \in F_1$, for all $j = \overline{1,n}$, respectively $\tilde{b} = ((b_1,...,b_4),(b_5,...,b_8))$. Thus, by Theorems 1-2, the equivalent form of model (14) is

$$\max_{\mathbf{x} \in \Gamma} \frac{1}{96} \sum_{j=1}^{n} \left[4(l_{j1} + r_{j8}) + 15(l_{j2} + r_{j7}) + 18(l_{j3} + r_{j6}) + 11(l_{j4} + r_{j5}) \right] x_j, \tag{16}$$

where $\Gamma = \bigcup_{i=1}^{4} \Gamma_i$, the sets Γ_i, $i = \overline{1,4}$ are disjoint and

$$\Gamma_1 = \left\{ \mathbf{x} \in \mathbb{R}^n \middle| \sum_{j=1}^{n} x_j = 1, \alpha_j \leq x_j \leq \beta_j \right\}, \tag{17}$$

$$\Gamma_i = \left\{ \mathbf{x} \in \mathbb{R}^n \middle| \sum_{j=1}^{n} \left[(1-h_{i-1}) l_{j,i-1} + h_{i-1} l_{ji} \right] x_j \geq (1-h_{i-1}) b_{10-i} + h_{i-1} b_{9-i}, \sum_{j=1}^{n} x_j = 1, \alpha_j \leq x_j \leq \beta_j \right\}, \forall i = \overline{2,4}. \tag{18}$$

So, the final form of the initial possibilistic model is a linear optimization problem. At this point, the solution can be obtained with well established techniques such as the simplex method.

4. APPLICATION TO THE PORTFOLIO SELECTION PROBLEM

4.1. The portfolio selection model

In the probabilistic framework, since the introduction of mean-variance portfolio selection models in the work of Markowitz [12, 13], variance has been accepted as a risk measure. However, over time, other measures of risk with better properties have been introduced. The most representative of them is the Value-at-Risk (VaR) [14, 16]. In terms of losses, the Value at Risk is defined as the amount of loss such that the probability of running a loss this large or even larger over a certain period of time, is limited. If our model formulation is considered in terms of gains (return), the definition of VaR can be written analogously. A Mean-VaR efficient portfolio [8], maximizes the portfolio total return and, simultaneously minimizes the VaR at a specified confidence level. Solving a bi-objective problem is a difficult task and even more as VaR is itself the optimal value of a minimization problem. Therefore, a value V of the minimum accepted return is specified and the following model is considered, namely,

$$\max_{\mathbf{x}\in\Psi}\left[E\left(\sum_{j=1}^{n} R_j x_j\right)\right], \tag{19}$$

where

$$\Psi=\left\{\mathbf{x}\in\mathbb{R}^n \,\middle|\, P\left(\sum_{j=1}^{n} R_j x_j < V\right)\le \varepsilon, \sum_{j=1}^{n} x_j = 1, \alpha_j \le x_j \le \beta_j, \forall j=\overline{1,n}\right\}. \tag{20}$$

Here P means “probability”, R_j, $j=\overline{1,n}$, are the return rates of security j expressed as random variables, x_j is the fraction of the total capital invested in the security j, α_j, β_j are the lower and upper limits of the investment in security j and ε is the given confidence level applied to the probabilistic constraint (chance constraint). The possibilistic portfolio selection model formulation corresponding to the previously mentioned probabilistic model (19) has the form

$$\max_{\mathbf{x}\in\Omega}\left[E\left(\sum_{j=1}^{n} x_j \tilde{a}_j\right)\right], \tag{21}$$

where

$$\Omega=\left\{\mathbf{x}\in\mathbb{R}^n \,\middle|\, \mathrm{Pos}\left(\sum_{j=1}^{n} x_j \tilde{a}_j \le \tilde{b}\right)\le \varepsilon, \sum_{j=1}^{n} x_j = 1, \alpha_j \le x_j \le \beta_j, \forall j=\overline{1,n}\right\}, \tag{22}$$

The return rates $\tilde{a}_j$ ($j=\overline{1,n}$), and the predetermined acceptable value of the minimum accepted return $\tilde{b}$ are modeled as *s-LR* fuzzy numbers and E represents the possibilistic mean value from Proposition 2.

4.2. Short considerations on the method

When solving the possibilistic portfolio optimization problem (21), the possibilistic constraint compels us to take into consideration the cases presented in Theorems 1 and 2 and consequently, the portfolio model (21) has a similar formulation to (16) as a linear optimization problem with the feasible set being a reunion of disjoint sets like in (17)–(18). Since writing the portfolio model as a linear optimization model is straightforward (only some replacements in the notation have to be done, e.g., $\widetilde{a}_j$ is replaced by R_j, $j=\overline{1,n}$) we omit it and, in the sequel, we refer to (16)-(18) as the portfolio model. The problem (16)–(18) can be solved by direct search following the steps: 1) Initialization; 2) Checking the feasibility of the current point; 3) Checking if the value of the objective function corresponding to the current iteration is superior to the previous one; 4) Testing for the stopping criteria; 5) Stopping, or performing a search for finding the next current point, incrementing the index and continuing the algorithm from Step 2. On the other hand, we can also consider four linear programming models and use the simplex algorithm for them. The simplex algorithm provides always an exact output (a unique optimal solution, multiple solutions or no solution) for all four problems, as opposed to the approximate solution given by the direct search method. Therefore, we consider the four linear programming models

$$\max_{\mathbf{x}\in\Gamma_i}\frac{1}{96}\sum_{j=1}^{n}\left[4\left(l_{j1}+r_{j8}\right)+15\left(l_{j2}+r_{j7}\right)+18\left(l_{j3}+r_{j6}\right)+11\left(l_{j4}+r_{j5}\right)\right]x_j \tag{23}$$

($i=\overline{1,4}$), each of the feasible sets being defined in (17), respectively, (18). The performed simulations on portfolios made of securities listed at the Bucharest Stock Exchange showed the effectiveness of our method.

5. CONCLUSIONS

In this work we propose a type of fuzzy numbers called *superior-LR-piecewise* (*s-LR*), we investigate their properties and we present and discussed a chance-constraint type optimization model with fuzzy constraints and objective function expressed through the possibilistic mean value. This generalizes some previous models in literature which only used trapezoidal fuzzy numbers. Compared with the classical approach based on trapezoidal numbers, our approach provides substantial benefits: using the theoretical results obtained in this paper, the portfolio optimization problem can be transformed into a special fuzzy optimization model and moreover, into a set of crisp linear optimization problems which can be solved by classical means such as the simplex algorithm. Practical applications of our model and method are numerous, particularly where in mathematical language is necessary to introduce a subjective point of view. But the fuzzy approach do not replace but rather complements probabilistic methods. The model discussed is open to such mixed approaches and to develop it is one of our future directions of research.

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