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THE MODIFIED EXPONENTIAL-POISSON DISTRIBUTION

Vasile PREDA[*], Eugenia PANAITESCU[**], Roxana CIUMARA[***]

[*] University of Bucharest, Faculty of Mathematics and Computer Science, Bucharest, Romania
[**] "Carol Davila" University of Medicine and Pharmacy,
Department of Medical Informatics and Biostatistics, Bucharest, Romania
[***] Academy of Economic Studies, Department of Mathematics, Bucharest, Romania
E-mail: e.panaitescu@yahoo.com

The two-parameter distribution known as exponential-Poisson (EP) distribution, which has decreasing failure rate, was introduced by Kus (2007). In this paper we generalize the EP distribution and show that the failure rate of the new distribution can be decreasing or increasing. The failure rate can also be upside-down bathtub shaped. We provide closed-form expressions for the density, cumulative distribution, survival and failure rate functions. The EM algorithm is used to determine the maximum likelihood estimates and the asymptotic varinces and covariances of these estimates are obtained.

Key words: Compounding; Failure rate; EM algorithm; Maximum likelihood estimates; Modified exponential-Poisson distribution.

1. INTRODUCTION

The exponential distribution (ED) provides a simple, elegant and close form solution to many problems in lifetime testing and reliability studies. However, the ED does not provide a reasonable parametric fit for some practical applications where the underlying hazard rates are nonconstant, presenting monotone shapes. In recent years, in order to overcome such a problem, new classes of models were introduced based on modifications of the ED. Gupta and Kundu (1999) proposed a generalized ED. This extended family can accommodate data with increasing and decreasing failure rate functions. Adamidis and Loukas (1998) introduced a distribution with decreasing failure rate. This distribution is known as exponential-geometric distribution and is obtained by compouding an exponential distribution with a geometric distribution. In the same fashion, Kus (2007) introduced a two-parameter distribution known as exponential-Poisson (EP) distribution, which has decreasing failure rate, by compouding an exponential distribution with a Poisson distribution. While Barreto-Souza and Cribari-Neto (2009) generalizes the distribution proposed by Kus (2007) by including a power parameter in his distribution.

In this paper, we propose a new distribution family based on the ED with increasing or decreasing failure rate function. Its genesis is stated on a parameterization scheme for a survival function proposed by Marshall and Olkin (1997).

The paper is organized as follows. Sections 1 and 2 present the new distribution and its properties and Section 3 outlines an EM-type algorithm for maximum likelihood estimation.

2. GENESIS OF THE DISTRIBUTION

Marshall and Olkin (1997) introduced a parameterization scheme for a distribution function $F(y,a)$ by defining another distribution function

$$F(y,a)=\frac{F(y)}{F(y)+a(1-F(y))},\ y\in\mathbb{R},\ a>0\,.$$

We use this parameterization to obtain the modified exponential distribution function $\hat{F}(x,\alpha,\beta)=\dfrac{1-e^{-\beta x}}{1-(1-\alpha)e^{-\beta x}}$, $x>0$, $\alpha,\beta>0$.

Let $W_1,W_2,W_3\ldots,W_Z$ be a random sample from modified exponential distribution with density $\hat{f}(w,\alpha,\beta)=\dfrac{\alpha\beta e^{-\beta w}}{\left(1-\bar{\alpha}e^{-\beta w}\right)^2}$, where $w,\alpha,\beta>0$ and $\bar{\alpha}=1-\alpha$, Z a zero truncated Poisson variable with probability function

$$P(z;\lambda)=\frac{e^{-\lambda}\lambda^z}{\Gamma(z+1)(1-e^{-\lambda})},\ z\in\mathbb{N},\ \lambda>0,$$

where $\Gamma(\cdot)$ is the gamma function and Z and W are independent.

Let $X=\min(W_1,W_2,\ldots,W_Z)$. Then $\hat{f}(x\,|\,z,\alpha,\beta)=\dfrac{\alpha^z\beta z e^{-\beta z x}}{\left(1-\bar{\alpha}e^{-\beta x}\right)^{z+1}}$ and the marginal probability density function of X is

$$f(x,\alpha,\beta,\lambda)=\frac{\alpha\beta\lambda e^{-\lambda-\beta x+\frac{\alpha\lambda e^{-\beta x}}{1-\bar{\alpha}e^{-\beta x}}}}{\left(1-\bar{\alpha}e^{-\beta x}\right)^2\left(1-e^{-\lambda}\right)},\qquad(1)$$

where $x>0$ and $\alpha,\beta,\lambda>0$.

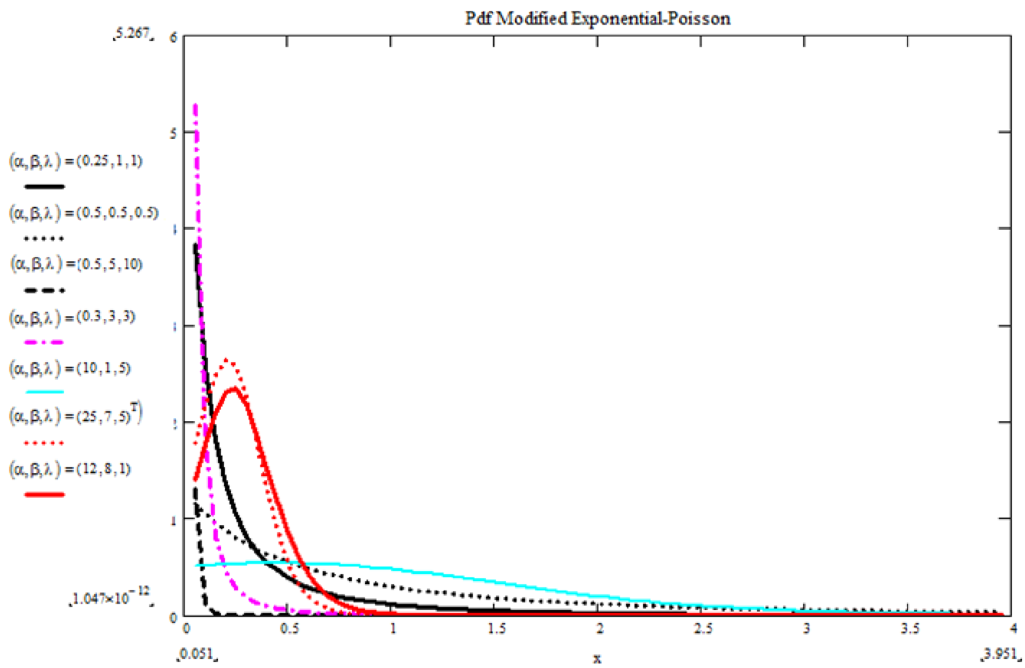

Fig. 1 – Probability density function of MEP distribution.

In the sequel, distribution of X will be referred to as the modified exponential-Poisson distribution (MEP) which is customary for such names to be given to the distribution arising via the operation of compounding in the literature. The EP distribution introduced by Kus (2007) is a particular case of the MEP distribution corresponding $\alpha=1$.

It can be seen that the MEP density function arising is bell-shaped for $\alpha > 2$ and $0 < \lambda < \min\left(\frac{2\bar{\alpha}^2}{\alpha}, \alpha - 2\right)$ with modal value $\frac{\alpha\beta\lambda t_0 e^{-\lambda - \frac{\alpha\lambda t_0}{(1-\bar{\alpha}t_0)}}}{(1-\bar{\alpha}t_0)^2(1-e^{-\lambda})}$ at $x_0 = -\frac{1}{\beta}\ln(t_0)$, where $t_0 = \frac{\alpha\lambda + \sqrt{\alpha^2\lambda^2 + 4\bar{\alpha}^2}}{2\bar{\alpha}^2}$. If the condition is false then the MEP density function is monotone decreasing (reverse J-shaped) with modal value $\frac{\alpha\beta\lambda e^{-\lambda - \frac{\alpha\lambda}{1-\bar{\alpha}}}}{(1-\bar{\alpha})^2(1-e^{-\lambda})}$ at $x = 0$.

MEP probability density function is displayed in Fig. 1 for selected parameter values.

1. PROPERTIES OF THE DISTRIBUTION

The distribution function is given by

$$F(x;\alpha,\beta,\lambda) = \frac{1 - e^{-\lambda + \frac{\alpha\lambda e^{-\beta x}}{1-\bar{\alpha}e^{-\beta x}}}}{1 - e^{-\lambda}}. \tag{2}$$

The q^{th} quantile x_q can be obtained from (2) as

$$x_q = \frac{1}{\beta}\ln\left(\left(\frac{\alpha\lambda}{\ln\left(\frac{1+(q-1)e^{\lambda}}{q}\right)} + \bar{\alpha}\right)\right).$$

In particular, the median is

$$x_{\frac{1}{2}} = \frac{1}{\beta}\ln\left(\left(\frac{\alpha\lambda}{\ln\left(\frac{1+e^{\lambda}}{2}\right)} + \bar{\alpha}\right)\right)$$

The r^{th} raw moment of the MEP distribution can be written as

$$E(X^r;\alpha,\beta,\lambda) = \frac{\alpha\lambda e^{-\lambda}\Gamma(r+1)}{\beta^r(1-e^{-\lambda})}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+m+1}{k}\frac{\alpha^m\bar{\alpha}^k\lambda^m}{m!(m+k+1)^{r+1}}.$$

Hence the mean and variance of MEP distribution are given, respectively, by

$$E(X;\alpha,\beta,\lambda) = \frac{\alpha\lambda e^{-\lambda}}{\beta(1-e^{-\lambda})}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+m+1}{k}\frac{\alpha^m\bar{\alpha}^k\lambda^m}{m!(m+k+1)^2},$$

$$\mathrm{Var}(X;\alpha,\beta,\lambda) = \frac{2\alpha\lambda e^{-\lambda}}{\beta^2(1-e^{-\lambda})}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+m+1}{k}\frac{\alpha^m\bar{\alpha}^k\lambda^m}{m!(m+k+1)^3} -$$

$$-\left(\frac{\alpha\lambda e^{-\lambda}}{\beta(1-e^{-\lambda})}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+m+1}{k}\frac{\alpha^m\bar{\alpha}^k\lambda^m}{m!(m+k+1)^2}\right)^2.$$

Using (1) and (2), the survival function (also known reliability function) and hazard function (also known as failure rate function) of MEP distribution are given, respectively, by

$$s(x;\alpha,\beta,\lambda)=\frac{1-e^{\frac{\alpha\lambda e^{-\beta x}}{1-\bar{\alpha}e^{-\beta x}}}}{1-e^{\lambda}}, \tag{3}$$

$$h(x;\alpha,\beta,\lambda)=\frac{\alpha\beta\lambda\left(1-e^{\lambda}\right)e^{-\lambda-\beta x+\frac{\alpha\lambda e^{-\beta x}}{1-\bar{\alpha}e^{-\beta x}}}}{\left(1-\bar{\alpha}e^{-\beta x}\right)^{2}\left(1-e^{-\lambda}\right)\left(1-e^{\frac{\alpha\lambda e^{-\beta x}}{1-\bar{\alpha}e^{-\beta x}}}\right)}. \tag{4}$$

We shall now show that the failure rate of the GEP distribution can be decreasing or increasing depending on the parameter values.

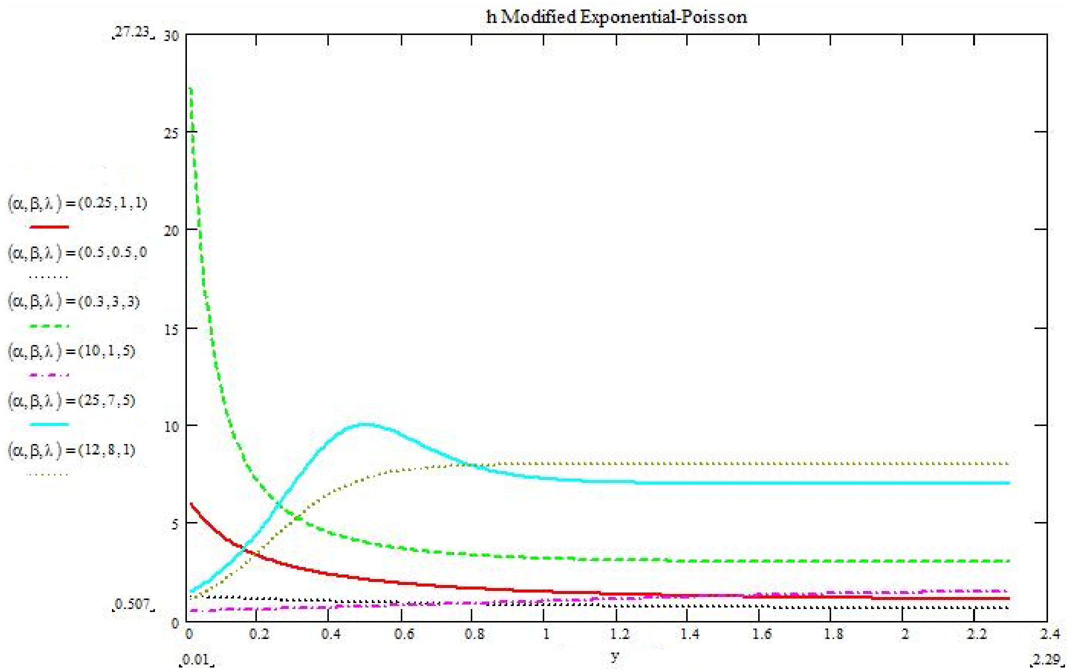

Fig. 2 – Hazard function of MEP distribution.

Define $\eta(x)=\dfrac{f^{'}(x)}{f(x)}$, where $f^{'}$ denotes the first derivative of f. It is straightforward to show that

$$\eta(x)=\frac{-\beta\left(1+\alpha\lambda e^{-\beta x}-\bar{\alpha}^{2}e^{-2\beta x}\right)}{\left(1-\bar{\alpha}e^{-\beta x}\right)^{2}}$$

and

$$\eta'(x)=-\beta^{2}e^{-\beta x}\,\frac{\left(\alpha\lambda-2\bar{\alpha}\right)\left(1-\bar{\alpha}e^{-\beta x}\right)-2\alpha\lambda}{\left(1-\bar{\alpha}e^{-\beta x}\right)^{3}}.$$

If $\alpha>1$ (then $\bar{\alpha}<0$) for $2\bar{\alpha}+\alpha\lambda>0$ and $x>\dfrac{1}{\beta}\ln\dfrac{\bar{\alpha}(2\bar{\alpha}-\alpha\lambda)}{2\bar{\alpha}+\alpha\lambda}$, we have $\eta'(x)>0$ and for $2\bar{\alpha}+\alpha\lambda>0$ and $0<x<\dfrac{1}{\beta}\ln\dfrac{\bar{\alpha}(2\bar{\alpha}-\alpha\lambda)}{2\bar{\alpha}+\alpha\lambda}$, we have $\eta'(x)<0$.

If $0 < \alpha < 1$ (then $\overline{\alpha} > 0$) for $2\overline{\alpha} - \alpha\lambda > 0$ and $0 < x < \frac{1}{\beta}\ln\frac{\overline{\alpha}(2\overline{\alpha} - \alpha\lambda)}{2\overline{\alpha} + \alpha\lambda}$, we have $\eta'(x) > 0$ and for $2\overline{\alpha} - \alpha\lambda > 0$ and $x > \frac{1}{\beta}\ln\frac{\overline{\alpha}(2\overline{\alpha} - \alpha\lambda)}{2\overline{\alpha} + \alpha\lambda}$, we have $\eta'(x) > 0$.

It follows from Theorem (b) of Glaser (1980) that if $\eta'(x) > 0$ the failure rate is increasing and if $\eta'(x) < 0$ the failure rate is decreasing.

The mean residual life of the MEP distribution is given by

$$m(x_0;\alpha,\beta,\lambda) = E\left(X - x_0 \mid X \geq x_0;\alpha,\beta,\lambda\right) = \frac{1}{\mathrm{e}^{\frac{\alpha\lambda\mathrm{e}^{-\beta x_0}}{1-\overline{\alpha}\mathrm{e}^{-\beta x_0}}} - 1}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{\alpha^m\overline{\alpha}^k\lambda^m\Gamma(m+j)}{m!\,j!\,\Gamma(m)(m+j)}\mathrm{e}^{-(m+j)x_0}.$$

2. ESTIMATION OF THE PARAMETERS

The log-likelihood function based on observed sample size n, $y_{obs} = (x_i; i = 1,2,...,n)$ from MEP distribution is given by

$$l_n(\alpha,\beta,\lambda;y_{obs}) = n\ln\alpha + n\ln\beta + n\ln\lambda - n\ln\left(1-e^{-\lambda}\right) - n\lambda - \beta\sum_{i=1}^{n}x_i + $$
$$+\alpha\lambda\sum_{i=1}^{n}\frac{\mathrm{e}^{-\beta x_i}}{1-\overline{\alpha}\mathrm{e}^{-\beta x_i}} - 2\sum_{i=1}^{n}\ln(1-\overline{\alpha}\mathrm{e}^{-\beta x_i})$$

and subsequently the associated gradients are found to be

$$\frac{\partial l_n(\alpha,\beta,\lambda;y_{obs})}{\partial\alpha} = \frac{n}{\alpha} + \lambda\sum_{i=1}^{n}\frac{(1-\mathrm{e}^{-\beta x_i})\mathrm{e}^{-\beta x_i}}{\left(1-\overline{\alpha}\mathrm{e}^{-\beta x_i}\right)^2} - 2\sum_{i=1}^{n}\frac{\mathrm{e}^{-\beta x_i}}{1-\overline{\alpha}\mathrm{e}^{-\beta x_i}}, \tag{5}$$

$$\frac{\partial l_n(\alpha,\beta,\lambda;y_{obs})}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}x_i - \alpha\lambda\sum_{i=1}^{n}\frac{x_i\mathrm{e}^{-\beta x_i}}{\left(1-\overline{\alpha}\mathrm{e}^{-\beta x_i}\right)^2} - 2\sum_{i=1}^{n}\frac{\overline{\alpha}x_i\mathrm{e}^{-\beta x_i}}{1-\overline{\alpha}\mathrm{e}^{-\beta x_i}}, \tag{6}$$

$$\frac{\partial l_n(\alpha,\beta,\lambda;y_{obs})}{\partial\lambda} = \frac{n}{\lambda} - n + \alpha\sum_{i=1}^{n}\frac{\mathrm{e}^{-\beta x_i}}{1-\overline{\alpha}\mathrm{e}^{-\beta x_i}} - \frac{n\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}. \tag{7}$$

We can use the first equation to express λ as a function of α and β and replace this form in the last two equations. The obtained system may be solving using an iteration scheme.

Newton-Raphson algorithm is one of the standard methods to determine the MLE of the parameters. To employ the algorithm, the second derivate of the log-likelihood are required for all iteration. EM algorithm is an iterative method by repeatedly replacing the missing data with estimated value and updating the parameter estimates. It is especially useful if the complete data set is easy to analyze. To start the algorithm, hypothetical complete-data distribution is defined with density function

$$f(x,z,\alpha,\beta,\lambda) = \frac{\alpha^z\beta z\mathrm{e}^{-\beta zx}}{\left(1-\overline{\alpha}\mathrm{e}^{-\beta x}\right)^{z+1}}\frac{\mathrm{e}^{-\lambda}\lambda^z}{\Gamma(z+1)(1-\mathrm{e}^{-\lambda})},$$

where $x > 0$, $z = 1, 2, \ldots, \beta, \alpha, \lambda > 0$.

Thus it is straightforward to verify that the E-step of an EM cycle requires the computation of the conditional expectation $\left(Z \mid X;\alpha^{(h)},\beta^{(h)},\lambda^{(h)}\right)$ where $\left(\alpha^{(h)},\beta^{(h)},\lambda^{(h)}\right)$ is the current estimate of (α,β,λ).

Using the fact that $P(z \mid x, \alpha, \beta, \lambda) = \dfrac{\alpha^{z-1}\lambda^{z-1} e^{-\beta x(z-1) - \frac{\alpha\lambda e^{-\beta x}}{1-\bar{\alpha}e^{-\beta x}}}}{\left(1-\bar{\alpha}e^{-\beta x}\right)^{z-1}\Gamma(z)}$, $z \in \mathbb{N}$, we find $E\left(Z \mid x; \alpha, \beta, \lambda\right) = 1 + \dfrac{\alpha\lambda e^{-\beta x}}{1-\bar{\alpha}e^{-\beta x}}$. The EM cycle is completed with M-step, which is complete data maximum likelihood over (α, β, λ), with missing Z's replaced by their conditional expectations $E(Z \mid x; \alpha, \beta, \lambda)$. Thus an EM iteration, taking $\left(\alpha^{(h)}, \beta^{(h)}, \lambda^{(h)}\right)$ into $\left(\alpha^{(h+1)}, \beta^{(h+1)}, \lambda^{(h+1)}\right)$ is given by

$$\alpha^{(h+1)} = n\left(\sum_{i=1}^{n} \frac{e^{-\beta^{(h)}x_i}\left(\lambda^{(h)}e^{-\beta^{(h)}x_i} - 2\bar{\alpha}^{(h)}e^{-\beta^{(h)}x_i} - \lambda^{(h)} + 2\right)}{\left(1-\bar{\alpha}^{(h)}e^{-\beta^{(h)}x_i}\right)^2}\right)^{-1},$$

$$\beta^{(h+1)} = n\left(\sum_{i=1}^{n} \frac{x_i\left(1-\left(\bar{\alpha}^{(h)}\right)^2 e^{-2\beta^{(h)}x_i} + \alpha^{(h)}\lambda^{(h)}e^{-\beta^{(h)}x_i}\right)}{\left(1-\bar{\alpha}^{(h)}e^{-\beta^{(h)}x_i}\right)^2}\right)^{-1},$$

$$\lambda^{(h+1)} = n\left(\frac{ne^{-\lambda^{(h)}}}{1-e^{-\lambda^{(h)}}} - \sum_{i=1}^{n}\left(\frac{1-e^{-\beta^{(h)}x_i}}{1-\bar{\alpha}^{(h)}e^{-\beta^{(h)}x_i}}\right)\right)^{-1}.$$

It can be seen that only a one-dimensional search such as Newton-Raphson is required for M-step of an EM cycle.

Applying the usual large sample approximation, the MLE of $\boldsymbol{\theta} = (\alpha, \beta, \lambda)$ can be treated as being approximately multivariate normal with mean $\boldsymbol{\theta}$ and variance-covariance matrix, which is the inverse of the expected information matrix $J_n(\boldsymbol{\theta}) = E(I_n; \boldsymbol{\theta})$, where $I_n = I(\boldsymbol{\theta}; y_{obs})$ is the observed information matrix with elements $\left(I_n\right)_{ij} = -\dfrac{\partial^2 l_n}{\partial\theta_i \partial\theta_j}$, $i, j = 1,2,3$, and the expectation is to be taken with respect to the distribution of X. Differentiating (5), (6) and (7), the elements of the symmetric, second-order observed information matrix are found to be

$$\left(I_n\right)_{11} = \frac{n}{\alpha^2} + 2\lambda\sum_{i=1}^{n}\frac{e^{-2\beta x_i}(1-e^{-\beta x_i})}{(1-\bar{\alpha}e^{-\beta x_i})^3} - 2\sum_{i=1}^{n}\frac{e^{-2\beta x_i}}{(1-\bar{\alpha}e^{-\beta x_i})^2},$$

$$\left(I_n\right)_{12} = \lambda\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}(1-e^{-\beta x_i} - \alpha e^{-\beta x_i})}{(1-\bar{\alpha}e^{-\beta x_i})^3} - 2\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}}{(1-\bar{\alpha}e^{-\beta x_i})^2},$$

$$\left(I_n\right)_{13} = -\sum_{i=1}^{n}\frac{(1-e^{-\beta x_i})e^{-\beta x_i}}{(1-\bar{\alpha}e^{-\beta x_i})^2},$$

$$\left(I_n\right)_{22} = \frac{n}{\beta^2} - \alpha\lambda\sum_{i=1}^{n}\frac{x_i^2 e^{-\beta x_i}(1+\bar{\alpha}e^{-\beta x_i})}{(1-\bar{\alpha}e^{-\beta x_i})^3} - 2\sum_{i=1}^{n}\frac{\bar{\alpha}x_i^2 e^{-\beta x_i}}{(1-\bar{\alpha}e^{-\beta x_i})^2},$$

$$\left(I_n\right)_{23} = \alpha\sum_{i=1}^{n}\frac{x_i\, e^{-\beta x_i}}{(1-\bar{\alpha}e^{-\beta x_i})^2},$$

$$\left(I_n\right)_{33} = \frac{n}{\lambda^2} - \frac{ne^{-\lambda}}{(1-e^{-\lambda})^2}.$$

The elements of the expected information matrix $J_n(\boldsymbol{\theta})$ are calculated by taking the expectation of $(I_n)_{ij}$, $i, j = 1, 2, 3$, with respect to the distribution of *X*, i.e the following expectation is required:

$$E_{t_1 t_2}^r = E\left(\frac{X^r e^{-bt_1 X}}{\left(1 - ae^{-bX}\right)^{t_2}}\right) = \frac{\alpha\lambda e^{-\lambda}\Gamma(r+1)}{\beta^r\left(1 - e^{-\lambda}\right)}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+m+t_2+1}{k}\frac{\alpha^m\overline{\alpha}^k\lambda^m}{m!\left(m+k+t_1+1\right)^{r+1}}.$$

Thus $J_n(\boldsymbol{\theta})$ are derived in the form

$$(J_n)_{11} = \frac{n}{\alpha^2} + 2\lambda E_{23}^0 - 2\lambda\sum_{i=1}^{n} E_{33}^0 - 2E_{22}^0,$$

$$(J_n)_{12} = \lambda E_{13}^1 - \lambda(\alpha+1)E_{23}^1 - 2E_{12}^1,$$

$$(J_n)_{13} = E_{22}^0 - E_{12}^0,$$

$$(J_n)_{22} = \frac{n}{\lambda^2} - \alpha\lambda\left(E_{13}^2 - E_{23}^2\right) - 2\overline{\alpha}E_{12}^2,$$

$$(J_n)_{23} = \alpha E_{12}^1,$$

$$(J_n)_{33} = \frac{n}{\lambda} - \frac{ne^{-\lambda}}{(1-e^{-\lambda})^2}.$$

The inverse of $J_n(\boldsymbol{\theta})$, evaluated at $\hat{\boldsymbol{\theta}}$, provides the asymptotic variance-covariance matrix of the MLEs. Alternative estimates can be obtained from the inverse of the observed information matrix since it is a consistent estimator of $J_n^{-1}(\boldsymbol{\theta})$.

Under conditions that are fulfilled for the parameter θ in the interior of the parameter space but not on the boundary, the asymptotic distribution of

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \text{ is } N_3\left(0, K^{-1}(\boldsymbol{\theta})\right),$$

where $\lim_{n\to\infty} J_n(\boldsymbol{\theta}) = K(\boldsymbol{\theta})$ is the unit information matrix. This asymptotic behaviour remains valid if $K(\boldsymbol{\theta})$ is replaced by the average sample information matrix evaluated at $\hat{\boldsymbol{\theta}}$, i.e., $J_n(\boldsymbol{\theta})$.

It is noteworthy that the multivariate normal distribution $N_3\left(0, J_n^{-1}(\boldsymbol{\theta})\right)$ can be used to construct confidence intervals for the parameters. In fact, an $(1-\gamma)\times 100\%$ ($0 < \gamma < \frac{1}{2}$) asymptotic confidence interval for the i-th parameter θ_i in θ is

$$ACI_i = \left(\hat{\theta}_i - z_{1-\gamma/2}\sqrt{\hat{J}^{\theta_i\theta_j}}, \hat{\theta}_i + z_{1-\gamma/2}\sqrt{\hat{J}^{\theta_i\theta_j}}\right),$$

where $\hat{J}^{\theta_i\theta_j}$ denotes the *i*-th diagonal element of $J_n^{-1}(\boldsymbol{\theta})$, $i = 1,2,3$ and $z_{1-\gamma/2}$ is the $1-\gamma/2$ standard normal quantile.

We shall now move to hypothesis testing inference on the parameters of the MEP law. Consider the partition $\boldsymbol{\theta} = \left(\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_1^T\right)$ of the MEP parameter vector and suppose we wish to test the hypothesis $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ against the alternative hypothesis $H_A : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$.

To that end, we can use the likelihood ratio (*LR*) test whose test statistic is given by $w = 2\left\{ l(\hat{\theta}) - l(\tilde{\theta})\right\}$, where $\hat{\boldsymbol{\theta}} = \left(\hat{\boldsymbol{\theta}}_1^{\mathrm{T}}, \hat{\boldsymbol{\theta}}_1^{\mathrm{T}}\right)$ and $\tilde{\boldsymbol{\theta}} = \left(\left(\boldsymbol{\theta}_1^{(0)}\right)^{\mathrm{T}}, \tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}\right)$ stand for the MLEs of θ under the null and the alternative hypotheses, respectively.

Under the null hypothesis, w is asymptotically (as $n \to \infty$) distributed as χ_k^2, where k is the dimension of the vector θ_1 of parameters of interest. We reject the null hypothesis at the nominal level γ ($0 < \gamma < 1$) if $w > \chi_{k,1-\gamma}^2$, where $\chi_{k,1-\gamma}^2$ is the $1-\gamma$ quartile of χ_k^2.

Using this test, one can decide between a MEP and an EP model, which can be done by testing $H_0 : \alpha = 1$ against $H_A : \alpha \neq 1$.

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