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COMPARATIVE ANALYSIS IN DYNAMICS OF THE 3-RRR PLANAR PARALLEL ROBOT

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Matrix relations for kinematics and inverse dynamics of the 3-RRR planar parallel robot are established in this paper. Three identical planar legs connecting to the moving platform are located in the same plane. Knowing the general motion of the platform, we develop first the inverse kinematics problem and determine the positions, velocities and accelerations of the robot's elements. The inverse dynamics problem is solved using the principle of virtual work, but it has been verified the results in the framework of the Lagrange equations with multipliers. Recursive equations offer expressions and graphs for the input powers of three revolute actuators and the internal forces in the joints.

Key words: Kinematics, Dynamics, Planar parallel robot, Lagrange equations, Virtual work.

LIST OF SYMBOLS

$a_{k,k-1}$ – orthogonal transformation matrix

$\varphi_{k,k-1}$ – relative rotation angle of T_k rigid body

$\vec{\omega}_{k,k-1}$ – relative angular velocity of T_k

$\tilde{\omega}_{k,k-1}$ – skew-symmetric matrix associated to the angular velocity $\vec{\omega}_{k,k-1}$

$\vec{r}_{k,k-1}$ – relative position vector of the centre A_k of joint

m_k , $\hat{J}_k$ – mass and symmetric matrix of tensor of inertia of T_k about the link-frame $x_k y_k z_k$

p_{10}^A, p_{10}^B, p_{10}^C – powers of three fixed revolute actuators

1. INTRODUCTION

Compared with serial manipulators, the followings are the potential advantages of parallel robots: higher kinematical precision, lighter weight and better stiffness, greater load bearing, stabile capacity and suitable position of arrangement of actuators [1]. Considerable efforts have been devoted to the kinematics and dynamic analysis of fully parallel manipulators. Among these, the class of manipulators known as Stewart-Gough platform focused great attention (Stewart [2]). They are used in flight simulators and more recently for Parallel Kinematics Machines. The prototype of Delta parallel robot (Clavel [3]; Tsai and Stamper [4]; Staicu [5]) as well as the Star parallel manipulator (Hervé and Sparacino [6]) are equipped with three motors which train on the mobile platform in a three-degrees-of-freedom general translation motion. Angeles [7], Wang and Gosselin [8] analysed the kinematics, dynamics and singularity loci of Agile Wrist spherical robot with three actuators.

A mechanism is said to be a *planar robot* if all the moving links in the mechanism perform the planar motions. In a planar linkage, the axes of all revolute joints must be normal to the plane of motion, while the direction of translation of a prismatic joint must be parallel to the plane of motion. Aradyfio and Qiao [9] examined the inverse kinematics solution for the three different 3-DOF planar parallel robots. Pennock and Kassner [10] present a kinematical study of a planar parallel robot, where a moving platform is connected to a fixed base by three links, each leg consisting of two binary links and three parallel revolute joints. Merlet [11] solved the forward pose kinematics problem for a broad class of planar parallel manipulators and Yang et al. [12] concentrate on the singularity analysis of a class of 3-RRR planar parallel robots developed in its laboratory.

A recursive method is introduced in the present paper, to reduce the number of equations and computation operations by using a set of matrices for kinematics and dynamics models of the planar 3-RRR parallel robot.

2. KINEMATICS ANALYSIS

Having a closed-loop structure, the planar parallel robot 3-RRR is a special symmetrical mechanism composed of three planar kinematical chains with identical topology, all connecting the fixed base to the moving platform (Fig. 1). In the actuation schema of the parallel robot ($\underline{R}RR$) with all revolute actuators installed on the fixed base, we consider the moving platform as the output link and the elements A_1A_2, B_1B_2, C_1C_2 as the input links. We attach a Cartesian frame $x_0y_0z_0(T_0)$ to the fixed base with its origin located at triangle centre O, the z_0 axis perpendicular to the base. Another mobile reference frame $x_Gy_Gz_G$ is attached to the moving platform (Fig.2).

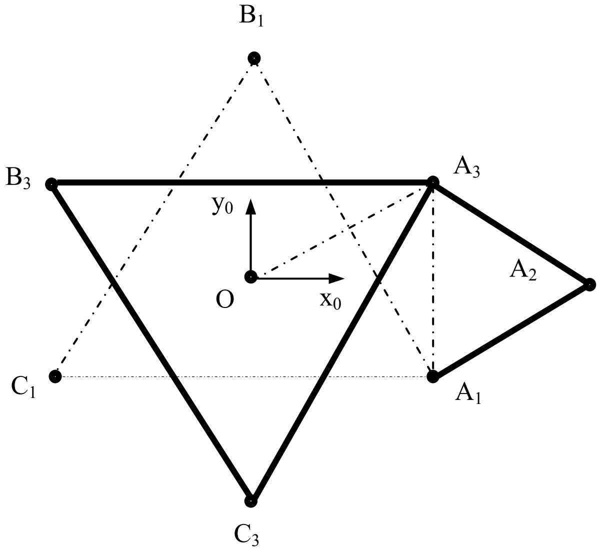

Fig. 1 – The planar 3-RRR parallel robot.

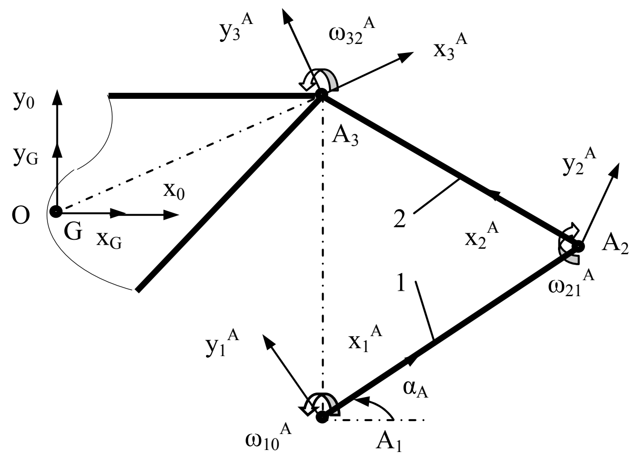

Fig. 2 – Kinematical scheme of first leg A of the mechanism.

Considering that the moving platform is initially located at a *central configuration*, we note that the relative rotation of T_k body with $\varphi_{k,k-1}$ angle must be always pointing about the direction of z_k axis. One of three active legs (for example leg A) consists of a fixed revolute joint, a moving crank **1** of length l_1, mass m_1 and tensor of inertia $\hat{J}_1$, which has rotation about z_1^A axis with the angle φ_{10}^A, the angular velocity $\omega_{10}^A = \dot{\varphi}_{10}^A$ and the angular acceleration $\varepsilon_{10}^A = \ddot{\varphi}_{10}^A$. A new element of the leg is a rigid rod **2** linked at the $x_2^A y_2^A z_2^A$ frame, having a relative rotation with the angle φ_{21}^A, velocity $\omega_{21}^A = \dot{\varphi}_{21}^A$ and acceleration $\varepsilon_{21}^A = \ddot{\varphi}_{21}^A$. It has the length l_2, mass m_2 and tensor of inertia $\hat{J}_2$. Finally, a revolute joint is introduced at the moving platform, which is schematised as an equilateral triangle congruent to the base with the edge $l = r\sqrt{3}$, mass m_3 and tensor of inertia $\hat{J}_3$ with respect to frame $A_3x_3^A y_3^A z_3^A$. At the central configuration, we also consider that all legs are symmetrically extended and that the angles giving the initial pose of the mechanism have following values

$$\alpha_A = \frac{\pi}{6},\ \alpha_B = \frac{5\pi}{6},\ \alpha_C = -\frac{\pi}{2}. \tag{2.1}$$

We call the matrix $a^{\varphi}_{k,k-1}$, for example, the orthogonal transformation 3×3 matrix of relative rotation with the angle $\varphi^A_{k,k-1}$ of link T^A_k around z^A_k axis. Using the rotation matrix $p^{\varphi}_{k,k-1} = \mathrm{rot}(z, \varphi^i_{k,k-1})$ and pursuing the independent legs one obtains the following transformation matrices

$$p_{10} = p^{\varphi}_{10}\theta^i_{\alpha},\ p_{21} = p^{\varphi}_{21}\theta_2\theta_1,\ p_{32} = p^{\varphi}_{32}\theta_2\theta_1\ \ (p = a,\ b, c;\quad i = A, B, C),, \tag{2.2}$$

where $\theta^i_{\alpha},\ \theta_1, \theta_2$ are three appropriate constant matrices. In the inverse geometric problem, the position of the mechanism is completely given through the coordinates x^G_0, y^G_0 of the mass centre G and the orientation angle ϕ of the movable frame $x_G y_G z_G$. The orthogonal rotation matrix of the moving platform from $x_0 y_0 z_0$ to $x_G y_G z_G$ reference system is $R = \mathrm{rot}(z, \phi)$.

Further, we suppose that the position vector of G centre $\vec{r}^{\,G}_0 = [x^G_0\ \ y^G_0\ \ 0]^{\mathrm{T}}$ and the orientation angle ϕ, which are expressed by following analytical functions can describe the general absolute motion of the moving platform in its *vertical plane*

$$x^G_0 = x^{G*}_0(1-\cos\frac{\pi}{3}t),\ y^G_0 = y^{G*}_0(1-\cos\frac{\pi}{3}t),\ \phi = \phi^*(1-\cos\frac{\pi}{3}t). \tag{2.3}$$

From the conditions $p^{\circ\mathrm{T}}_{30} p_{30} = R$, $p_{30} = p_{32}p_{21}p_{10}$, $\overset{\circ}{p}_{30} = p_{30}(t=0) = \theta^i_{\alpha}$ $\big(\text{with } i = A,\ B,\ C;\quad p = a,\ b,\ c\big)$, concerning the orientation of the platform one, the first relations between angles of rotation are obtained

$$\varphi^A_{10} - \varphi^A_{21} + \varphi^A_{32} = \varphi^B_{10} - \varphi^B_{21} + \varphi^B_{32} = \varphi^C_{10} - \varphi^C_{21} + \varphi^C_{32} = \phi. \tag{2.4}$$

Six variables $\varphi^A_{10},\ \varphi^A_{21}, \varphi^B_{10},\ \varphi^B_{21}, \varphi^C_{10},\ \varphi^C_{21}$ will be determined from several vector-loop equations, as follows

$$\vec{r}^{\,A}_{10} + \sum_{k=1}^{2} a^{\mathrm{T}}_{k0}\vec{r}^{\,A}_{k+1,k} + a^{\mathrm{T}}_{30}\vec{r}^{\,GA}_3 = \vec{r}^{\,B}_{10} + \sum_{k=1}^{2} b^{\mathrm{T}}_{k0}\vec{r}^{\,B}_{k+1,k} + b^{\mathrm{T}}_{30}\vec{r}^{\,GB}_3 = \vec{r}^{\,C}_{10} + \sum_{k=1}^{2} c^{\mathrm{T}}_{k0}\vec{r}^{\,C}_{k+1,k} + c^{\mathrm{T}}_{30}\vec{r}^{\,GC}_3 = \vec{r}^{\,G}_0, \tag{2.5}$$

where:

$$\vec{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \tilde{u}_3 = \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$\vec{r}^{\,A}_{10} = 0.5r[\sqrt{3}\ \ -1\ \ 0]^{\mathrm{T}},\quad \vec{r}^{\,B}_{10} = r[0\ \ 1\ \ 0]^{\mathrm{T}},\quad \vec{r}^{\,C}_{10} = 0.5r[-\sqrt{3}\ \ -1\ \ 0]^{\mathrm{T}},\quad \vec{r}^{\,i}_{21} = r\vec{u}_1, \vec{r}^{\,i}_{32} = r\vec{u}_1,\quad \vec{r}^{\,Gi}_3 = -r\vec{u}_1. \tag{2.6}$$

In the inverse kinematics modelling we compute first the linear and angular velocities of each leg in terms of the angular velocity $\vec{\omega}^G_0 = \dot{\phi}\vec{u}_3$ and the centre's velocity $\vec{v}^{\,G}_0 = \dot{\vec{r}}^{\,G}_0$ of the moving platform. The kinematics of the elements of the leg *A* are characterized by the velocities of the centres of joints A_k and the absolute angular velocities:

$$\vec{v}^{\,A}_{10} = \vec{0},\ \vec{v}^{\,A}_{20} = \dot{\varphi}^A_{10} a_{21}\tilde{u}_3\vec{r}^{\,A}_{21},\ \vec{v}^{\,A}_{30} = a_{32}\vec{v}^{\,A}_{20} + (\dot{\varphi}^A_{21} - \dot{\varphi}^A_{10})a_{32}\tilde{u}_3\vec{r}^{\,A}_{32}.$$
$$\vec{\omega}^A_{10} = \dot{\varphi}^A_{10}\vec{u}_3, \vec{\omega}^A_{20} = a_{21}\vec{\omega}^A_{10} + \vec{\omega}^A_{21} = (\dot{\varphi}^A_{21} - \dot{\varphi}^A_{10})\vec{u}_3,\ \vec{\omega}^A_{30} = a_{32}\vec{\omega}^A_{20} + \vec{\omega}^A_{32} = (\dot{\varphi}^A_{10} - \dot{\varphi}^A_{21} + \dot{\varphi}^A_{32})\vec{u}_3. \tag{2.7}$$

Equations of geometrical constraints (2.4) and (2.5) can be derivate with respect to time to obtain the following *matrix conditions of connectivity* [13]

$$\omega_{10}^{A}\vec{u}_{j}^{\mathrm{T}}a_{10}^{\mathrm{T}}\tilde{u}_{3}\{\vec{r}_{21}^{A}+a_{21}^{\mathrm{T}}\vec{r}_{32}^{A}+a_{21}^{\mathrm{T}}a_{32}^{\mathrm{T}}\vec{r}_{3}^{GA}\}+\omega_{21}^{A}\vec{u}_{j}^{\mathrm{T}}a_{20}^{\mathrm{T}}\tilde{u}_{3}\{\vec{r}_{32}^{A}+a_{32}^{\mathrm{T}}\vec{r}_{3}^{GA}\}+\omega_{32}^{A}\vec{u}_{j}^{\mathrm{T}}a_{30}^{\mathrm{T}}\tilde{u}_{3}\hat{r}_{3}^{GA}=\vec{u}_{j}^{\mathrm{T}}\dot{\vec{r}}_{0}^{G}\quad(j=1,2),$$
$$\omega_{10}^{A}-\omega_{21}^{A}+\omega_{32}^{A}=\dot{\phi}. \tag{2.8}$$

From these equations, we obtain the relative velocities ω_{10}^{A}, ω_{21}^{A}, ω_{32}^{A} as functions of basic velocities $\dot{x}_{0}^{G}$, $\dot{y}_{0}^{G}$, $\dot{\phi}$.

Now, let us assume that the robot has successively *virtual motions* determined by three sets of angular velocities $\omega_{10a}^{Av}=1$, $\omega_{10a}^{Bv}=0$, $\omega_{10a}^{Cv}=0$, $\omega_{10b}^{Av}=0$, $\omega_{10b}^{Bv}=1$, $\omega_{10b}^{Cv}=0$ and $\omega_{10c}^{Av}=0$, $\omega_{10c}^{Bv}=0$, $\omega_{10c}^{Cv}=1$. All characteristic virtual velocities are expressed as functions of the pose of the mechanism by the general kinematical equations (2.8). As for the accelerations $\varepsilon_{10}^{A},\varepsilon_{21}^{A},\varepsilon_{32}^{A}$ of the robot, new conditions of connectivity are obtained:

$$\varepsilon_{10}^{A}\vec{u}_{j}^{\mathrm{T}}a_{10}^{\mathrm{T}}\tilde{u}_{3}\{\vec{r}_{21}^{A}+a_{21}^{\mathrm{T}}\vec{r}_{32}^{A}+a_{21}^{\mathrm{T}}a_{32}^{\mathrm{T}}\vec{r}_{3}^{GA}\}+\varepsilon_{21}^{A}\vec{u}_{j}^{\mathrm{T}}a_{20}^{\mathrm{T}}\tilde{u}_{3}\{\vec{r}_{32}^{A}+a_{32}^{\mathrm{T}}\vec{r}_{3}^{GA}\}+$$
$$+\varepsilon_{32}^{A}\vec{u}_{j}^{\mathrm{T}}a_{30}^{\mathrm{T}}\tilde{u}_{3}\hat{r}_{3}^{GA}=\vec{u}_{j}^{\mathrm{T}}\ddot{\vec{r}}_{0}^{G}-\omega_{10}^{A}\omega_{10}^{A}\vec{u}_{j}^{\mathrm{T}}a_{10}^{\mathrm{T}}\tilde{u}_{3}\,\tilde{u}_{3}\{\vec{r}_{21}^{A}+a_{21}^{\mathrm{T}}\vec{r}_{32}^{A}+a_{21}^{\mathrm{T}}a_{32}^{\mathrm{T}}\vec{r}_{3}^{GA}\}-$$
$$-\omega_{21}^{A}\omega_{21}^{A}\vec{u}_{j}^{\mathrm{T}}a_{20}^{\mathrm{T}}\tilde{u}_{3}\,\tilde{u}_{3}\{\vec{r}_{32}^{A}+a_{32}^{\mathrm{T}}\vec{r}_{3}^{GA}\}-\omega_{32}^{A}\omega_{32}^{A}\vec{u}_{j}^{\mathrm{T}}a_{30}^{\mathrm{T}}\tilde{u}_{3}\tilde{u}_{3}\vec{r}_{3}^{GA}- \tag{2.9}$$
$$-2\omega_{10}^{A}\omega_{21}^{A}\vec{u}_{j}^{\mathrm{T}}a_{10}^{\mathrm{T}}\tilde{u}_{3}a_{21}^{\mathrm{T}}\,\tilde{u}_{3}\{\vec{r}_{32}^{A}+a_{32}^{\mathrm{T}}\vec{r}_{3}^{GA}\}-2\omega_{10}^{A}\omega_{32}^{A}\vec{u}_{j}^{\mathrm{T}}a_{10}^{\mathrm{T}}\tilde{u}_{3}a_{21}^{\mathrm{T}}a_{32}^{\mathrm{T}}\tilde{u}_{3}\vec{r}_{3}^{GA}-$$
$$-2\omega_{21}^{A}\omega_{32}^{A}\vec{u}_{j}^{\mathrm{T}}a_{20}^{\mathrm{T}}\tilde{u}_{3}a_{32}^{\mathrm{T}}\tilde{u}_{3}\vec{r}_{3}^{GA},\quad\varepsilon_{10}^{A}-\varepsilon_{21}^{A}+\varepsilon_{32}^{A}=\ddot{\phi}\quad(j=1,\ 2).$$

The derivatives of the relations (2.7) give the accelerations $\vec{\gamma}_{k0}^{A}$ and $\vec{\varepsilon}_{k0}^{A}$.

3. DYNAMICS EQUATIONS

Three different methods could lead to the same results concerning the input torques. The first one is the Newton-Euler approach, which consists to apply the free-body diagram procedure for each body [14]. The second method is based on the Lagrange formalism, which introduces scalar multipliers for each closure equation and the third method for the dynamics analysis is based on the principle of virtual work [15].

3.1. Principle of virtual work

Knowing the kinematics state of each link as well as the external forces acting on the robot, one applies first the principle of virtual work for the inverse dynamic problem in order to establish some matrix relations. Three electric motors that generate the moments $\vec{m}_{10}^{A}=m_{10}^{A}\vec{u}_{3}$, $\vec{m}_{10}^{B}=m_{10}^{B}\vec{u}_{3}$, $\vec{m}_{10}^{C}=m_{10}^{C}\vec{u}_{3}$ oriented about fixed axes, simultaneously control the motion of the moving platform. The parallel robot can artificially be transformed in a set of three open chains by introducing the corresponding constraint conditions.

The wrench of two vectors $\vec{f}_{k}^{*A}=9.81m_{k}^{A}a_{k0}\vec{u}_{3}$, $\vec{m}_{k}^{*A}=9.81m_{k}^{A}\tilde{r}_{k}^{CA}a_{k0}\vec{u}_{3}$ evaluates the action of the weight $m_{k}^{A}\vec{g}$ and eventually of other external and internal forces applied to the same element T_{k}^{A} of the mechanism. We compute also the force of inertia $\vec{f}_{k0}^{inA}$ and the resulting moment of inertia forces $\vec{m}_{k0}^{inA}$ of an arbitrary rigid body T_{k}^{A} of mass m_{k}^{A} and tensor of inertia $\hat{J}_{k}^{A}$ with respect to the centre of its first joint:

$$\vec{f}_{k0}^{inA}=-m_{k}^{A}\{\vec{\gamma}_{k0}^{A}+(\tilde{\omega}_{k0}^{A}\tilde{\omega}_{k0}^{A}+\tilde{\varepsilon}_{k0}^{A})\vec{r}_{k}^{CA}\},\ \vec{m}_{k0}^{inA}=-\{m_{k}^{A}\tilde{r}_{k}^{CA}\vec{\gamma}_{k0}^{A}+\hat{J}_{k}^{A}\vec{\varepsilon}_{k0}^{A}+\tilde{\omega}_{k0}^{A}\hat{J}_{k}^{A}\vec{\omega}_{k0}^{A}\}\quad(k=1,2,3). \tag{3.1}$$

By intermediate of the conditions of connectivity (2.8), the absolute virtual velocities $\vec{v}_{k0}^{v}$, $\vec{\omega}_{k0}^{v}$ are related to a set of independent *relative virtual velocities*. The fundamental principle of the virtual work states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism. Total virtual work contributed by the first input torque $\vec{m}_{10}^{A}$, for example, inertia forces and moments of inertia forces and by the wrench of known external or internal forces can be written in a compact form. Applying *the fundamental equations of the parallel robots dynamics* [16], a compact matrix relation results for the first torque

$$m_{10}^{A} = \vec{u}_3^{\mathrm{T}} \{ \vec{M}_1^{A} + \omega_{21a}^{Av} \vec{M}_2^{A} + \omega_{32a}^{Av} \vec{M}_3^{A} + \omega_{21a}^{Bv} \vec{M}_2^{B} + \omega_{21a}^{Cv} \vec{M}_2^{C} \} . \tag{3.2}$$

Two recursive relations generate the vectors

$$\vec{F}_k^{A} = \vec{F}_{k0}^{A} + a_{k+1,k}^{\mathrm{T}} \vec{F}_{k+1}^{A},\ \vec{M}_k^{A} = \vec{M}_{k0}^{A} + a_{k+1,k}^{\mathrm{T}} \vec{M}_{k+1}^{A} + \tilde{r}_{k+1,k}^{A} a_{k+1,k}^{\mathrm{T}} \vec{F}_{k+1}^{A}$$
$$\vec{F}_{k0}^{A} = -\vec{f}_{k0}^{inA} - \vec{f}_k^{*A},\ \vec{M}_{k0}^{A} = -\vec{m}_{k0}^{inA} - \vec{m}_k^{*A} \quad (k = 1,2,3) . \tag{3.3}$$

3.2. Lagrange equations

A solution of the dynamics problem of the 3-RRR planar parallel robot can be developed based on the Lagrange equations with constraints. The generalized coordinates are represented by 12 independent parameters:

$$q_1 = x_0^G ,\, q_2 = y_0^G ,\, q_3 = z_0^G ,\, q_4 = \varphi_{10}^A ,\, q_5 = \varphi_{21}^A ,\, q_6 = \varphi_{32}^A$$
$$q_7 = \varphi_{10}^B ,\, q_8 = \varphi_{21}^B ,\, q_9 = \varphi_{32}^B ,\, q_{10} = \varphi_{10}^C ,\, q_{11} = \varphi_{21}^C ,\, q_{12} = \varphi_{32}^C . \tag{3.4}$$

The Lagrange equations with nine multipliers $\lambda_1,\ \lambda_2,\ \ldots ,\lambda_9$ will be expressed by 12 differential relations [17]

$$\frac{\mathrm{d}}{\mathrm{d}t} \{ \frac{\partial L}{\partial \dot{q}_k} \} - \frac{\partial L}{\partial q_k} = Q_k + \sum_{s=1}^{9} \lambda_s c_{sk} \quad (k = 1,\ 2,\ldots,\ 12) , \tag{3.5}$$

which contain following 12 generalized forces $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 0$, $Q_4 = m_{10}^A$, $Q_5 = 0$, $Q_6 = 0$, $Q_7 = m_{10}^B$, $Q_8 = 0$, $Q_9 = 0$, $Q_{10} = m_{10}^C$, $Q_{11} = 0$, $Q_{12} = 0$.

A number of nine kinematical conditions of constraint are given from the relations (2.8):

$$\sum_{k=1}^{12} c_{sk} \dot{q}_k = 0 \qquad (s = 1,\, 2,\ldots,9) \cdot \tag{3.6}$$

The general Lagrange function $L = L_p + \sum_{\nu=1}^{2} (L_\nu^A + L_\nu^B + L_\nu^C)$ is expressed as analytical function of the generalized coordinates and their first derivatives with respect to time

$$L_p = \frac{1}{2} m_p \vec{v}_p^{\mathrm{T}} \vec{v}_p - m_p g z_0^G ,\, L_1^i = \frac{1}{2} \vec{\omega}_{10}^{i\mathrm{T}} \hat{J}_1^i \vec{\omega}_{10}^i - m_1^i g \vec{u}_2^{\mathrm{T}} p_{10}^{\mathrm{T}} \vec{r}_1^{Ci} ,$$
$$L_2^i = \frac{1}{2} m_2^i \vec{v}_{20}^{i\mathrm{T}} \vec{v}_{20}^i + \frac{1}{2} \vec{\omega}_{20}^{i\mathrm{T}} \hat{J}_3^i \vec{\omega}_{20}^i + m_2^i \vec{v}_{20}^{i\mathrm{T}} \tilde{\omega}_{20}^i \vec{r}_2^{Ci} - m_2^i g \vec{u}_2^{\mathrm{T}} p_{10}^{\mathrm{T}} (\vec{r}_{21}^i + p_{21}^{\mathrm{T}} \vec{r}_2^{Ci}) . \tag{3.7}$$

The first derivatives of orthogonal matrices $p_{k,k-1}$ are computed as follows:

$$\dot{p}_{k,k-1} = \dot{\varphi}_{k,k-1}^i \tilde{u}_3^{\mathrm{T}} p_{k,k-1} ,\ \dot{p}_{k,k-1}^{\mathrm{T}} = \dot{\varphi}_{k,k-1}^i p_{k,k-1}^{\mathrm{T}} \tilde{u}_3 ,\ \frac{\partial p_{k,k-1}}{\partial \varphi_{k,k-1}^i} = \tilde{u}_3^{\mathrm{T}} p_{k,k-1} ,\ \frac{\partial p_{k,k-1}^{\mathrm{T}}}{\partial \varphi_{k,k-1}^i} = p_{k,k-1}^{\mathrm{T}} \tilde{u}_3 . \tag{3.8}$$

A long calculus about the partial derivatives and the expressions $\frac{\mathrm{d}}{\mathrm{d}t} \{ \frac{\partial L}{\partial \dot{q}_k} \}$ leads to an algebraic system of 12 relations. In the direct or inverse dynamics problem, after elimination of the nine multipliers, finally we obtain same expressions (3.2) for the three input torques required by the three revolute actuators.

4. CALCULUS OF INTERNAL JOINT FORCES

4.1. Principle of virtual work

Above compact relations (3.2) and (3.3) can be also applied to calculate any joint force or joint torque by cutting successively each joint and considering several appropriate virtual displacements.

For the calculus of two external forces acting in joint A_1, for example, we consider two different successive virtual displacements as follows

$$\vec{v}_{10a}^{Av} = a_{10}\vec{u}_1 ,\ \omega_{10}^{Av} = 0 ,\ \omega_{10a}^{Bv} = 0 ,\ \omega_{10a}^{Cv} = 0 ,\ \vec{v}_{10a}^{Av} = a_{10}\vec{u}_2 ,\ \omega_{10a}^{Av} = 0 , \omega_{10a}^{Bv} = 0 ,\ \omega_{10a}^{Cv} = 0 . \tag{4.1}$$

Using the matrix conditions of connectivity (2.8), we determine two new sets of independent relative virtual velocities. Finally, the recursive relations (3.2) give the two joint forces:

$$\begin{aligned} f_{10x}^{A} &= \vec{u}_1^{\mathrm{T}} a_{10}^{\mathrm{T}} \vec{F}_1^{A} + \vec{u}_3^{\mathrm{T}} \{\omega_{21a}^{Av}\vec{M}_2^{A} + \omega_{32a}^{Av}\vec{M}_3^{A} + \omega_{21a}^{Bv}\vec{M}_2^{B} + \omega_{21a}^{Cv}\vec{M}_2^{C}\} , \\ f_{10y}^{A} &= \vec{u}_2^{\mathrm{T}} a_{10}^{\mathrm{T}} \vec{F}_1^{A} + \vec{u}_3^{\mathrm{T}} \{\omega_{21a}^{Av}\vec{M}_2^{A} + \omega_{32a}^{Av}\vec{M}_3^{A} + \omega_{21a}^{Bv}\vec{M}_2^{B} + \omega_{21a}^{Cv}\vec{M}_2^{C}\} . \end{aligned} \tag{4.2}$$

Analogously, the joint force f_{21}^{A} in A_2 is quickly determined if we suppose others two virtual displacement in the robot's kinematics, starting from the basic virtual velocities

$$\vec{v}_{21a}^{Av} = a_{21}\vec{u}_1 \text{ or } \vec{v}_{21a}^{Av} = a_{21}\vec{u}_2 . \tag{4.3}$$

4.2. Lagrange equations

For the planar mechanical system represented by the set of 12 variables $(q) = (q_1, q_2, \ldots\ldots, q_{12})$, the Lagrange equations are completed with following differential relation [18]

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\frac{\partial L_A}{\partial \dot{w}}\} - \frac{\partial L_A}{\partial w} = f_{10y}^{A} + \sum_{s=1}^{9} \lambda_s c_{sk} \quad (k = 1,\ 2, \ldots,\ 13) , \tag{4.4}$$

where the vertical external force f_{10y}^{A} in the joint A_1, for exampla, as new generalized force can be found if a new fictitious mobility in accord with the joint is considered.

Supposing $\vec{r}_{10w}^{A} = \vec{r}_{10}^{A} + w\vec{u}_2$, $\vec{v}_{10w}^{A} = \dot{w}a_{10}\vec{u}_2$, $\vec{\gamma}_{10w}^{A} = \ddot{w}a_{10}\vec{u}_2$, we determine a new expression for the Lagrange function L_A and we replace it in the formula (4.4). Considering again the mechanism, the joint force is obtained in following definitive form

$$f_{10y}^{A} = \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L_A}{\partial \dot{w}} \right) \right]_{\substack{w=0 \\ \dot{w}=0 \\ \ddot{w}=0}} - (m_1^{A} + m_2^{A})g - \lambda_{13} . \tag{4.5}$$

5. EXAMPLE

As application, let us consider a 3-RRR planar robot which has the following characteristics:

$$x_0^{G*} = -0.025\mathrm{m}, y_0^{G*} = 0.025\mathrm{m}, \phi^* = \frac{\pi}{12}, r = 0.3\mathrm{m}, l = r\sqrt{3}, l_1 = l_2 = 0.3\mathrm{m}, \Delta t = 3\mathrm{s}, m_1 = 3\mathrm{kg}, m_2 = 1.5\mathrm{kg}, m_3 = 5\mathrm{kg}.$$

Assuming that there is no external force and moment acting on the moving platform, the numerical study is based on the computation of the power required by each actuator p_{10}^{A}, p_{10}^{B}, p_{10}^{C} during the platform's evolution.

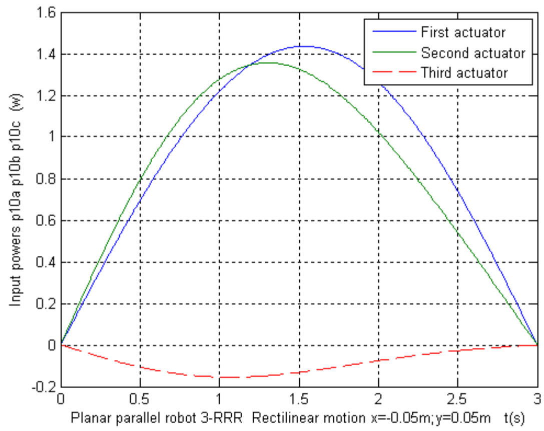

Fig. 3 – Powers p_{10}^{A}, p_{10}^{B}, p_{10}^{C} of three actuators.

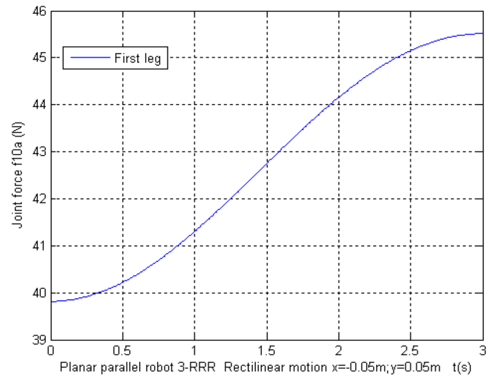

Fig. 4 – Force f_{10}^{A} in the external joint A_1

Using the MATLAB software, a computer program was developed to solve the inverse dynamics of the planar parallel robot. To illustrate the algorithm, it is assumed that for a period of three second the platform starts at rest from a central configuration and rotates or moves along rectilinear directions.

Following examples are solved to illustrate the simulation. If the platform's centre G moves along a *rectilinear planar trajectory* without rotation of platform, the powers required by the actuators A_1, B_1, C_1 are calculated by the program and plotted versus time (Fig. 3). Concerning the force f_{10}^{A} in the external joint A_1, for example, this is given as follows (Fig. 4). For the second example we consider the *rotation motion* of the moving platform about z_0 axis with variable angular acceleration while all other positional parameters are held equal to zero. The powers of the actuators (Fig. 5) and the internal joint force f_{21}^{A} in A_2 (Fig. 6), for example, are also determined by the program and plotted versus time.

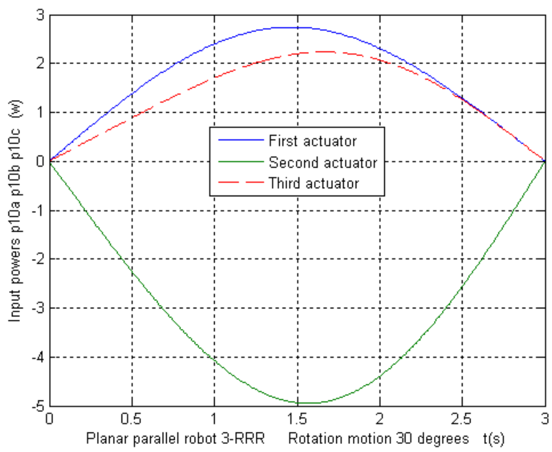

Fig. 5 – Powers p_{10}^{A}, p_{10}^{B}, p_{10}^{C} of three actuators.

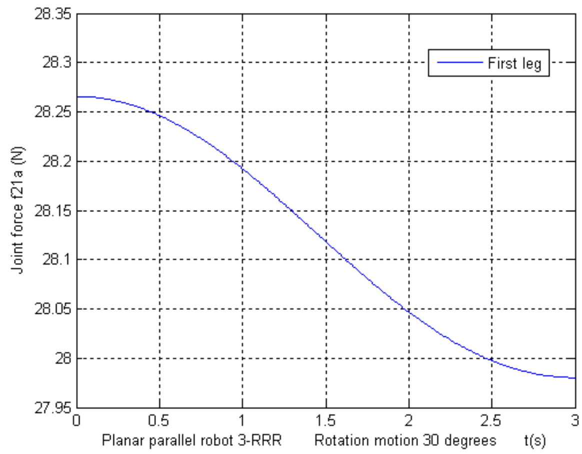

Fig. 6 – Force f_{21}^{A} in the internal joint A_2 .

6. CONCLUSIONS

In the inverse kinematics analysis some exact relations that give in real-time the position, velocity and acceleration of each element of the planar parallel robot have been established in the present paper.

The dynamics model takes into consideration the masses and forces of inertia introduced by all component elements of the parallel mechanism. Based on the principle of virtual work or the Lagrange equations, the approach establishes also a direct determination of the time-history evolution for all forces in external and internal joints.

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