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# NEW METHOD FOR CALCULATING RESPONSE OF A PERIODIC STRUCTURE DUE TO MOVING LOADS

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The paper herein proposes a new method to study the response of a discretely supported rail, as a result of moving loads. The method bases on the rail Green's functions and the basic features of the ballasted track - namely to be a periodic and damped structure - and it takes into account an arbitrary number of moving loads travelling along the rail. Numerical simulations focus on the case of two moving harmonic loads and feature significant coupling between the loading points.

*Key words:* Rail, Moving loads, Timoshenko beam, Green's functions.

## 1. INTRODUCTION

The problem of loads travelling along a periodic structure is frequently come across in many important engineering systems, mainly those in the railway transport field. Here, we refer to the wheel/rail [1, 2] and pantograph-catenary [3, 4] systems, where the periodic structure is either the rail discretely supported on sleepers and the ballast bed, or the overhead wire suspended by the stoppers via droppers and the messenger wire and also the steady arms. The accurate estimation of the response of such structures due to moving loads is critical for a correct prediction of the dynamic behaviour in the systems above.

Dynamics of periodic structures under moving loads has been studied by a number of researchers employing different methods. Mead and Jezequel [5, 6], and Sheng et al. [7] based their research on the Fourier-series technique. Floquet theorem was applied by Krzyzynski and Popp [8], and Krzyzynski [9] with the purpose of inquiring into the issue of the vertical wave propagation in the track and its response under a moving harmonic force. Chebli et al. [10] calculated the response of periodic structures subjected to moving forces by applying the Floquet decomposition, which allows the restriction of the analysis to a generic cell. Metrikine, Bosch [11] and Belotserkovskiy [12] used periodicity condition to simulate the dynamic response of a two-level catenary or the response of an infinite periodic string to a moving load, respectively. Belotserkovskiy [13] explored the same method in the case of an infinite periodic Euler-Bernoulli beam subjected to a moving load as well. Nordborg [14] developed the Green's functions method to study the rolling noise. To this end, the receptance of the rail has been calculated from a unit stationary harmonic load, and then inversed to obtain the time-dependent stiffness of the track.

Following an unbeaten route, the Green's functions method was first applied by Mazilu [15] and latter by Mazilu et al. [16] to investigate many aspects regarding the response of a discretely supported rail due to moving harmonic load, including the Doppler effect. The original approach is based on *the Green's matrix of the track*, which contains the response of the rail along a length span, following a unit moving impulse force. The starting point of the numerical approach is the rail receptance and then, by applying the inverse Fourier transform, the space-time rail Green's functions are calculated step by step, considering the moving impulse force and assembled in the Green's track matrix. This method has been used to study the vertical interaction between a moving wheel and a discretely supported rail [17]. In the paper herein, we generalize this method involving an arbitrary number of moving loads that uniformly travel along the rail.

## 2. MECHANICAL MODEL

In order to present the new method for calculating the response of a periodic structure supporting many moving loads, let's take a look at the mechanical model of an infinite rail discretely supported on sleepers-ballast-subgrade system, as shown in Fig. 2.1 [16]. The rail is considered a uniform infinite Timoshenko beam and the sleepers are assumed as rigid bodies. In addition, the inertial effect due to the ballast bed enters the equation. Three directional Kelvin-Voigt systems are in use to model the visco-elastic feature of the rail pad and ballast, and a mixed Kelvin-Voigt/Maxwell system for the subgrade.

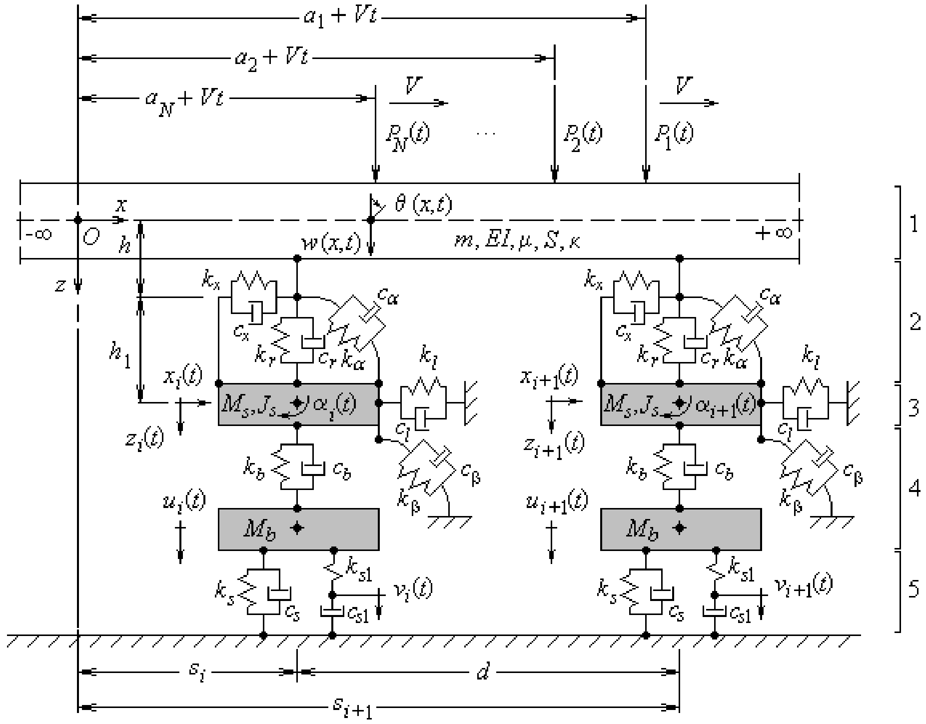


Fig. 2.1.

The parameters for the rail are: the specific mass per length unit $m$, the cross-sectional area $S$, the area moment of inertia $I$, the density $\rho$, the Young's modulus $E$, the shear modulus $\mu$ the shear coefficient $\kappa$ and the distance from the centroid of the cross-section to the rail foot $h$. The loss factor of the rail is not included. The motion of the rail is described by the column vector $\mathbf{q} = [w(x,t)\ \theta(x,t)]^{\mathrm{T}}$, where $w(x,t)$ and $\theta(x,t)$ stands for the vertical displacement, and the rotation of the cross-section respectively; $x$ is the coordinate along the rail and $t$ time.

The parameters for the rail support are as follows: the elastic constants $k_x$, $k_r$ and $k_\alpha$ and the damping constants $c_x$, $c_r$ and $c_\alpha$ of the rail pad, the mass $M_s$ and the mass-moment of inertia $J_s$ of the sleeper, the sleeper bay $d$, the distance between the semi-sleeper centroid and the rail pad $h_1$, the mass $M_b$, the elastic constants $k_l$, $k_b$ and $k_\beta$ and the corresponding damping constants $c_l$, $c_b$ and $c_\beta$ for the ballast and the elastic constants $k_s$ and $k_{s1}$ and the damping constants $c_s$ and $c_{s1}$ for the subgrade.

The rail is subjected to a $N$ uniformly moving loads with speed $V$, which starts from the initial positions corresponding to the distances $a_i$, $i = 1 \div N$ from the inertial frame $Oxz$.

The track's differential equations of motion may be formulated in a matrix-like form as

$$\begin{gathered}\mathbf{T}_{x,t}\mathbf{q} - \sum_{i=-\infty}^{\infty}(\mathbf{A}_t\mathbf{q}_i - \mathbf{B}_t\mathbf{q}_i^s)\delta(x - s_i) = \left[ -\sum_{i=1}^{N} P_i(t)\delta(x - Vt - a_i) \quad 0 \right]^{\mathrm{T}}, \\ \mathbf{C}_t\mathbf{q}_i^s = \mathbf{B}_t^{\mathrm{T}}\mathbf{q}_i,\end{gathered} \tag{2.1}$$

where $\mathbf{q}_i = [w(s_i,t)\ \theta(s_i,t)]^{\mathrm{T}}$ contains the rail displacements above the sleeper $i$, the column vector $\mathbf{q}^s_i = [x_i(t)\ z_i(t)\ \alpha_i(t)\ v_i(t)\ v_{i1}(t)]^{\mathrm{T}}$ includes the displacements corresponding to the support $i$, and $\delta(.)$ is the Dirac's delta function. Also, $\mathbf{T}_{x,t}$ stands for the Timoshenko's matrix operator and $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{C}_t$ are the matrix differentials,

$$\mathbf{A}_t = \begin{bmatrix} c_r \frac{\mathrm{d}}{\mathrm{d}t} + k_r & 0 \\ 0 & c_\theta \frac{\mathrm{d}}{\mathrm{d}t} + k_\theta \end{bmatrix},$$

$$\mathbf{B}_t = \begin{bmatrix} 0 & c_r \frac{\mathrm{d}}{\mathrm{d}t} + k_r & 0 & 0 & 0 \\ -h\left(c_x \frac{\mathrm{d}}{\mathrm{d}t} + k_x\right) & 0 & \Delta c_\alpha \frac{\mathrm{d}}{\mathrm{d}t} + \Delta k_\alpha & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_t = \begin{bmatrix} D_x & 0 & h_1\left(c_x \frac{\mathrm{d}}{\mathrm{d}t} + k_x\right) & 0 & 0 \\ 0 & D_z & 0 & -c_b \frac{\mathrm{d}}{\mathrm{d}t} - k_b & 0 \\ h_1\left(c_x \frac{\mathrm{d}}{\mathrm{d}t} + k_x\right) & 0 & D_\alpha & 0 & 0 \\ 0 & -c_b \frac{\mathrm{d}}{\mathrm{d}t} - k_b & 0 & D_v & -k_{s1} \\ 0 & 0 & 0 & -k_{s1} & D_{v1} \end{bmatrix}, \tag{2.2}$$

$$\mathbf{T}_{x,t} = \begin{bmatrix} \kappa\mu S \frac{\partial^2}{\partial x^2} - m \frac{\partial^2}{\partial t^2} & -\kappa\mu S \frac{\partial}{\partial x} \\ \kappa\mu S \frac{\partial}{\partial x} & EI \frac{\partial^2}{\partial x^2} - \kappa\mu S - \rho I \frac{\partial^2}{\partial t^2} \end{bmatrix},$$

where

$$D_x = M_s \frac{\mathrm{d}^2}{\mathrm{d}t^2} + c_x \frac{\mathrm{d}}{\mathrm{d}t} + k_x,\ D_z = M_s \frac{\mathrm{d}^2}{\mathrm{d}t^2} + (c_r + c_b)\frac{\mathrm{d}}{\mathrm{d}t} + k_r + k_b,\ D_{v1} = c_{s1} \frac{\partial}{\partial t} + k_{s1},$$
$$D_\alpha = J_s \frac{\mathrm{d}^2}{\mathrm{d}t^2} + (c_\alpha + h_1^2 c_x)\frac{\mathrm{d}}{\mathrm{d}t} + k_\alpha + h_1^2 k_x,\ D_v = M_b \frac{\mathrm{d}^2}{\mathrm{d}t^2} + (c_b + c_s)\frac{\mathrm{d}}{\mathrm{d}t} + k_b + k_s + k_{s1}, \tag{2.3}$$
$$c_\theta = c_\alpha + h^2 c_x,\ k_\theta = k_\alpha + h^2 k_x,\ \Delta c_\alpha = c_\alpha - h h_1 c_x,\ \Delta k_\alpha = k_\alpha - h h_1 k_x.$$

All initial conditions are null and the boundary conditions are null as well

$$\mathbf{q}(x,t) = [0\quad 0]^{\mathrm{T}},\ \mathbf{q}_i^s(t) = [0\quad 0\quad 0\quad 0\quad 0]^{\mathrm{T}},$$
$$\lim_{|x-Vt|\to\infty} \mathbf{q}(x,t) = [0\quad 0]^{\mathrm{T}},\ \lim_{i\to\pm\infty} \mathbf{q}_i^s(t) = [0\quad 0\quad 0\quad 0\quad 0]^{\mathrm{T}}. \tag{2.4}$$

According to the convolution theorem and considering the causality condition of the track model, the solution of the problem may be given as

$$\mathbf{q}(x,t) = \int_{-\infty}^{\infty}\int_0^t \mathbf{g}(x,\xi,t-\tau)\sum_{i=1}^{N} P_i(\tau)\delta(\xi - V\tau - a_i)\mathrm{d}\tau\mathrm{d}\xi = \sum_{i=1}^{N}\int_0^t \mathbf{g}(x,V\tau - a_i,t-\tau)P_i(\tau)\mathrm{d}\tau, \tag{2.5}$$

where the column vector

$$\mathbf{g}(x,\xi,t-\tau) = \left[g^w(x,\xi,t-\tau)\quad g^\theta(x,\xi,t-\tau)\right]^{\mathrm{T}}, \tag{2.6}$$

contains the space-time Green's functions of the rail, which give the response of the rail in the $x$ section at $t$–$\tau$ moment, due to a unit impulse force applied at the $\tau$ moment in rail section $\xi$.

Applying the inverse Fourier transform, the space-time Green's functions of the rail are calculated from the receptances of the track

$$\mathbf{g}(x,\xi,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{G}(x,\xi,\omega)\exp(\mathrm{i}\omega t)\mathrm{d}\omega\,, \tag{2.7}$$

where

$$\mathbf{G}(x,\xi,\omega)=\left[G^{w}(x,\xi,\omega)\quad G^{0}(x,\xi,\omega)\right]^{\mathrm{T}}, \tag{2.8}$$

are the receptances and represent the rail's response in the $x$ section caused by a unit harmonic force at an angular $\omega$ frequency, occurring in the $\xi$ rail's section.

The rail's displacements at loading points are the most interesting values. Taking $x = Vt + a_k$ in (2.5), we obtain the rail's displacement at the $k$ loading point

$$w(Vt+a_k,t)=\sum_{i=1}^{N}\int_{0}^{t}g^{w}(Vt+a_k,V\tau-a_i,t-\tau)P_i(\tau)\mathrm{d}\tau\,,\quad k=1\div N. \tag{2.9}$$

For the numeric calculus, a certain $t_0, t_1, \ldots, t_n$ (with $t_0 = 0$, $t_n = t$ and $\Delta t = t_j - t_{j\text{-}1}$ where $j = 1 \div n$) time partition is considered. The equation (2.9) for the rail displacements under moving loads may be written under the recurrent form:

$$w(Vt_n+a_k,t_n)=\sum_{i=1}^{N}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}g^{w}(Vt_n+a_k,V\tau-a_i,t_n-\tau)P_i(\tau)\mathrm{d}\tau\,,\quad k=1\div N. \tag{2.10}$$

Assuming that in the $[t_{i\text{-}1}, t_i]$ time interval, the moving loads $P_i(\tau)$ and the Green's functions will have a linear variation, we will have

$$w(Vt_n+a_k,t_n)=\Delta t\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{g^{w}_{k,i,j-1}P_{ij}+g^{w}_{k,i,j}P_{ij-1}}{2}+\frac{(g^{w}_{k,i,j}-g^{w}_{k,i,j-1})(P_{ij}-P_{ij-1})}{3}\,,\ k=1\div N, \tag{2.11}$$

where

$$g^{w}_{k,i,j}=g^{w}(Vt_n+a_k,t_j-a_i,t_n-t_j)\,,\ P_{ij}=P_i(t_j)\,,\quad k,i=1\div N, j=1\div n. \tag{2.12}$$

The space-time Green's functions of the rail are levelled off in time and, because of that, their norms are practically "concentrated" within a particular time interval $T$. Therefore, they have to be calculated only for an adequate time interval, including the receptances in the central zone of the track model, where the periodic feature of the track may be closely monitored. Further, they may be used to simulate the track's dynamics for any particular section based on the periodicity feature. To this purpose, the integrals in Eq. (2.7) are calculated using a numerical method and meeting the requirements in Eq. (2.9). In other words, we have $N\times N$ functions

$$g^{w}_{k,i}=g^{w}(Vt+a_k,V\tau-a_i,t-\tau)\,,\quad k,i=1\div N, \tag{2.13}$$

for $0 \le t-\tau \le T$ and subsequently, the same matrices number has to be calculated according to the method of the Green's matrix of the track.

## 3. NUMERICAL APPLICATION

In this section, numerical results in a particular ballasted track model under two moving loads are presented, using the new method previously described.

The parameters for the rail are: $m$ = 60 kg/m, $S$ = 7.69 $\cdot 10^{-3}$ m$^2$, $I$ = 30.55$\cdot 10^{-6}$ m$^4$, $h_0$ = 0.08 m, $\rho$ = 7,850 kg/m$^3$, $E$ = 210 GPa, $\mu$ = 81 GPa and $\kappa$ = 0.4.

The parameters for the discrete support are the following: $M_s$ = 145 kg, $J_s$ = 1.28 kgm$^2$, $M_b$ = 2,500 kg, $d$ = 0.6, $h_1$ = 0.116 m, $k_x$ = 50 MN/m, $k_r$ = 280 MN/m, $k_\alpha = k_r c^2/3$ = 597 kNm/rad (2$c$ = 160 mm, rail pad width), $k_u$ = 39.6 MN/m, $k_b$ = 120 MN/m, $k_s$ = 60 MN/m, $k_{s1}$ = 100 MN/m, $c_x$ = 10 kNs/m, $c_r$ = 30 kNs/m, $c_\alpha = c_r c^2/3$ = = 64 Nms/rad, $c_u$ = 52 kNs/m, $c_b$ = 70 kNs/m, $c_s$ = 150 kNs/m and $c_{s1}$ = 600 kNs/m. The distance between the two moving loads is 1.8 m.

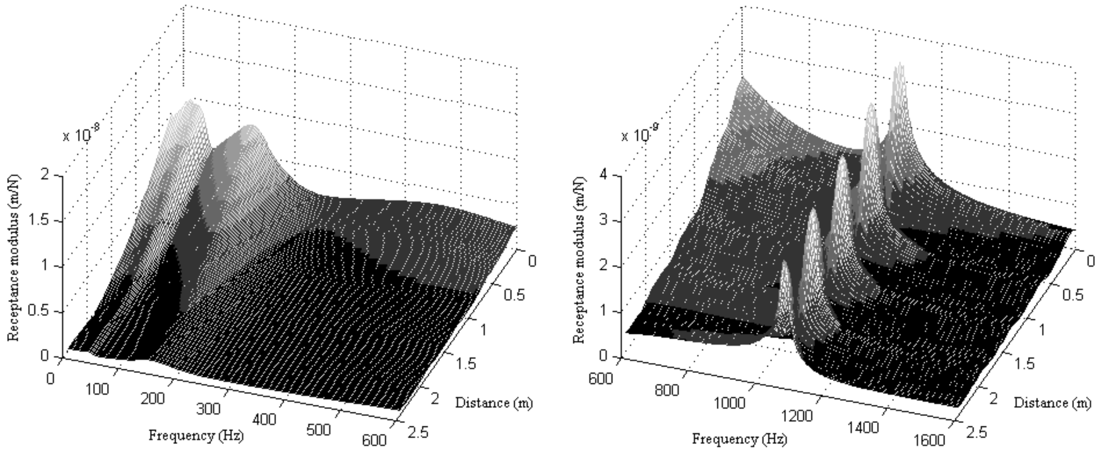


Fig. 3.1.

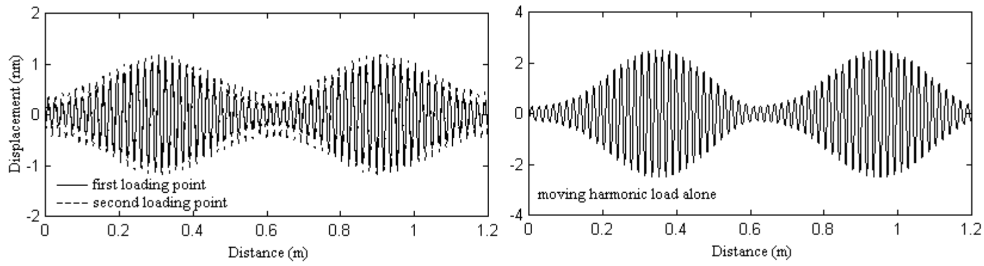


Fig. 3.2.

As for the numerical simulation, the track model length is 50 sleeper bays and this length satisfies the mandatory criterion to maintain the periodic feature of the track in the central area of the model. The rail receptance in this area is solely used to calculate all the matrices in the equation (2.13).

Figure 3.1 shows the cross receptance of the rail along a distance of 2.5 m from the loading point, when the unit harmonic excitation acts at mid span. At the loading point (distance = 0), one can observe the well-known peaks corresponding to the resonance frequencies, due to the rail and sleepers that have a dynamic behavior, similar to the one in a two-degree-of-freedom system. In addition, the rail receptance reaches as low as 50 Hz, because of the ballast influence and the subgrade, as in the experimental results [18]. The "pinned-pinned resonance" appears at the frequency of 1,074 Hz, which reflects the first bending mode of the rail on the sleepers. The cross receptance drastically slows down along the rail for all frequencies, excepting the range of the pinned-pinned resonance. The conclusion is that the influence of the second moving force will be significant in this frequency range.

Figure 3.2 presents the rail displacements at the moving points, for two moving harmonic unit loads. The moving harmonic loads are in phase at the frequency of 1,067 Hz, which is within the range of the pinned-pinned resonance. The velocity of the moving loads is 20 m/s. The displacement of the rail under a moving harmonic unit load alone is also presented for comparison. One may notice that the displacement of the rail under the leading load is lower than the one under the trailer load. Actually, the amplitudes are 1.03 nm and 1.18 nm respectively, at the mid span. These values are significantly lower than the amplitude of the rail under a moving harmonic load alone, which reaches the value of 2.51 nm when the moving load travels on the mid span.

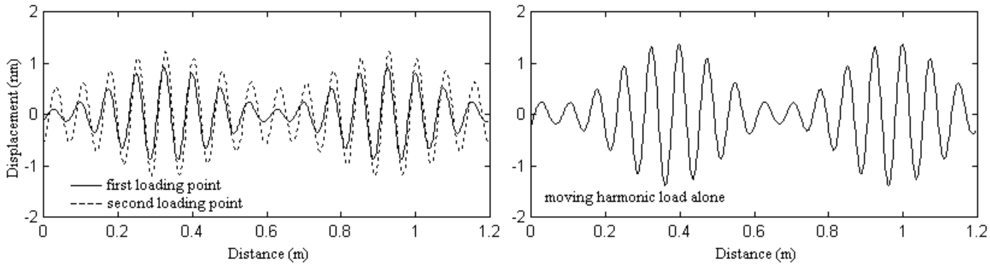


Fig. 3.3.

Figure 3.3 shows the previous case of the rail under the two moving harmonic loads, at the speed of 80 m/s. Here, the difference between the displacements of the two moving loads is higher. The amplitudes at the mid span are 0.90 nm for the leading point and 1.20 nm for the trailing. Both amplitudes are lower than the one deriving from the moving harmonic load alone, with the magnitude of 1.37 nm. Very important difference may be pointed out when the moving loads travel on the sleepers.

## 4. CONCLUSIONS

In this paper, a new method is suggested to help calculating the response of a periodically supported rail under moving loads. This method generalizes the Green's matrix of the track method, successfully applied for both the rail's response due to moving load and the moving wheel. The new method is at hand for the case of the arbitrary moving loads. The method relies on the rail Green's functions and the features of the track supposed to be a periodic and damped structure.

In order to detail the application of the method, a complex model for the ballasted track has been considered. The rail is an infinite Timoshenko beam and the model of the rail's periodic support consists of two three-directional Kelvin-Voigt systems for the rail pad and the ballast and a mixed Kelvin-Voight/Maxwell system for the subgrade. The inertia of the sleeper and ballast block enters the support model as well. The track model gives a high accuracy prediction for an extended range of frequency.

The numerical application focuses on the case of two harmonic loads travelling on the rail. The most interesting situation relates to the pinned-pinned resonance when the bending waves of the rail travel along the track. The numerical simulations have shown that there are significant influences between the loading points for both low and high velocities.

Further research will aim to develop this method, so that the issue of the interaction between a wheels tandem and the rail be finally understood and solved.

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