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A NEW INEQUALITY OF SHANNON TYPE

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In this paper we extend some properties of the parametric version of the Shannon inequality, using some result of Furuta [2] and the characterization of the convexifiable function.

Key words: Entropy, Uncertainty, Shannon inequality, Relative operator entropy, P-power mean, Convexifiable function, Operator on Hilbert space.

1. INTRODUCTION

Claude E. Shannon (1948) intended to investigated a new mathematical model of communication system using information theory. One of the most important innovation of this model was to consider the components of a communications system as probabilistic entities. Shannon proposed a quantitative measure of the amount of information supplied by a probabilistic experiment, based on the classical Boltzmann's entropy (1896). In this conception, the amount of information is strongly connected to the amount of uncertainly. In fact, the information is equal to the removed uncertainly.

The classical entropy of a system was first defined by Clausius (1864) as a function of some other macro-coordinates that can be measured directly. Clausius' entropy is a non-probabilistic concept. In thermodynamics, the entropy was defined as a function of positions and velocities of all particles included in the physical system, and Boltzmann noticed its probabilistic meaning. In analogy with Boltzmann's probabilistic expression of classical entropy, Shannon introduced in 1948 the abstract entropy as a measure of information or uncertainly of arbitrary probabilistic experiments. We observe that Shannon entropy is independent from classical entropy because we cannot obtain, transmit or keep in store information of any kind without an increase in the total entropy of the isolated system in witch we act.

Even if Boltzmann made no explicit reference to information, he noticed that the entropy of a physical system can be considered as a measure of the disorder in the system. In a physical case having many degrees of freedom the number measuring the disorder of the system measures also the uncertainly concerning the states of the individual particles.

In information theory, the entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

2. MEASURES OF UNCERTAINTY, ENTROPIES.

In information theory, more than 30 measures of entropies generalizing Shannon's entropy, as parametric, trigonometric and weighted entropies have been introduced.

For the first time, in 1961, Renyi had the idea of parametric entropy, by relaxing the additive requirements of Shannon's entropy. He extends the entropy to a continuous family of entropy measures that obey

$$H_q(P)=\frac{1}{1-q}\ln\sum_{i=1}^{N}{p_i}^q .$$

The parametric entropies involve one and two scalar parameters. Other researchers like Havrda and Charvát (1967), Arimoto (1971) , Sharma and Mittal (1975) etc. were interested in other kinds of expressions generalizing Shannon's entropy.

Based on the same motivations of Rényi, Aczél and Daróczy in 1963 generalized the entropy of order r by changing some postulates expression. So, *entropy of order (r, s)*

$$H_{r,s}(P) = (s-r)^{-1}\log\left(\frac{\sum_{i=1}^{n} p_i^{\,r}}{\sum_{i=1}^{n} p_i^{\,s}}\right),\ r \neq s, r > 0, s > 0.$$

Later, a lot of researchers studied entropy from different points of view (Havrda, Charvát (1967) Sharma, Tajena, 1975) and showed that it is more natural to consider expression $\sum_{i=1}^{n} p_i^{\,r}$ as an information measure, so define *entropy of degree s* and it is generalization with two parameters, *entropy of degree (r, s)*

$$H^{r,s}(P) = (2^{1-r} - 2^{1-s})^{-1}\sum_{i=1}^{n} p_i^{\,r} - p_i^{\,s},\ r \neq s,\ r > 0,\ s > 0.$$

In 1971, Arimoto considered generalized *f*-entropy, involving a real function f with some conditions, *entropy of kind t*, proving some important results

$${}_tH(P) = (2^{t-1} - 1)^{-1}\left[\left(\sum p_i^{\,1/t}\right)^{t} - 1\right],\ t \neq 1, t > 0.$$

A few years later (1975), Sharma and Mittal were able to generalize the entropies $H_r(P)$, $H^s(P)$, ${}_tH(P)$ and introduced *entropy of order r and degree s*

$$H_r^s(P) = (2^{1-s} - 1)^{-1}\left[\left(\sum_{i=1}^{n} p_i^{\,r}\right)^{\frac{s-1}{r-1}} - 1\right],\ r \neq 1, s \neq 1, r > 0.$$

Sharma and Taheja, in same period obtain other expression of entropy, even trigonometric one

$$H'(P) = -2^{r-1}\sum_{i=1}^{n} p_i^{\,r}\log p_i, r > 0,$$

$$H_{trig}(P) = -\frac{2^{r-1}}{\sin s}\sum_{i=1}^{n} p_i^{\,r}\sin(s\log p_i), s \neq k\pi, k = 0, 1,\ldots, r > 0.$$

Ferreri, in 1980, obtain an interesting parametric expression of entropic type

$$H_\lambda(P) = 1 + \frac{1}{\lambda}\log(1+\lambda) - \frac{1}{\lambda}\sum(1+\lambda p_i)\log(1+\lambda p_i),\quad \lambda > 0.$$

The idea of weighted entropies started by Belis and Guaisu (1968)

$$H_W(P) = -\sum_{i=1}^{n} w_i p_i \log p_i,\ \ w_i > 0, i = 1,2,\ldots n,\ \ \sum_{i=1}^{n} p_i = 1.$$

In 1979 Picard introduced some generalized weighted measures.

In the following we start with the concept of Shannon entropy, i.e., the measure of information (or uncertainly) supplied by a probabilistic experiment of any kind. So, we use the entropy of a random variable

X, which takes on a finite number of arbitrary values x_k (k = 1, 2, …, m), with probabilities $p_k > 0$, $\sum_{k=1}^{m} p_k = 1$, then the entropy of the random variable X will be

$$H(X) = -\sum_{k=1}^{m} p_k \log p_k$$

and similarly if the random variable X takes on denumerably many values.

3. SOME PRELIMINARY RESULTS FOR SHANNON'S INEQUALITY

Let $p = \{p_1, …, p_n\}$ and $q = \{q_1, …, q_n\}$ be two probability vectors. First Shannon's inequality asserts that

$$\sum_{i=1}^{n} p_i \log \frac{1}{p_i} \le \sum_{i=1}^{n} p_i \log \frac{1}{q_i}, \text{ with } \sum_{i=1}^{n} p_i = 1 \text{ and } \sum_{i=1}^{n} q_i = 1,$$

result sometimes called the *fundamental lemma of information theory*.

We remark that the above inequality can be written using "divergence" like following inequality

$$\sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \ge 0 .$$

Some extension of Shannon inequality was given in [7]

$$\sum_{i=1}^{n} p_i \log \frac{1}{p_i} \le \sum_{i=1}^{n} p_i \log \frac{1}{q_i} + \log\alpha \text{ and} \int_I p(x) \log \frac{1}{p(x)} \mathrm{d}x \le \int_I p(x) \log \frac{1}{q(x)} \mathrm{d}x + \log \alpha.$$

Also[7], if I is a finite or countable set of integers and $\{p_i|\ i\in I\}$ and $\{q_i|\ i\in I\}$ sets of positive numbers such that $\sum_{i\in I} p_i = 1$ and $\alpha = \sum_{i\in I} q_i < \infty$, under the condition $\sum_{i=1}^{n} p_i \log \frac{1}{q_i}$ finite, then

$$\sum_{i=1}^{n} p_i \log \frac{1}{p_i} \text{ is finite, too,}$$

and

$$0 < \sum_{i=1}^{n} p_i \log \frac{1}{p_i} < \sum_{i=1}^{n} p_i \log \frac{1}{q_i} + \log\alpha .$$

If additionally $\sum_{i\in I} \frac{p_i^2}{q_i} < \infty$, then

$$0 < \sum_{i=1}^{n} p_i \log \frac{1}{q_i} - \sum_{i=1}^{n} p_i \log \frac{1}{p_i} + \log\alpha \le \log\left[\alpha \sum_{i=1}^{n} \frac{p_i^2}{q_1}\right] \le \frac{1}{\ln b}\left[\alpha \sum_{i=1}^{n} \frac{p_i^2}{q_1} - 1\right],$$

with equality if and only if $q_i = p_i$ for all $i\in I$.

Kapur and Kumar [5] supposed that $\sum_{i\in I} p_i = a$ and $\sum_{i\in I} q_i = b$, with p_i, $q_i \ge 0$ for all I = 1, …, n and proved that

$$\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i} \geq a\log\frac{a}{b} \qquad \text{(if } a = b = 1 \Rightarrow \text{ Shannon's inequality).}$$

Also, Kapur and Kumar proved a generalization of Renyi inequality (obtained for $a = b = 1$):

$$\frac{1}{1-\alpha}\left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - \sum_{i=1}^{n} p_i\right] \geq \frac{1}{1-\alpha}\left[a^{\alpha}b^{1-\alpha} - a\right].$$

4. SOME RESULTS FROM NONCOMMUTATIVE INFORMATION THEORY

We shall introduce the parametric extension of Shannon inequality and its reverse one in Hilbert space operators.

Let T be a bounded linear operator on a Hilbert space H. We say that T is positive, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$. Also T is said to be strictly positive, $T > 0$, if T is invertible and positive.

In [3] was introduced the relative operator entropy, $S(A|B)$ for $A, B > 0$, an extension of the usual operator entropy $S(A|I) = -A \log A$.

$$S(A \mid B) = A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2} = S_0(A \mid B). \tag{1}$$

Definition 4.1. Let A, B be strictly positive operators. For any real number p is defined a new operator [2] considered the generalized parametric entropy

$$S_p(A \mid B) = A^{1/2}(A^{-1/2}BA^{-1/2})^p \log(A^{-1/2}BA^{-1/2})A^{1/2}, \tag{2}$$

where $S_0(A|B)$ is like above.

Definition 4.2. Let A, B be strictly positive operators. For any real number λ in [6] is defined a new operator $A\#_{\lambda}B$, a generalized mean operator type

$$A \#_{\lambda} B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\lambda} A^{1/2}, \tag{3}$$

where $A\#_{p}B$ for $p\in[0, 1]$ is p-power mean.

Remark 4.3. In the context of the above notation we remark that $S_1(A|B) = -S(B|A)$ and moreover $S_p(A|B) = -S_{1-p}(B|A)$ for any real number p. The original Shannon inequality can be rewritten as

$$0 \geq \sum_{i=1}^{n} p_i \log\frac{q_i}{p_i} = \sum_{i=1}^{n} p_i^{1/2}\left(\log p_i^{-1/2} q_i p_i^{-1/2}\right)p_i^{1/2} = \sum_{i=1}^{n} S(p_i \mid q_i),$$

with operator version

$$0 \geq \sum_{i=1}^{n} S(A_i \mid B_i).$$

So, there is following correspondence between the original case of the Shannon inequality

$$0 \geq \sum_{i=1}^{n} p_i \log\frac{q_i}{p_i} \geq -\log\sum_{i=1}^{n}\frac{p_i^2}{q_i}, \quad \text{for } p_i, q_i > 0, \quad \text{with} \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1 \tag{4}$$

and operator version of Shannon inequality

$$0 \geq \sum_{i=1}^{n} S(A_i \mid B_i) \geq -\log\sum_{i=1}^{n} A_i B_i^{-1} A_i \quad \text{for } A_i, B_i > 0, \quad \text{with} \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = I. \tag{5}$$

In [2], Furuta proved Shannon inequality for two sequence of strictly positive operators on a Hilbert space H $\{A_1, \ldots, A_n\}$, $\{B_1, \ldots, B_n\}$ such that $\sum_{i=1}^{n} A_i \#_p B_i \le I$ and $p\in[0, 1]$ using properties of two operator concave function $f(t) = \log t$ and $f(t) = -t\log t$.

$$\begin{aligned}
\sum_{j=1}^{n} S_{p+1}\left(A_j \mid B_j\right) &\ge \left[\sum_{j=1}^{n}\left(A_j \#_{p+1} B_j\right)+\left(I-\sum_{j=1}^{n}\left(A_j \#_p B_j\right)\right)\right]\log\left[\sum_{j=1}^{n}\left(A_j \#_{p+1} B_j\right)+\left(I-\sum_{j=1}^{n}\left(A_j \#_p B_j\right)\right)\right] \ge \\
&\ge \log\left[\sum_{j=1}^{n}\left(A_j \#_{p+1} B_j\right)+\left(I-\sum_{j=1}^{n}\left(A_j \#_p B_j\right)\right)\right] \ge \sum_{j=1}^{n} S_p\left(A_j \mid B_j\right) \ge \\
&\ge -\log\left[\sum_{j=1}^{n}\left(A_j \#_{p-1} B_j\right)+\left(I-\sum_{j=1}^{n}\left(A_j \#_p B_j\right)\right)\right] \ge \\
&\ge -\left[\sum_{j=1}^{n}\left(A_j \#_{p-1} B_j\right)+\left(I-\sum_{j=1}^{n}\left(A_j \#_p B_j\right)\right)\right]\log\left[\sum_{j=1}^{n}\left(A_j \#_{p-1} B_j\right)+\left(I-\sum_{j=1}^{n}\left(A_j \#_p B_j\right)\right)\right] \ge \\
&\ge \sum_{j=1}^{n} S_{p-1}\left(A_j \mid B_j\right).
\end{aligned} \tag{6}$$

From the definition of $S_p(A|B)$ for $A, B > 0$ and any real number p by an easy calculation we have

$$\frac{\mathrm{d}}{\mathrm{d}p}[S_p(A \mid B)] = A^{1/2}(A^{-1/2}BA^{-1/2})^p[\log(A^{-1/2}BA^{-1/2})]^2 A^{1/2} \ge 0\,.$$

So, we observe that $S_p(A|B)$ is an increasing function of p and it is reasonable to point out the decreasing order of previous expression (6).

Definition 4.4. Let F be an operator over Hilbert space H. F is a convex (concave) operator over $J \subset H$ if F satisfy

$$F(\alpha x+\beta y) \le (\ge)\ \alpha F(x)+\beta F(y), \tag{7}$$

for any $\alpha, \beta \in R$, with $\alpha + \beta = 1$, $\alpha, \beta \ge 0$ and $x, y \in J$.

Definition 4.5. Consider an operator F over $J \subset H$, a Hilbert space. We say that F is an convexifiable operator if there exists some real number α such that for $A\in B(H)$ the new operator

$$\varphi(A,\alpha) = F(A) - \frac{1}{2}\alpha \cdot A^* A \tag{8}$$

is convex over J.

Theorem 4.6 [4]. *Let f be a continuous, real function on an interval J. Then f is operator concave on J if and only if verifies inequality*

$$F(C^*AC+t_0(I-C^*C)) \ge C^*f(A)C+f(t_0)(I-C^*C) \tag{9}$$

for an operator C with $\|C\| \le 1$ and a self-adjoint operator A with $\sigma(A) \subseteq J$ and for a fixed real number $t_0 \in J$.

Moreover, the previous inequality can be reformulated as

$$f\left(\sum_{j=1}^{n} C_j^{*} A_j C_j + t_0\left(I-\sum_{j=1}^{n} C_j^{*} C_j\right)\right) \ge \sum_{j=1}^{n} C_j^{*} f(A_j) C_j + f(t_0)\left(I-\sum_{j=1}^{n} C_j^{*} C_j\right), \tag{10}$$

if we consider n operators C_j with $\sum_{j=1}^{n} C_j^{*} C_j \le I$ and self-adjoint operators A_j with $\sigma(A_j) \subseteq J$ for $j = 1, 2, ..., n$ and for a fixed real number $t_0 \in J$.

5. A NEW INEQUALITY OF SHANNON TYPE. THE MAIN RESULT.

In this paper we obtain more precise estimations of parametric expression of operatorial characterization of Shannon inequality. For that we use the characterization of operator convexifiable function.

Let F_α be a parametric function defined by *B(H)* the set of continuous function from Hilbert space *H* to *H*, with expression

$$F_\alpha(A) = A\ln A + \alpha \cdot A^* A, \tag{11}$$

where $A \in H$, operator strict pozitiv.

The definition domain of this function is

$$\text{D:}\quad \alpha < 0 \text{ and } 2\alpha \cdot A + I \le 0, \text{ for pozitive operators } A \in H. \tag{12}$$

(from the sign of second derivative of F_α and $A > 0$). Now, we can aplly Theorem 2.14 [1]. Thus, for a strict positive operator C on a Hilbert space H such that $\|C\| \le 1$ and a self-adjoint operator A with $\sigma(A) \subseteq J$, where J is the definition set of F_α, we have

$$F_\alpha(C^*AC + t_0(I - C^*C)) \ge C^* F_\alpha(A)C + F_\alpha(t_0)(I - C^*C),$$

with $t_0 \in J$.

Replacing F_α we obtain

$$\begin{aligned}&(C^*AC + t_0(I - C^*C))\ln\left(C^*AC + t_0(I - C^*C)\right) + \\ &+\alpha\left(C^*AC + t_0(I - C^*C)\right)\left(C^*AC + t_0(I - C^*C)\right)^{\mathrm{T}} \ge \\ &\ge C^*\left(A\ln A + \alpha A^2\right)C + (t_0\ln t_0 + \alpha t_0^2)(I - C^*C)\end{aligned}$$

Finally,

$$\begin{aligned}&(C^*AC + t_0(I - C^*C))\ln\left(C^*AC + t_0(I - C^*C)\right) \ge \\ &\ge C^*(A\ln A)C + (t_0\ln t_0)\left(I - C^*C\right) + \alpha[C^*A^2C + t_0^2\left(I - C^*C\right) - \left(C^*AC + t_0(I - C^*C)\right)^2].\end{aligned} \tag{13}$$

Problem. Let $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ be two sequences of strictly positive operators on Hilbert space *H,* such that $\sum_{i=1}^{n} A_j \#_p B_j \le I$. We are interested in analyzing the parametric extension of Shannon inequality and we want to response to answer the question: what assumption we need to have the relations

$$U \le \sum_{j=1}^{n} S_{p+1}\left(A_j \mid B_j\right) \le U + \delta, \tag{14}$$

$$V \ge \sum_{j=1}^{n} S_{p+1}\left(A_j \mid B_j\right) \ge V - \gamma, \tag{15}$$

where

$$U \approx \left[\sum_{j=1}^{n} A_j \#_{p+1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] \ln\left[\sum_{j=1}^{n} A_j \#_{p+1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right],$$

and

$$V \approx -\left[\sum_{j=1}^{n} A_j \#_{p-1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] \ln\left[\sum_{j=1}^{n} A_j \#_{p-1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right].$$

Indeed, we can consider a convexifiable function F_α, a family of n strictly positive operators$\{C_j\}$ defined on a Hilbert space with $\sum_{j=1}^{n} C_j^* C_j \le I$ and a family of self-adjoint operators $\{X_1, ..., X_n\}$ with $\sigma(X_i)$ in the domain of F_α, (verifying (12)). Then, as in (13) we obtain

$$\left(\sum_{j=1}^{n} C_j^* X_j C_j + t_0\left(I - \sum_{j=1}^{n} C_j^* C_j\right)\right)\ln\left(\sum_{j=1}^{n} C_j^* X_j C_j + t_0\left(I - \sum_{j=1}^{n} C_j^* C_j\right)\right) \ge$$
$$\ge \sum_{j=1}^{n} C_j^*\left(X_j \ln X_j\right)C_j + (t_0 \ln t_0)\left(I - \sum_{j=1}^{n} C_j^* C_j\right) + \alpha\phi(t_0), \tag{16}$$

where

$$\phi(t_0) = \sum_{j=1}^{n} C_j^* X_j^2 C_j + t_0^2\left(I - \sum_{j=1}^{n} C_j^* C_j\right) - \left(\sum_{j=1}^{n} C_j^* X_j C_j + t_0(I - \sum_{j=1}^{n} C_j^* C_j)\right)^2. \tag{17}$$

If we consider two families of n strictly positive operators $\{A_j\}_j$, $\{B_j\}_j$, we can consider $X_j = \left(A_j^{-1/2} B_j A_j^{-1/2}\right)^q > 0$ for a real number q and $C_j = \left(A_j^{-1/2} B_j A_j^{-1/2}\right)^{p/2} A_j^{1/2}$ with its conjugate $C_j^* = A_j^{1/2}\left(A_j^{-1/2} B_j A_j^{-1/2}\right)^{p/2}$, for a real number $p \in [0, 1]$. We obtain that $\sum_{j=1}^{n} C_j^* C_j = \sum_{j=1}^{n} A_j \#_p B_j \le I$. Now, the previous inequality (16) become,

$$\left[\sum_{j=1}^{n} A_j \#_{p+q} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right]\ln\left[\sum_{j=1}^{n} A_j \#_{p+q} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] \ge$$
$$\ge q\sum_{j=1}^{n} S_{p+q}(A_j \mid B_j) + (t_0 \ln t_0)(I - \sum_{j=1}^{n} A_j \#_p B_j) + \alpha \cdot \Phi(t_0, q), \tag{18}$$

for fixed real number $t_0 > 0$. Putting $q = 1$ and $q = -1$ we obtain

$$\sum_{j=1}^{n} S_{p+1}\left(A_j \mid B_j\right) \le \left[\sum_{j=1}^{n} A_j \#_{p+1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right]\ln\left[\sum_{j=1}^{n} A_j \#_{p+1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] -$$
$$- (t_0 \ln t_0)(I - \sum_{j=1}^{n} A_j \#_p B_j) - \alpha \cdot \Phi(t_0, 1), \tag{19}$$

$$\sum_{j=1}^{n} S_{p-1}\left(A_j \mid B_j\right) \ge -\left[\sum_{j=1}^{n} A_j \#_{p-1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right]\ln\left[\sum_{j=1}^{n} A_j \#_{p-1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] +$$
$$+ (t_0 \ln t_0)(I - \sum_{j=1}^{n} A_j \#_p B_j) + \alpha \cdot \Phi(t_0, -1), \tag{20}$$

with Φ

$$\Phi(t_0, 1) = (I - \sum_{j=1}^{n} A_j \#_p B_j) \cdot t_0^2 - [\sum_{i=1}^{n} S_{p+1}(A_j \mid B_j) + t_0(I - \sum_{j=1}^{n} A_j \#_p B_j)]^2 + \sum_{i=1}^{n} S_{p+2}(A_j \mid B_j) > 0, \tag{21}$$

$$\Phi(t_0,-1)=(I-\sum_{j=1}^{n}A_j\#_p B_j)\cdot t_0^{\,2}-[\sum_{i=1}^{n}S_{p-1}(A_j\mid B_j)+t_0(I-\sum_{j=1}^{n}A_j\#_p B_j)]^2+\sum_{i=1}^{n}S_{p-2}(A_j\mid B_j)>0. \tag{22}$$

In [2], Furuta finds a majoration for $\sum_{j=1}^{n}S_{p-1}\left(A_j\mid B_j\right)$ and a minoration for $\sum_{j=1}^{n}S_{p+1}\left(A_j\mid B_j\right)$. From (12) we have $\alpha<0$ and using (19), (20) and [2] we obtain that $\Phi(t_0,1)>0$ and $\Phi(t_0,-1)>0$.

Proposition 5.1. *Let $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ be two sequences of strictly positive operators on the Hilbert space H such that $\sum_{j=1}^{n}A_j\#_p B_j=I$. The parametric extension of Shannon inequality states the relations*

$$\left[\sum_{j=1}^{n}A_j\#_{p+1}B_j\right]\ln\left[\sum_{j=1}^{n}A_j\#_{p+1}B_j\right]\le\sum_{j=1}^{n}S_{p+1}\left(A_j\mid B_j\right)\le\left[\sum_{j=1}^{n}A_j\#_{p+1}B_j\right]\ln\left[\sum_{j=1}^{n}A_j\#_{p+1}B_j\right]-\alpha\cdot\Phi_1(1,1) \tag{23}$$

$$-\left[\sum_{j=1}^{n}A_j\#_{p-1}B_j\right]\ln\left[\sum_{j=1}^{n}A_j\#_{p-1}B_j\right]+\alpha\cdot\Phi_1(1,-1)\le\sum_{j=1}^{n}S_{p-1}\left(A_j\mid B_j\right)\le-\left[\sum_{j=1}^{n}A_j\#_{p-1}B_j\right]\ln\left[\sum_{j=1}^{n}A_j\#_{p-1}B_j\right], \tag{24}$$

if $\{A_j\}$ and $\{B_j\}$ verifying (12) *and $B_j=\lambda\cdot A_j$, for all j, with* $1<\lambda<\mathrm{e}$.

Proof. In this case, (21) and (22) become

$$\Phi_1(1,1)=\sum_{i=1}^{n}S_{p+2}(A_j\mid B_j)-(\sum_{i=1}^{n}S_{p+1}(A_j\mid B_j))^2, \tag{25}$$

$$\Phi_1(1,-1)=\sum_{i=1}^{n}S_{p-2}(A_j\mid B_j)-(\sum_{i=1}^{n}S_{p-1}(A_j\mid B_j))^2. \tag{26}$$

If we substitute $B_j=\lambda\cdot A_j$, we obtain $\sum_{j=1}^{n}A_j\#_p B_j=\lambda^p\sum_{j=1}^{n}A_j=I$, so $\sum_{j=1}^{n}A_j=\lambda^{-p}$. Then

$$\Phi_1(1,1)=\lambda^{p+2}\ln\lambda\sum_{i=1}^{n}A_j-(\lambda^{p+1}\ln\lambda\sum_{i=1}^{n}A_j)^2>0$$

and

$$\Phi_1(1,-1)=\lambda^{p-2}\ln\lambda\sum_{i=1}^{n}A_j-(\lambda^{p-1}\ln\lambda\sum_{i=1}^{n}A_j)^2>0$$

for $1<\lambda<\mathrm{e}$.

Indeed, $-\alpha\cdot\Phi_1(1,1)\ge0$ and $\alpha\cdot\Phi_1(1,-1)\le0$

Corollary 5.2. *If additionally we have* $\alpha=0$, *then*

$$S_{p+1}=\left[\sum_{j=1}^{n}A_j\#_{p+1}B_j\right]\ln\left[\sum_{j=1}^{n}A_j\#_{p+1}B_j\right], \tag{27}$$

$$S_{p-1}-\left[\sum_{j=1}^{n}A_j\#_{p-1}B_j\right]\ln\left[\sum_{j=1}^{n}A_j\#_{p-1}B_j\right]. \tag{28}$$

Open problem. In the same context we are interested to obtain (for $t_0=1$)

$$\left[\sum_{j=1}^{n} A_j \#_{p+1} B_j + \left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] \ln\left[\sum_{j=1}^{n} A_j \#_{p+1} B_j + \left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] \le \sum_{j=1}^{n} S_{p+1}\left(A_j \mid B_j\right) \le$$

$$\le \left[\sum_{j=1}^{n} A_j \#_{p+1} B_j + \left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] \ln\left[\sum_{j=1}^{n} A_j \#_{p+1} B_j + \left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] - \alpha \cdot \Phi(1,1),$$

and

$$-\left[\sum_{j=1}^{n} A_j \#_{p-1} B_j + \left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] \ln\left[\sum_{j=1}^{n} A_j \#_{p-1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] + \alpha \cdot \Phi(1,-1) \le$$

$$\le \sum_{j=1}^{n} S_{p-1}\left(A_j \mid B_j\right) \le -\left[\sum_{j=1}^{n} A_j \#_{p-1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right] \ln\left[\sum_{j=1}^{n} A_j \#_{p-1} B_j + t_0\left(I - \sum_{j=1}^{n} A_j \#_p B_j\right)\right],$$

where

$$\Phi(1,1) = (I - \sum_{j=1}^{n} A_j \#_p B_j)(\sum_{j=1}^{n} A_j \#_p B_j) - 2 \cdot \sum_{j=1}^{n} S_{p+1}(A_j \mid B_j) + (\sum_{j=1}^{n} A_j \#_p B_j)(\sum_{j=1}^{n} S_{p+1}(A_j \mid B_j)) +$$

$$+ (\sum_{j=1}^{n} S_{p+1}(A_j \mid B_j))(\sum_{j=1}^{n} A_j \#_p B_j) + \sum_{i=1}^{n} S_{p+2}(A_j \mid B_j) - (\sum_{i=1}^{n} S_{p+1}(A_j \mid B_j))^2,$$

and

$$\Phi(1,-1) = (I - \sum_{j=1}^{n} A_j \#_p B_j)(\sum_{j=1}^{n} A_j \#_p B_j) - 2 \cdot \sum_{j=1}^{n} S_{p-1}(A_j \mid B_j) + (\sum_{j=1}^{n} A_j \#_p B_j)(\sum_{j=1}^{n} S_{p-1}(A_j \mid B_j)) +$$

$$+ (\sum_{j=1}^{n} S_{p-1}(A_j \mid B_j))(\sum_{j=1}^{n} A_j \#_p B_j) + \sum_{i=1}^{n} S_{p-2}(A_j \mid B_j) - (\sum_{i=1}^{n} S_{p-1}(A_j \mid B_j))^2.$$

Also using [8, 9, 10, 11] on this line we can considered some optimization problems.

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