THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 11, Number 3/2010, pp. 253–260

CALCULUS OF JOINT FORCES USING LAGRANGE EQUATIONS AND PRINCIPLE OF VIRTUAL WORK

Ion STROE, Ştefan STAICU

"Politehnica" University of Bucharest, Romania
E-mail: ion.stroe@gmail.com

Lagrange equations and the principle of virtual work are used to study the motion of a system under the action of known external and internal forces. If an internal force has to be found, a supplementary mobility is considered in the system. The corresponding internal force for new mobility is found for zero mobility. General kinematics problem of systems of rigid bodies with constraints is first presented in paper. Based on a comparative analysis of the Lagrange formalism and the principle of virtual work, the models allow solving a large number of problems in multi-bodies systems dynamics.

Key words: Kinematicsş Multi-bodies system, Constraint, dynamics.

LIST OF SYMBOLS

$a_{k,k-1}$ – orthogonal transformation matrix

$\varphi_{k,k-1}$ – relative rotation angle of T_k rigid body

$\vec{\omega}_{k,k-1}$ – relative angular velocity of T_k

$\tilde{\omega}_{k,k-1}$ – skew-symmetric matrix associated to the angular velocity $\vec{\omega}_{k,k-1}$

m_k, $\hat{J}_k$ – mass and symmetric matrix of tensor of inertia of T_k about the link-frame $x_k y_k z_k$

$m_{k,k-1}$, $f_{k,k-1}$ – active torques and joint forces

1. INTRODUCTION

The dynamics of multi-bodies systems is complicated by existence of closed-loop chains. Difficulties commonly encountered in dynamics modelling include problematic issues such as: complicated kinematical structure with possess of large number of passive degrees of freedom, dominance of inertial forces over the frictional and gravitational components and the problem linked to the solution of the inverse dynamics.

In the context of the real-time control, neglecting the frictions forces and considering the gravitational effects, the relevant objective of the multi-bodies dynamics is to determine the input torques or forces and the external and internal joint forces. Upon to now, several methods have been applied to formulate the forward and inverse dynamics, which could provide the same results concerning these torques or forces. The first one is using the Newton-Euler classical procedure, the second one applies the Lagrange's equations and multipliers formalism and the third one is based on the principle of virtual work [1, 2, 3].

Lu [4] uses the virtual method to determine, in spatial parallel structures, the generalized forces of the actuators and relates them to the real forces that they exert. Geike and Mc Phee [5] proposed a general approach which could determine the inverse dynamic solutions for a planar 3-RRR parallel manipulator and a spatial 6-DOF parallel mechanism.

2. KINEMATICS OF SYSTEMS OF RIGID BODIES

Let two bodies (T_i) and (T_j) be with constrained motions by a coupling mechanism which is made precise by two points O_i and O_j (Fig.1). The motion of the body (T_i) with respect the inertial reference frame $O_0x_0y_0z_0(T_0)$ is determined by the position vector $\vec{r}_0^{C_i} = \overrightarrow{O_0C_i}$ of mass center C_i and by the transformation matrix a_{i0} which gives the attitude of the frame $C_ix_iy_iz_i$ jointed with (T_i) body with respect the fixed reference frame. In the same way are defined position vector $\vec{r}_0^{C_j} = \overrightarrow{O_0C_j}$ and matrix a_{j0} for the body (T_j).

A free body (T_i) or (T_j), has six degrees of freedom. But, the number of degrees of freedom is reduced by the number of constrains which are imposed by several coupling mechanisms. Coupling mechanisms between (T_i) and (T_j) imposes restrictions on relative motion of (T_i) body with respect to (T_j).

If a general motion of bodies (T_i) and (T_j) with respect the fixed reference frame $O_0x_0y_0z_0$ are known, then the relative motion of the body (T_i) with respect (T_j) can be determined by the relative position vector

$$\vec{r}_{ij} = \vec{r}_0^{C_i} - \vec{r}_0^{C_j} \tag{2.1}$$

and by the matrix a_{ij} which gives the relative attitude of (T_i) body versus (T_j) body.

Starting from the frame $C_jx_jy_{ji}z_j(T_j)$ and ending to the frame $C_ix_iy_iz_i(T_i)$, we can evaluate nine parameters $\alpha_{11} = \vec{i}_j^{\mathrm{T}}\vec{i}_i$, $\alpha_{12} = \vec{j}_j^{\mathrm{T}}\vec{i}_i$, $\alpha_{13} = \vec{k}_j^{\mathrm{T}}\vec{i}_i$, $\alpha_{21} = \vec{i}_j^{\mathrm{T}}\vec{j}_i$, $\alpha_{22} = \vec{j}_j^{\mathrm{T}}\vec{j}_i$, $\alpha_{23} = \vec{k}_j^{\mathrm{T}}\vec{j}_i$, $\alpha_{31} = \vec{i}_j^{\mathrm{T}}\vec{k}_i$, $\alpha_{32} = \vec{j}_j^{\mathrm{T}}\vec{k}_i$, $\alpha_{33} = \vec{k}_j^{T}\vec{k}_i$ giving the relative orientation of the mobile axes x_i, y_i, z_i with respect the frame of x_j, y_j, z_j axes: These parameters constitute the contents of an orthogonal matrix of transformation a_{ij}.

We note that the projections in the frame $C_ix_iy_iz_i$ of a vector $\vec{r}_j$ known in the space of the $C_jx_jy_jz_j$ frame are given by the matrix relation $\vec{r}_i = a_{ij}\vec{r}_j$ and that the matrix a_{ij} could be easily determined using two absolute matrices: $a_{ij} = a_{i0}a_{j0}^{\mathrm{T}}$. Since all rotations take place successively about the moving coordinate axes x_0, y', z_i, the general rotation matrix a_{ij} is obtained by multiplying three transformation matrices, as follows

$$a_1^i = \mathrm{rot}(x_0, \varphi_{1i}),\ a_2^i = \mathrm{rot}(y', \varphi_{2i}),\ a_3^i = \mathrm{rot}(z_i, \varphi_{3i}),\ a_{ij} = a_3^ia_2^ia_1^i. \tag{2.2}$$

The absolute angular velocity $\vec{\omega}_{i0}$ of the body (T_i), for example, fixed in the frame $C_ix_iy_iz_i$ is a vector *associated* to the skew symmetric matrix $\tilde{\omega}_{i0} = a_{i0}\dot{a}_{i0}^{\mathrm{T}}$ as follows

$$\vec{\omega}_{i0} = \dot{\varphi}_{1i}a_3^ia_2^i\vec{u}_1 + \dot{\varphi}_{2i}a_3^i\vec{u}_2 + \dot{\varphi}_{3i}\vec{u}_3, \tag{2.3}$$

where $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are three orthogonal unit vectors.

Now, considering a kinematical chain $T_0, T_1, ..., T_j, ..., T_{k-1}, T_k, ..., T_i, ..., T_n$, the motions of the compounding elements are characterized by the following skew symmetric matrices

$$\tilde{\omega}_{k0} = a_{k,k-1}\tilde{\omega}_{k-1,0}a_{k,k-1}^{\mathrm{T}} + \omega_{k,k-1}\tilde{u}_3,\ \omega_{k,k-1} = \dot{\varphi}_{k,k-1}, \tag{2.4}$$

which are associated to the absolute angular velocities given by the recursive relations

$$\vec{\omega}_{k0} = a_{k,k-1}\vec{\omega}_{k-1,0} + \omega_{k,k-1}\vec{u}_3. \tag{2.5}$$

Following relation give the absolute velocity $\vec{v}_{k0}$ of the origin O_k of the frame (T_k)

$$\vec{v}_{k0} = a_{k,k-1}\vec{v}_{k-1,0} + a_{k,k-1}\tilde{\omega}_{k-1,0}\vec{r}_{k,k-1} + v_{k,k-1}\vec{u}_3. \tag{2.6}$$

Some recursive relations give also the angular acceleration $\vec{\varepsilon}_{k0}^{A}$ and the absolute acceleration $\vec{\gamma}_{k0}$ of the joint O_k

$$\vec{\varepsilon}_{k0}^{A} = a_{k,k-1}\vec{\varepsilon}_{k-1,0}^{A} + \varepsilon_{k,k-1}^{A}\vec{u}_3 + \omega_{k,k-1}^{A}a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}a_{k,k-1}^{\mathrm{T}}\vec{u}_3$$

$$\vec{\gamma}_{k0}^{A} = a_{k,k-1}\left[\vec{\gamma}_{k-1,0}^{A} + a_{k,k-1}\left(\tilde{\omega}_{k-1,0}^{A}\tilde{\omega}_{k-1,0}^{A} + \tilde{\varepsilon}_{k-1,0}^{A}\right)\ \vec{r}_{k,k-1}^{A}\right] + 2v_{k,k-1}^{A}a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}a_{k,k-1}^{\mathrm{T}}\vec{u}_3 + \gamma_{k,k-1}^{A}\vec{u}_3 . \tag{2.7}$$

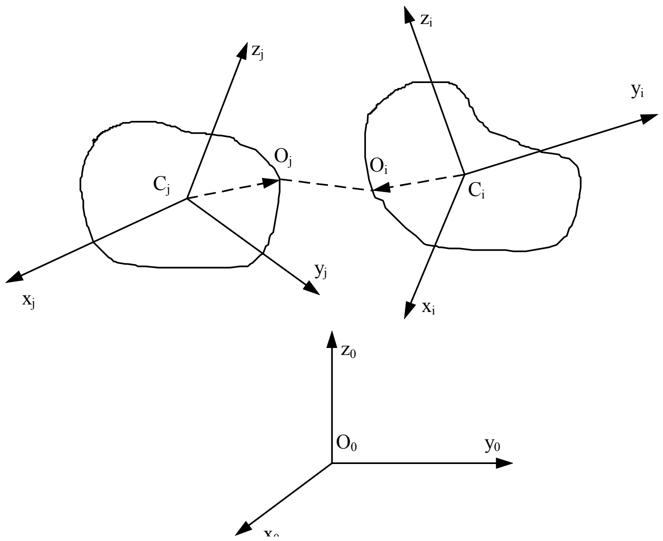

Fig.1 – System of rigid bodies.

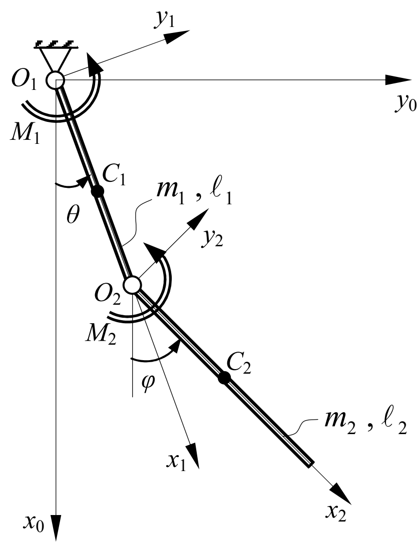

Fig. 2 – Composed pendulum.

3. DYNAMICS OF SYSTEMS OF RIGID BODIES

3.1. Lagrange equations

When constraints are functions of coordinates, the motion of the systems can be studied with Lagrange equations for holonomic systems with dependent variables, while if others conditions of constraint are expressed by velocities, the motion is described with Lagrange equations for non-holonomic systems.

For a non-holonomic rheonomic system, the general equations of Lagrange corresponding to a system of h generalized coordinates [6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right) - \frac{\partial E}{\partial q_k} = Q_k + \sum_{i=1}^{p}\lambda_i a_{ik} \qquad k = 1,2,\cdots,h \tag{3.1}$$

are completed with conditions of constraint

$$\sum_{k=1}^{h} a_{ik}\dot{q}_k + b_i = 0 \qquad i = 1,2,\cdots p . \tag{3.2}$$

Solving the system of equations (3.1) and (3.2), the generalized coordinates q_k and the Lagrange multipliers λ_i are immediately obtained. In the case of a holonomic system when the constraint relations are expressed in an implicit form, we have

$$a_{ik} = \frac{\partial \Phi_i}{\partial q_k}, \ \Phi_i\left(q_1,\cdots,q_h,t\right) = 0, \quad i = 1,2,\cdots p . \tag{3.3}$$

Defining the analytical function $U_{\varphi} = \sum_{i=1}^{p} \lambda_i \Phi_i$, above differential equations become

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right) - \frac{\partial E}{\partial q_k} = Q_k + \frac{\partial U_{\Phi}}{\partial q_k}, \quad k = 1,2,\cdots,h. \tag{3.4}$$

Starting from these h differential equations with p relations of constraint, we determine just the generalized coordinates q_k and the Lagrange multipliers λ_i [7].

3.2. Principle of virtual work

A solution of the dynamics problem of a multi-bodies mechanism provided with constraints can be developed based on the fundamental principle of virtual work. Knowing the position and kinematics state of each link as well as the external forces acting on the mechanism, we applies the principle of virtual work for the inverse dynamic problem in order to establish some matrix relations giving the input forces or torques required in a given motion, using a recursive procedure [8].

The spatial mechanism can artificially be transformed in a set of some open chains subject to the constraints. This is possible by cutting each joint and taking its effect into account by introducing the corresponding constraint conditions.

The force of inertia of an arbitrary rigid body T_k , for example, and the resulting moment of the forces of inertia

$$\begin{aligned} \vec{f}_{k0}^{in} &= -m_k\left[\vec{\gamma}_{k0} + \left(\tilde{\omega}_{k0}\tilde{\omega}_{k0} + \tilde{\varepsilon}_{k0}\right)\ \vec{r}_k^{\,C}\right], \\ \vec{m}_{k0}^{in} &= -\ [m_k \tilde{r}_k^C \vec{\gamma}_{k0} + \hat{J}_k \vec{\varepsilon}_{k0} + \tilde{\omega}_{k0}\hat{J}_k \vec{\omega}_{k0}] \end{aligned} \tag{3.5}$$

are determined with respect to the centre of joint O_k . On the other hand, the wrench of two vectors $\vec{f}_k^{\,*}$ and $\vec{m}_k^{\,*}$ evaluates the influence of the action of the weight $m_k \vec{g}$ and of other external and internal forces applied to the same element T_k of the mechanism, for example

$$\vec{f}_k^{\,*} = 9.81\, m_k a_{k0} \vec{u}_3\,,\ \vec{m}_k^{\,*} = 9.81\, m_k \tilde{r}_k^C a_{k0} \vec{u}_3\,. \tag{3.6}$$

Considering successive independent virtual motions of the mechanical system, *virtual displacements and velocities* should be compatible with the virtual motions imposed by all kinematical constraints and joints at a given instant in time.

The fundamental principle of the virtual work states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism.

Assuming that frictional forces at the joints are negligible, the virtual work produced by the forces of constraint at the joints is zero. So, the virtual powers contributed by active force $\vec{f}_{q,q-1}$ and actuator torque $\vec{m}_{q,q-1}$, known external forces and moments $\vec{f}_{\tau}^{\,*}$ and $\vec{m}_{\tau}^{\,*}$ and by inertia forces and moments of inertia forces $\vec{f}_{\tau}^{\,in}$ and $\vec{m}_{\tau}^{\,in}$, can be written as follows [9]:

$$v_{q,q-1}^{v} f_{q,q-1} + \omega_{q,q-1}^{v} m_{q,q-1} = \vec{u}_3^{\,T} \sum_{\tau=1}^{n} \{v_{\tau,\tau-1}\vec{F}_{\tau} + \omega_{\tau,\tau-1}\vec{M}_{\tau}\}\,, \tag{3.7}$$

where

$$\begin{gathered} \vec{F}_{\tau} = \vec{F}_{\tau 0} + a_{\tau+1,\tau}^{T}\vec{F}_{\tau+1},\ \vec{M}_{\tau} = \vec{M}_{\tau 0} + a_{\tau+1,\tau}^{T}\vec{M}_{\tau+1} + \tilde{r}_{\tau+1,\tau}a_{\tau+1,\tau}^{T}\vec{F}_{\tau+1} \\ \vec{F}_{\tau 0} = -\vec{f}_{\tau}^{\,*} - \vec{f}_{\tau}^{\,in},\ \vec{M}_{\tau 0} = -\vec{m}_{\tau}^{\,*} - \vec{m}_{\tau}^{\,in}\,. \end{gathered} \tag{3.8}$$

The dynamics model expressed by the recursive matrix equations (3.7) and (3.8) represents the explicit dynamics equations of a multi-bodies constrained system.

4. CALCULUS OF INTERNAL JOINT FORCES

4.1. Lagrange equations

For a mechanical system with h degrees of freedom represented by the set of independent variables $(q)=(q_1,\ q_2,\ .....,q_h)$, the Lagrange equations are expressed in following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right)-\frac{\partial E}{\partial q_k}=\frac{\partial U}{\partial q_k}+\Re_{k+1},\ \ k=1,2,\ ...,\ h,\ h+1\,, \tag{4.1}$$

where the joint force $\Re_{k+1}=Q_{h+1}^{*}$ as new generalized force can be found if a new fictitious mobility in accord with the joint is considered. So, considering again the mechanism, the reaction is easily obtained from (4.1) in following definitive form

$$\Re_{h+1}=\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_{h+1}}\right)-\frac{\partial E}{\partial q_{h+1}}-\frac{\partial U}{\partial q_{h+1}}\right]_{\substack{q_{h+1}=0\\ \dot{q}_{h+1}=0\\ \ddot{q}_{h+1}=0}}. \tag{4.2}$$

4.2. Principle of virtual work

Above compact relations (3.7) and (3.8) can be also applied to calculate any joint force or joint torque by cutting successively each joint O_k and writing the formulae as follows

$$\begin{aligned} f_{k,k-1}^{x}&=\vec{u}_1^{\mathrm{T}}a_{k0}^{\mathrm{T}}\vec{F}_k\ ,\ f_{k,k-1}^{y}=\vec{u}_2^{\mathrm{T}}a_{k0}^{\mathrm{T}}\vec{F}_k\ ,\ f_{k,k-1}^{z}=\vec{u}_3^{\mathrm{T}}a_{k0}^{\mathrm{T}}\vec{F}_k\\ m_{k,k-1}^{x}&=\vec{u}_1^{\mathrm{T}}a_{k0}^{\mathrm{T}}\vec{M}_k\ ,\ m_{k,k-1}^{y}=\vec{u}_2^{\mathrm{T}}a_{k0}^{\mathrm{T}}\vec{M}_k\ ,\ m_{k,k-1}^{z}=\vec{u}_3^{\mathrm{T}}a_{k0}^{\mathrm{T}}\vec{M}_k\ . \end{aligned} \tag{4.3}$$

5. EXAMPLE

The two degrees-of-freedom system of a planar composed pendulum is considered as example (Fig. 2). We consider that the pendulum is initially located at a vertical position, where the two rods are not rotated with respect to the fixed base.

The first link consists of a moving crank O_1O_2 of length l_1, mass m_1 and tensor of inertia $\hat{J}_1$ with respect $O_1x_1y_1z_1$ frame, which has rotation about z_1 axis with the angle $\theta=\varphi_{10}$, the angular velocity $\omega_{10}=\dot{\varphi}_{10}$ and the angular acceleration $\varepsilon_{10}=\ddot{\varphi}_{10}$. A second element of the leg is a rigid rod jointed in O_2 and linked at the $O_2x_2y_2z_2$ frame, having a relative rotation with the angle $\varphi_{21}=\varphi-\theta$, angular velocity $\omega_{21}=\dot{\varphi}_{21}$ and angular acceleration $\varepsilon_{21}=\ddot{\varphi}_{21}$. It has the length l_2, mass m_2 and tensor of inertia $\hat{J}_2$.

Starting from the reference origin O_1 and pursuing the two rods one obtains the following transformation matrices

$$a_{k,k-1}=\mathrm{rot}(z_k,\varphi_{k,k-1})\,,\quad k=1,\ 2;\quad a_{20}=a_{21}a_{10}\,. \tag{5.1}$$

Concerning the kinematics of this system we note following position vectors, velocities and accelerations

$$\vec{r}_{10}=\vec{0},\quad \vec{r}_{21}=[l_1\ \ 0\ \ 0]^{\mathrm{T}},\quad \vec{r}_1^{\,C}=[l_1/2\ \ 0\ \ 0]^{\mathrm{T}},\quad \vec{r}_2^{\,C}=[l_2/2\ \ 0\ \ 0]^{\mathrm{T}},\quad \vec{r}_{20}=\vec{r}_{10}+a_{10}^{\mathrm{T}}\vec{r}_{21}$$
$$\vec{\omega}_{10}=\dot{\varphi}_{10}\vec{u}_3,\quad \tilde{\omega}_{10}=\dot{\varphi}_{10}\tilde{u}_3,\quad \vec{v}_{10}=\vec{0},\quad \vec{\varepsilon}_{10}=\ddot{\varphi}_{10}\vec{u}_3,\quad \tilde{\varepsilon}_{10}=\ddot{\varphi}_{10}\tilde{u}_3,\quad \vec{\gamma}_{10}=\vec{0}$$
$$\vec{\omega}_{21}=\dot{\varphi}_{21}\vec{u}_3,\quad \tilde{\omega}_{21}=\dot{\varphi}_{21}\tilde{u}_3,\quad \vec{v}_{21}=\vec{0},\quad \vec{\varepsilon}_{21}=\ddot{\varphi}_{21}\vec{u}_3,\quad \tilde{\varepsilon}_{21}=\ddot{\varphi}_{21}\tilde{u}_3,\quad \vec{\gamma}_{21}=\vec{0} \tag{5.2}$$
$$\vec{\omega}_{20}=a_{21}\vec{\omega}_{10}+\vec{\omega}_{21},\quad \tilde{\omega}_{20}=a_{21}\tilde{\omega}_{10}a_{21}^{\mathrm{T}}+\tilde{\omega}_{21},\quad \vec{v}_{20}=a_{21}(\vec{v}_{10}+\tilde{\omega}_{10}\vec{r}_{21})+\vec{v}_{21}$$
$$\vec{\varepsilon}_{20}=a_{21}\vec{\varepsilon}_{10}+\vec{\varepsilon}_{21},\quad \tilde{\varepsilon}_{20}=a_{21}\tilde{\varepsilon}_{10}a_{21}^{\mathrm{T}}+\tilde{\varepsilon}_{21},\quad \vec{\gamma}_{20}=a_{21}\{\vec{\gamma}_{10}+(\tilde{\omega}_{10}\tilde{\omega}_{10}+\tilde{\varepsilon}_{10})\vec{r}_{21}\}+\vec{\gamma}_{21}\,.$$

5.1. Lagrange equations

Two generalized coordinates of the open system are represented by the independent variables

$$q_1=\theta=\varphi_{10},\ q_2=\varphi-\theta=\varphi_{21}\,. \tag{5.3}$$

The Lagrange's equations will be expressed by two differential relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right)-\frac{\partial E}{\partial q_k}=\frac{\partial U}{\partial q_k}+Q_k\quad (k=1,2)\,, \tag{5.4}$$

which contain two active torques M_1 and M_2 as generalized forces $Q_1=M_1$, $Q_2=M_2$.

The general expression of the Lagrange function $L=E+U=L_1+L_2$ is expressed as analytical functions of the generalized coordinates and their first derivatives with respect to time

$$L_1=\frac{1}{2}\vec{\omega}_{10}^{\mathrm{T}}\hat{J}_1\,\vec{\omega}_{10}+m_1 g\vec{u}_1^{\mathrm{T}}a_{10}^{\mathrm{T}}\vec{r}_1^{\,C},$$
$$L_2=\frac{1}{2}m_2\vec{v}_{20}^{\mathrm{T}}\vec{v}_{20}+\frac{1}{2}\vec{\omega}_{20}^{\mathrm{T}}\hat{J}_2\vec{\omega}_{20}+m_2\vec{v}_{20}^{\mathrm{T}}\tilde{\omega}_{20}\vec{r}_2^{\,C}+\ m_2 g\vec{u}_1^{\mathrm{T}}a_{10}^{\mathrm{T}}(\vec{r}_{21}+a_{21}^{\mathrm{T}}\vec{r}_2^{\,C})\,. \tag{5.5}$$

The first derivatives of orthogonal matrices $a_{k,k-1}$ are computed as follows:

$$\dot{a}_{k,k-1}=\dot{\varphi}_{k,k-1}\tilde{u}_3^{\mathrm{T}}a_{k,k-1},\ \dot{a}_{k,k-1}^{\mathrm{T}}=\dot{\varphi}_{k,k-1}a_{k,k-1}^{\mathrm{T}}\tilde{u}_3,\ \frac{\partial a_{k,k-1}}{\partial\varphi_{k,k-1}}=\tilde{u}_3^{\mathrm{T}}a_{k,k-1},\ \frac{\partial a_{k,k-1}^{\mathrm{T}}}{\partial\varphi_{k,k-1}}=a_{k,k-1}^{\mathrm{T}}\tilde{u}_3\ \ (k=1,2)\,. \tag{5.6}$$

After a calculus about the partial derivatives of the functions (5.5) and the derivatives with respect to time, finally we obtain the expressions for the input torques

$$M_1=(\frac{1}{3}m_1+m_2)l_1^2\ddot{\varphi}_{10}+\frac{1}{3}m_2 l_2^2(\ddot{\varphi}_{10}+\ddot{\varphi}_{21})+\frac{1}{2}m_2 l_1 l_2(2\ddot{\varphi}_{10}+\ddot{\varphi}_{21})\cos\varphi_{21}-$$
$$-\frac{1}{2}m_2 l_1 l_2\dot{\varphi}_{21}(2\dot{\varphi}_{10}+\dot{\varphi}_{21})\sin\varphi_{21}+\frac{1}{2}(m_1+2m_2)gl_1\sin\varphi_{10}+\frac{1}{2}m_2 g l_2\sin(\varphi_{10}+\varphi_{21}), \tag{5.7}$$
$$M_2=\frac{1}{3}m_2 l_2^2(\ddot{\varphi}_{10}+\ddot{\varphi}_{21})+\frac{1}{2}m_2 l_1 l_2(\ddot{\varphi}_{10}\cos\varphi_{21}+\dot{\varphi}_{10}^2\sin\varphi_{21})+\frac{1}{2}m_2 g l_2\sin(\varphi_{10}+\varphi_{21})\,.$$

We suppose a fictitious vertical displacement x of the joint O_1. Now, the position of the system is evaluated by three generalised coordinates: φ_{10}, φ_{21} and the added variable x. Considering $\vec{r}_{10}=x\vec{u}_1$, $\vec{v}_{10}=\dot{x}\vec{u}_1$, $\vec{\gamma}_{10}=\ddot{x}\vec{u}_1$, we determine a new expression for the Lagrange function $L=E+U$ and we replace it in the formula (4.2); it results the expression of the vertical joint force

$$f_{10}^{x}=-(m_1+m_2)g-\frac{1}{2}(m_1+2m_2)l_1(\ddot{\varphi}_{10}\sin\varphi_{10}+\dot{\varphi}_{10}^2\cos\varphi_{10})-$$
$$-\frac{1}{2}m_2 l_2[(\ddot{\varphi}_{10}+\ddot{\varphi}_{21})\sin(\varphi_{10}+\varphi_{21})+(\dot{\varphi}_{10}+\dot{\varphi}_{21})^2\cos(\varphi_{10}+\varphi_{21})]. \tag{5.8}$$

5.2. Principle of virtual work

Starting from the matrix relations (3.5) and (3.6), we determine the wrench for the inertia forces and the weights of two rods with respect two joints O_1 and O_2. Replacing successively in the formulae (3.8), we obtain the vectors

$$\vec{F}_{10} = m_1\{\vec{\gamma}_{10} + (\tilde{\omega}_{10}\tilde{\omega}_{10} + \tilde{\varepsilon}_{10})\vec{r}_1^{\,C}\} - m_1 g a_{10}\vec{u}_1 \, , \vec{F}_{20} = m_2\{\vec{\gamma}_{20} + (\tilde{\omega}_{20}\tilde{\omega}_{20} + \tilde{\varepsilon}_{20})\vec{r}_2^{\,C}\} - m_2 g a_{20}\vec{u}_1 \, ,$$
$$\vec{M}_{10} = m_1\tilde{r}_1^{C}\vec{\gamma}_{10} + \hat{J}_1\vec{\varepsilon}_{10} + \tilde{\omega}_{10}\hat{J}_1\vec{\omega}_{10} - m_1 g\tilde{r}_1^{C} a_{10}\vec{u}_1 \, , \; \vec{M}_{20} = m_2\tilde{r}_2^{C}\vec{\gamma}_{20} + \hat{J}_2\vec{\varepsilon}_{20} + \tilde{\omega}_{20}\hat{J}_2\vec{\omega}_{20} - m_2 g\tilde{r}_2^{C} a_{20}\vec{u}_1 \tag{5.9}$$
$$\vec{F}_2 = \vec{F}_{20} \, , \; \vec{F}_1 = \vec{F}_{10} + a_{21}^{\mathrm{T}}\vec{F}_2 \, , \; \vec{M}_2 = \vec{M}_{20} \, , \; \vec{M}_1 = \vec{M}_{10} + a_{21}^{\mathrm{T}}\vec{M}_2 + \tilde{r}_{21}a_{21}^{\mathrm{T}}\vec{F}_2 \, .$$

Using the explicit dynamics equations of the multi-bodies constrained systems (3.7), the active torques and the joint forces in external and internal joints O_1 and O_2 are quickly calculated

$$m_{10}^{z} = \vec{u}_3^{\mathrm{T}} a_{10}^{\mathrm{T}}\vec{M}_1 \, , \; f_{10}^{x} = \vec{u}_1^{\mathrm{T}} a_{10}^{\mathrm{T}}\vec{F}_1 \, , \; f_{10}^{y} = \vec{u}_2^{\mathrm{T}} a_{10}^{\mathrm{T}}\vec{F}_1 \, ,$$
$$m_{21}^{z} = \vec{u}_3^{\mathrm{T}} a_{20}^{\mathrm{T}}\vec{M}_2 \, , \; f_{21}^{x} = \vec{u}_1^{\mathrm{T}} a_{20}^{\mathrm{T}}\vec{F}_2 \, , \; f_{21}^{y} = \vec{u}_2^{\mathrm{T}} a_{20}^{\mathrm{T}}\vec{F}_2 \, . \tag{5.10}$$

We remark a good agreement between the expressions for the torques m_{10}^{z}, m_{21}^{z} and the vertical joint force f_{10}^{x}, for example, and the expressions (5.7), (5.8) above given using the Lagrange formalism.

In the inverse dynamics, we suppose that the history of the rotations motions of two rods are known by following functions

$$\varphi_{10}(t) = \varphi_{10}^{*}[1 - \cos(\frac{\pi}{6}t)] \, , \; \varphi_{21}(t) = \varphi_{21}^{*}\left[1 - \cos\left(\frac{\pi}{6}t\right)\right] . \tag{5.11}$$

For simulation purposes let us consider a composed pendulum which has the following characteristics:

$$l_1 = 0.50 \text{ m} \, , \; l_2 = 0.75 \text{ m}, \; m_1 = 0.25 \text{ kg}, \; m_2 = 1 \text{ kg}, \; \varphi_{10}^{*} = \frac{\pi}{12} \, , \; \varphi_{21}^{*} = \frac{\pi}{6} \, .$$

Using the MATLAB software, a computer program was developed to solve the inverse dynamics of the composed pendulum. To illustrate the algorithm, it is assumed that for a period of six second the rods start at rest from initial position and rotate about its revolute joints. The active torques (Fig. 3, Fig. 4), the horizontal joint force of first rod (Fig. 5) and the vertical joint force of second rod (Fig. 6) are calculated by the program and plotted *versus* time.

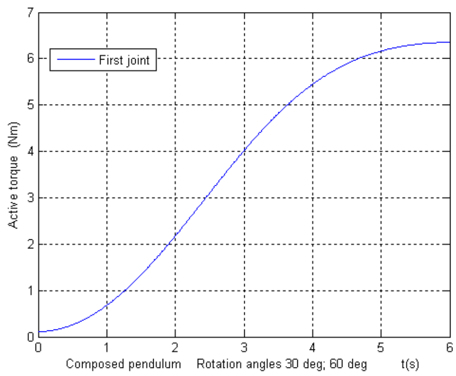

Fig. 3 – Active torque of first rod.

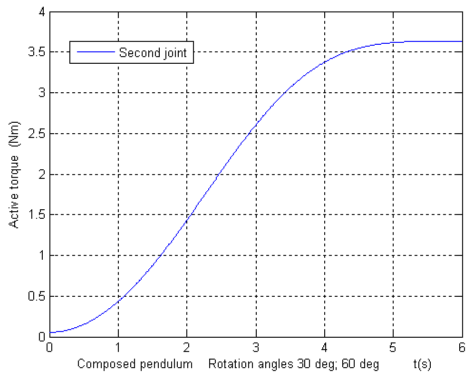

Fig. 4 – Active torque of second rod.

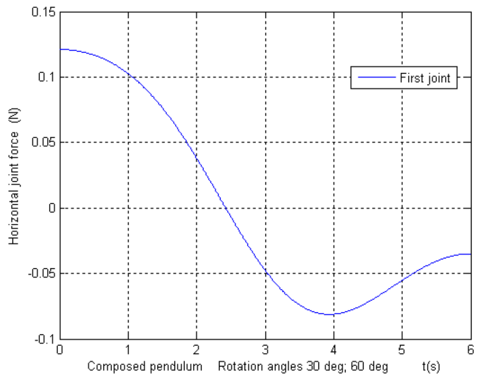

Fig. 5 – Horizontal joint force of first rod.

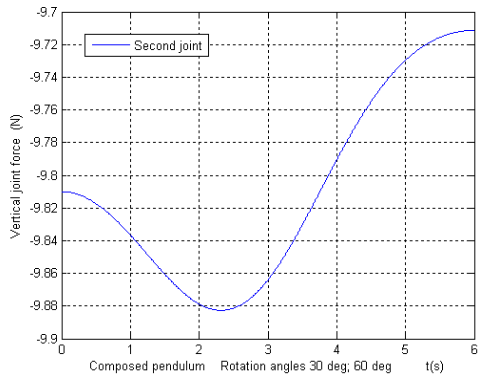

Fig. 6 – Vertical joint force of second rod.

6. CONCLUSIONS

In the kinematics analysis some exact relations that give in real-time the position, velocity and acceleration of each element of a multi-bodies system with constraints have been established in the present paper. The dynamics model takes into consideration the masses and forces of inertia introduced by all links of the mechanism. Based on the Lagrange equations or the principle of virtual work, the approach establishes a direct determination of the time-history evolution for all forces in external and internal joints.

REFERENCES

1. TSAI, L.-W., *Robot analysis*: *The Mechanics of Serial and Parallel Manipulators*, Wiley, New York, 1999.
2. BROUWER, D., CLEMENCE, G.M., *Methods of Celestial Mechanics*, Academic Press, New York, London, 1961.
3. DUBOSIN, G.N., *Nebesnaia mehanica*, Moskva, 1968.
4. LU, Y., *Using virtual work theory and CAD functionalities for solving active force and passive force of spatial parallel manipulators*, Mechanism and Machine Theory, Elsevier, **42**, pp. 839–858, 2007.
5. GEIKE, T., Mc PHEE, J., *Inverse dynamic analysis of parallel manipulators with full mobility*, Mechanism and Machine Theory, Elsevier, **38**, pp. 549–562, 2003.
6. STROE, I., *Miscarea generala a vehiculului spatial stationar la o latitudine arbitrara*, Stud. Cerc. Mec. Apl., **51,** *6*, 1992.
7. VOINEA, R., STROE, I., *A general method for kinematics pairs synthesis*, Mechanism and Machine Theory, Elsevier, **30**, *3*, pp. 461–470, 1995.
8. STAICU, S., ZHANG, D., *A novel dynamic modelling approach for parallel mechanisms analysis*, Robotics and Computer-Integrated Manufacturing, Elsevier, **24**, *1*, pp.167-172, 2008.
9. STAICU, S., LIU, X-J., LI, J., *Explicit dynamics equations of the constrained robotic systems*, Nonlinear Dynamics, Springer, **58**, *1–2*, pp. 217–235, 2009.

Received May 3, 2010