SOME INEQUALITIES FOR CONVEXIFIABLE FUNCTION 
WITH APPLICATIONS

Adriana CLIM
Bucharest University, Academiei 14, Bucharest, Romania
E-mail:clim.adriana@gmail.com

Some function becomes convex after adding to it a quadratic term. In this paper we extend some properties of convex function to convexifiable case of operators.

Key words: Convexity; Convexifiable function; Continuous function; Jensen inequality; Operator on Hilbert space.

1. INTRODUCTION

Convex functions are often used in applied mathematics. They have many uses in optimization and numerical methods. Using convexity, one can also study non-convex problems in two directions: transformation of arbitrary continuous functions to convex-like function and transformation of mathematical programs with such functions to equivalent programs.

For a given function $f: \mathbb{R} \rightarrow \mathbb{R}$, defined on a bounded convex set $J$ of a real line $\mathbb{R}$, we can construct the convex function by adding simple quadratic term $\alpha \cdot x^T \cdot x$ to $f$ where $\alpha$ is a sufficiently large non-negative number. The numerical value of the "convexifier" ($\alpha$) depends on the function $f$ and the interval where $f$ is "convexified". For $\alpha<0$ the quadratic term is strictly convex, so $f$ is called "weakly convex"[7]. Also, if there is $\alpha$, convexifier of $f$, then there are a lot of such values $\alpha^* \leq \alpha$ which are also convexifiers.

Therefore every convexifiable function $f$ can be written as the sum of a convex function $f(x) = \frac{\alpha}{2} \cdot x^T \cdot x$ and a concave quadratic term $\frac{\alpha}{2} \cdot x^T \cdot x$, for every $\alpha$ which is sufficiently small.

Convexifiable functions have been studied also on $\mathbb{R}^n$ and characterized using the fact that for continuous functions a class of convexifiable function is large: beside convex and twice continuously differentiable functions, also continuously differentiable functions with Lipschitz derivative. In [10] Zlobec showed that there exist continuously differentiable functions and also differentiable Lipschitz functions that can not be convexified.

Here we extend some of results from [5] to convexifiable case of convex operator.

2. SOME PRELIMINARY RESULTS

Definition 2.1. If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function of $n$ variables defined on a convex set $J$, $J \subseteq \mathbb{R}^n$ then the function is said to be convex (concave) on $J$ if

$$f(\lambda x + (1-\lambda)y) \leq (\geq) \ \lambda f(x) + (1-\lambda)f(y) \ \forall x, y \in J, \ (\forall) \ \lambda \in [0, 1].$$

(1)

Theorem 2.2 [2]. If $f$ is a continuous, real function on an interval $J$, the following conditions are equivalent:

(i) $f$ is operator concave.
(ii) \( f(C^*AC + t_0(I - C^*C)) \geq C^*f(A)C + f(t_0)(I - C^*C) \) for an operator \( C \) with \( ||C|| \leq 1 \) and a self-adjoint operator \( A \) with \( \sigma(A) \subseteq J \) and for fixed real number \( t_0 \in J \).

(iii) \( f \left( \sum_{j=1}^{n} C_j^* A_j C_j + t_0 \left( I - \sum_{j=1}^{n} C_j^* C_j \right) \right) \geq \sum_{j=1}^{n} C_j^* f(A_j) C_j + f(t_0) \left( I - \sum_{j=1}^{n} C_j^* C_j \right) \) for operators \( C_j \) with \( \sum_{j=1}^{n} C_j^* C_j \leq I \) and self-adjoint operator \( A_j \) with \( \sigma(A_j) \subseteq J, j = 1, 2, \ldots, n \), and for a fixed real number \( t_0 \in J \).

(iv) \( f \left( \sum_{j=1}^{n} C_j^* A_j C_j \right) \geq \sum_{j=1}^{n} C_j^* f(A_j) C_j \) for operators \( C_j \) with \( \sum_{j=1}^{n} C_j^* C_j = I \), self-adjoint operator \( A_j \) with \( \sigma(A_j) \subseteq J, j = 1, 2, \ldots, n \), where \( n \geq 2 \).

(v) \( f(PAP + t_0(I - P)) \geq P \cdot f(A) \cdot P + f(t_0)(I - P) \) for a projection \( P \) and a self-adjoint operator \( A \) with \( \sigma(A) \subseteq J \) and for a fixed real number \( t_0 \in J \).

**Definition 2.3.** [8] Given a continuous \( f : \mathbb{R}^n \to \mathbb{R} \) defined on a convex set \( J \subseteq \mathbb{R}^n \), consider the parametric function \( \varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) defined by

\[
\varphi(x, \alpha) = f(x) - \frac{1}{2} \alpha x^T x, \quad (2)
\]

where \( x^T \) is the transposed of \( x \). If \( \varphi(x, \alpha) \) is a convex function on \( J \) for some \( \alpha = \alpha^* \), then \( \varphi(x, \alpha) \) is a convexification of \( f \) and \( \alpha^* \) is its convexifier on \( J \). Function \( f \) is convexifiable if it has a convexification.

**Remark 2.4.** If \( \alpha \) is a convexifier of \( f \), then so is every \( \alpha^* \leq \alpha \).

**Theorem 2.5** [9]. If \( f \) is a continuous function \( f : \mathbb{R}^n \to \mathbb{R} \) defined on a convex set \( J \subseteq \mathbb{R}^n \) then \( f \) is convex if and only if \( f \) is mid-point convex, i.e.,

\[
f \left( \frac{x + y}{2} \right) \leq \frac{1}{2} \left( f(x) + f(y) \right), \quad \forall \, x, y \in J. \quad (3)
\]

**Remark 2.6.** Every convex function defined on a convex set from Euclidean space is mid-point convex. Over non-Euclidean space (e.g. the scalar field of rational numbers) we can construct a non-convex mid-point convex function.

With every continuous function \( f : \mathbb{R}^n \to \mathbb{R} \) we can associate a particular function \( \psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \). We denote the norm of \( u \in \mathbb{R}^n \) by \( ||u|| = (u^T u)^{1/2} \).

**Remark 2.7** [10]. Given a continuous function \( f : \mathbb{R}^n \to \mathbb{R} \) and a compact convex set \( J \) in \( \mathbb{R}^n \) the mid-point acceleration function of \( f \) on \( J \) is the function

\[
\psi(x, y) = \frac{4}{||x - y||^2} \left[ f(x) + f(y) - 2f \left( \frac{x + y}{2} \right) \right], \quad (\forall \, x, y \in J, x \neq y). \quad (4)
\]

**Remark 2.8.** (Justification of function’s name). If we take \( x, y \) in \( J \) then their mid point is \( 1/2(x+y) \) and also is \( x+1/2(y-x) \). Using the notation \( \Delta x = 1/2(y-x) \), the mid point can be written as \( x + \Delta x \), which is the same as \( y - \Delta x \). Then the distance from \( x \) and \( x + \Delta x \), i.e. \( ||\Delta x|| \), so the average displacement of \( f \) at \( x \) in the direction of mid-point \( x + \Delta x \), over distance is \( \Delta f(x) = (f(x + \Delta x) - f(x)) / ||\Delta x|| \).

This is repeated at the mid-point and \( y \), so we obtain \( \Delta f(x + \Delta x) = (f(y) - f(x + \Delta x)) / ||\Delta x|| \). Hence the average “displacement of the displacement”, i.e. the “acceleration” is

\[
[\Delta f(x + \Delta x) - \Delta f(x)] / ||\Delta x|| = \psi(x, y).
\]
Theorem 2.9 [10]. Given a continuous function $f: \mathbb{R}^n \to \mathbb{R}$ on a compact convex set $J$ in $\mathbb{R}^n$, function $f$ is convexifiable on $J$ if and only if its mid-point acceleration function is bounded on $J$.

Proof. From $f$ convexifiable we have $\varphi(x, \alpha) = f(x) - \frac{1}{2\alpha} ax^T x$ convex for some $\alpha$. But for $\varphi$ we have $\varphi((x+y, \alpha)/2) \leq 1/2(\varphi(x, \alpha) + \varphi(y, \alpha))$, $x, y \in J$.

After substitution, this is:

$$2f((x+y)/2) - f(x+y) \leq \alpha \{1/4 \cdot \|x\|^2 + 2 \cdot (x, y) + \|y\|^2 \} - 1/2 \cdot \{\|x\|^2 + \|y\|^2 \} = -\alpha/4 \cdot \|x - y\|^2.$$

So, finally $\alpha \leq \psi(x, y)$, for every $x, y \in J, x \neq y$.

Theorem 2.10 [11]. [Jensen’s inequality for convexifiable functions]. If $f$ is a convexifiable function on a bounded nontrivial convex set $J \subset \mathbb{R}^n$, and $\alpha$ is its convexifier, then

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) \leq \sum_{i=1}^{p} \lambda_i f(x_i) - \frac{\alpha}{2} \sum_{i,j=1}^{p} \lambda_i \lambda_j \|x_i - x_j\|^2$$

for every set of points $\{x_i\}_{i=1}^{p}$ from $J$ and all real scalar $\lambda_i \geq 0$ with $i = 1, 2, ..., p$ and $\sum_{i=1}^{p} \lambda_i = 1$.

Definition 2.11. Let $F$ be an operator over Hilbert space $H$. $F$ is convex (concave) operator over $J \subset H$ if

$$F(\alpha x + \beta y) \leq \alpha F(x) + \beta F(y) \quad (6)$$

for any real $\alpha, \beta$ with $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $x, y \in J$.

Definition 2.12. Let $F$ be an operator over $J \subset H$, a Hilbert space. We say that $F$ is convexifiable operator if exists some real number $\alpha$ so that for $A \in B(H)$ the new operator

$$\varphi(A, \alpha) = F(A) - \frac{1}{2} \alpha A^T A$$

is convex over $J$.

Remark 2.13. (generalization of Jensen inequality). Let $A, B$ be self-adjoint operators with $\sigma(A) \subseteq J$ and $\sigma(B) \subseteq J$. If $f$ is an convexifiable operator on an interval $J$ then for $s, t \geq 0$, $s + t = 1$ we have

$$f(s \cdot A + t \cdot B) \leq s \cdot f(A) + t \cdot f(B) - \alpha \left( s \cdot t \cdot \| A - B \|^2 \right).$$

Proof. If $f$ is convexifiable with $\alpha$ its convexifier then there is a convex operator $\varphi$ such that $\varphi(C, \alpha) = f(C) - \alpha/2 C^T C$.

If we apply Jensen’s inequality for convex function to $\varphi$, for $A, B, s, t$ with $s + t = 1$ we have

$$\varphi(sA + tB) \leq s \varphi(A) + t \varphi(B).$$

After substitutions, the inequality is

$$f(sA + tB) \leq s \cdot f(A) + t \cdot f(B) - \alpha \left( sA^2 + tB^2 - (sA + tB)^2 \right).$$

Finally, from $s + t = 1$ the conclusion follows.

3. OPERATOR INEQUALITIES FOR CONVEXIFIABLE CASE

Theorem 2.14. The following conditions are equivalent for an operator $F: J \to \mathbb{R}, J \subset \mathbb{R}$.
Some inequalities for convexifiable function with applications

i1. $F$ is a convexifiable operator with $\alpha$ its convexifier.

i2. For an operator $C$ with $\|C\| \leq 1$ and a self-adjoint operator $A$ with $\sigma(A) \subseteq J$ and for fixed real number $t_0 \in J$, the operator $F$ with its convexifier $\alpha$ satisfy

$$F(C^*AC + t_0(I - C^*C)) \leq C^*F(A)C + F(t_0)(I - C^*C) + \frac{\alpha}{2}D_1,$$

where

$$D_1 = (C^*AC + t_0(I - C^*C))^2 - C^*A^2C - t_0^2(I - C^*C).$$

i3. For operators $C_j$ with $\sum_{j=1}^n C_j^*C_j \leq I$ and self-adjoint operators $A_j$ with $\sigma(A_j) \subseteq J$ and for fixed real number $t_0 \in J$, the operator $F$ with its convexifier $\alpha$ satisfy

$$F\left(\sum_{j=1}^n C_j A_j C_j + t_0\left(I - \sum_{j=1}^n C_j^*C_j\right)\right) \leq \sum_{j=1}^n C_j^*F(A_j)C_j + F(t_0)\left(I - \sum_{j=1}^n C_j^*C_j\right) + \frac{\alpha}{2}D_2,$$

where $\alpha$ is its convexifier and

$$D_2 = \left(\sum_{j=1}^n C_j^*A_jC_j + t_0\left(I - \sum_{j=1}^n C_j^*C_j\right)\right)^2 - \sum_{j=1}^n C_j^*A_j^2C_j - t_0^2\left(I - \sum_{j=1}^n C_j^*C_j\right).$$

i4. If we have a particular case when operators $C_j$ satisfy condition $\sum_{j=1}^n C_j^*C_j = I$ then for self-adjoint operators $A_j$ with $\sigma(A_j) \subseteq J$ for $j = 1, 2, \ldots, n$, and for fixed real number $t_0 \in J$, $F$ verify the inequality

$$F\left(\sum_{j=1}^n C_j^*A_jC_j + t_0\left(I - \sum_{j=1}^n C_j^*C_j\right)\right) \leq \sum_{j=1}^n C_j^*F(A_j)C_j + \frac{\alpha}{2}\left[\sum_{j=1}^n C_j^*A_jC_j\right]^2 - \sum_{j=1}^n C_j^*A_j^2C_j.$$

i5. If we consider an operator projection $P$ then for a self-adjoint operator $A$ with $\sigma(A) \subseteq J$ and for fixed real number $t_0 \in J$, the operator $F$ with its convexifier $\alpha$ satisfy the inequality

$$F(PAP + t_0(I - P)) \leq P \cdot F(A) \cdot P + F(t_0)(I - P) + \frac{-\alpha}{2}\left[P A^2 P - PAP + (t_0^2 - t_0)(I - P)\right].$$

Proof. The equivalence will be done following i1 $\Rightarrow$ i2 $\Rightarrow$ i3 $\Rightarrow$ i4 $\Rightarrow$ i1 and i2 $\Rightarrow$ i5 $\Rightarrow$ i1.

i1 $\Rightarrow$ i2 From definition, if $F$ is convexifiable then there is some real number $\alpha$ so that new operator

$$\varphi(A, \alpha) = F(A) - \frac{\alpha}{2}A^TA$$

is convex. For every convex function the inequality holds (Theorem 2.2)

$$\varphi(C^*AC + t_0(I - C^*C), \alpha) \leq C^*\varphi(A, \alpha)C + \varphi(t_0, \alpha)(I - C^*C).$$

So, we have

$$F(C^*AC + t_0(I - C^*C)) - \frac{\alpha}{2}\left(C^*AC + t_0(I - C^*C)\right)^T\left(C^*AC + t_0(I - C^*C)\right) \leq C^*\left(F(A) - \frac{\alpha}{2}A^TA\right)C + \left(F(t_0) - \frac{\alpha}{2}t_0^2\right)(I - C^*C).$$
Since
\[(C^*AC + t_0(I - C^*C))^T (C^*AC + t_0(I - C^*C)) = \]
\[= \left( (C^*AC)^T + t_0(I - C^*C)^T \right) \left( C^*AC + t_0(I - C^*C) \right) = \left( C^*AC + t_0(I - C^*C) \right)^2 \]
we obtain
\[F(C^*AC + t_0(I - C^*C)) - \frac{\alpha}{2} \left( C^*AC + t_0(I - C^*C) \right)^2 \leq \]
\[\leq C^*F(A)C - \frac{\alpha}{2} (C^*A^2C) + \left( F(t_0) - \frac{\alpha}{2} t_0^2 \right) (I - C^*C). \]
So,
\[F(C^*AC + t_0(I - C^*C)) \leq C^*F(A)C + F(t_0)(I - C^*C) + \]
\[\quad + \frac{\alpha}{2} \left( C^*AC + t_0(I - C^*C) \right)^2 - C^*A^2C - t_0^2(I - C^*C). \]
i2 \Rightarrow i3. For a convex function \(\varphi\) we can prove the inequality (Theorem 2.2)
\[\varphi \left( \sum_{j=1}^{n} C_j^*A_jC_j + t_0(I - \sum_{j=1}^{n} C_j^{*2}C_j), \alpha \right) \leq \sum_{j=1}^{n} C^*\varphi(A, \alpha)C + \varphi(t_0, \alpha)(I - \sum_{j=1}^{n} C_j^{*2}C_j), \text{ i.e.} \]
\[F \left[ \sum_{j=1}^{n} C_j^*A_jC_j + t_0 \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right) \right] - \frac{\alpha}{2} \sum_{j=1}^{n} C_j^*A_jC_j + t_0 \left[ I - \sum_{j=1}^{n} C_j^{*2}C_j \right]^T \]
\[\leq \sum_{j=1}^{n} C_j^* \left( F(A_j) - \frac{\alpha}{2} A_j^T A_j \right) C_j + \left( F(t_0) - \frac{\alpha}{2} t_0^2 \right) \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right). \]
But,
\[\left( \sum_{j=1}^{n} C_j^*A_jC_j + \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right) \right)^T \left( \sum_{j=1}^{n} C_j^*A_jC_j + \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right) \right) = \left( \sum_{j=1}^{n} C_j^*A_jC_j + \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right) \right)^2. \]
So
\[F \left[ \sum_{j=1}^{n} C_j^*A_jC_j + t_0 \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right) \right] \leq \]
\[\leq \sum_{j=1}^{n} C^*F(A_j)C + \frac{\alpha}{2} \left( \sum_{j=1}^{n} C_j^*A_jC_j + \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right) \right)^2 - \frac{\alpha}{2} \sum_{j=1}^{n} C_j^{*2}C_j + \left( F(t_0) - \frac{\alpha}{2} t_0^2 \right) \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right). \]
If we distribute the term, we obtain
\[F \left[ \sum_{j=1}^{n} C_j^*A_jC_j + t_0 \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right) \right] \leq \sum_{j=1}^{n} C_j^*F(A_j)C_j + f(t_0) \left( I - \sum_{j=1}^{n} C_j^{*2}C_j \right) + \frac{\alpha}{2} D_2, \]
where \(D_2\) is defined by (13).
i3 ⇒ i4. If we have $\sum_{j=1}^{n} C_j^* C_j = I$ then for a convexifiable operator, from (12) and (13) we have

$$F\left(\sum_{j=1}^{n} C_j^* A_j C_j\right) \leq \sum_{j=1}^{n} C_j^* F(A_j) C_j + \frac{\alpha}{2} \left(\sum_{j=1}^{n} C_j^* A_j C_j \right)^2 - \sum_{j=1}^{n} C_j^* A_j^2 C_j.$$ 

i4 ⇒ i1. If we take real numbers $C_1, C_2$ and $C_3 = 0$ for $i \geq 3$, we obtain $C_1^2 + C_2^2 = 1$ and (14) became:

$$F(C_1^2 A_1 + C_2^2 A_2) \leq C_1^2 F(A_1) + C_2^2 F(A_2) + \alpha/2(C_1^2 A_1 + C_2^2 A_2)^2 - C_1^2 A_1^2 - C_2^2 A_2^2,$$

the convexifiable definition of $F$.

i2 ⇒ i5. If we consider $C=P$, a projection operator in (10) and (11), then $F$ satisfy the relation

$$F(P^* A P + t_0(I - P^* P)) \leq P^* F(A) P + F(t_0)(I - P^* P) + \frac{\alpha}{2} D,$$

with

$$D = (P^* A P + t_0(I - P^* P))^2 - P^* A^2 P - t_0^2(I - P^* P).$$

If $P$ is a projection then $P^2=P$ and $P^*=P$ so we obtain

$$D = (PAP + t_0(I - P))^2 - P^2 A^2 P - t_0^2(I - P) = PAP + t_0(I - P)^2 - P^2 A^2 P - t_0^2(I - P) = PAP - P^2 A^2 P - t_0^2(I - P).$$

Finally we have (15).

i5 ⇒ i1. For the self-adjoint operators $C, D$ with $\sigma(C), \sigma(D) \subseteq J$ and $\lambda \in [0,1]$ we construct some new operators over $H \oplus H$

$$X = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, \quad U = \begin{pmatrix} \lambda^{1/2}I & -(1-\lambda)^{1/2}I \\ -(1-\lambda)^{1/2}I & \lambda^{1/2}I \end{pmatrix}, \quad P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$ 

We have that $\sigma(X) = \sigma(C) \cup \sigma(D) \subseteq J$, $X$ is an self-adjoint operator, $U$ is unitary and $P$ is projection. Now, relative to $F$, we proceed as in [1]. Since $\sigma(\lambda C+(1-\lambda)D) \subseteq J$ we get

$$F(\lambda C+(1-\lambda)D) \leq \lambda F(C) + (1-\lambda) F(D) + \alpha/2(\lambda C+(1-\lambda)D)^2 - [\lambda C+(1-\lambda)D]^2.$$

**Remark 2.15.** On the line of papers [3, 4, 5, 6] we can formulate a problem of such type.

**REFERENCES**


*Received May 12, 2010*