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SOME INEQUALITIES FOR CONVEXIFIABLE FUNCTION WITH APPLICATIONS

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Some function becomes convex after adding to it a quadratic term. In this paper we extend some properties of convex function to convexifiable case of operators.

Key words: Convexity; Convexifiable function; Continuous function; Jensen inequality; Operator on Hilbert space.

1. INTRODUCTION

Convex functions are often used in applied mathematics. They have many uses in optimization and numerical methods. Using convexity, one can also study non-convex problems in two directions: transformation of arbitrary continuous functions to convex-like function and transformation of mathematical programs with such functions to equivalent programs.

For a given function $f: R \to R$, defined on a bounded convex set J of a real line R, we can construct the convex function by adding simple quadratic term $\alpha \cdot x^T \cdot x$ to f where α is a sufficiently large non-negative number. The numerical value of the „convexifier" (α) depends on the function f and the interval where f is „convexified". For $\alpha<0$ the quadratic term is strictly convex, so f is called „weakly convex"[7]. Also, if there is α, convexifier of f, then there are a lot of such values $\alpha^* \le \alpha$ which are also convexifiers.

Therefore every convexifiable function f can be written as the sum of a convex function $f(x) - \frac{\alpha}{2} \cdot x^T \cdot x$ and a concave quadratic term $\frac{\alpha}{2} \cdot x^T \cdot x$, for every α which is sufficiently small.

Convexifiable functions have been studied also on R^n and characterized using the fact that for continuous functions a class of convexifiable function is large: beside convex and twice continuously differentiable functions, also continuously differentiable functions with Lipschitz derivative. In [10] Zlobec showed that there exist continuously differentiable functions and also differentiable Lipschitz functions that can not be convexified.

Here we extend some of results from [5] to convexifiable case of convex operator.

2. SOME PRELIMINARY RESULTS

Definition 2.1. If $f: R^n \to R$ is a continuous function of n variables defined on a convex set J, $J \subseteq R^n$ then the function is said to be convex (concave) on J if

$$f(\lambda x+(1-\lambda)y) \le (\ge)\ \lambda f(x) + (1-\lambda)f(y)\ (\forall)\ x, y \in J,\ (\forall)\lambda \in [0, 1]. \quad (1)$$

Theorem 2.2 [2]. *If f is a continuous, real function on an interval J, the following conditions are equivalent:*

(i) *f is operator concave.*

(ii) $f(C^*AC+t_0(I-C^*C)) \geq C^*f(A)C + f(t_0)(I-C^*C)$ *for an operator* C *with* $\|C\|\leq 1$ *and a self-adjoint operator* A *with* $\sigma(A)\subseteq J$ *and for fixed real number* $t_0\in J$.

(iii) $f\left(\sum_{j=1}^{n} C_j^* A_j C_j + t_0\left(I - \sum_{j=1}^{n} C_j^* C_j\right)\right) \geq \sum_{j=1}^{n} C_j^* f(A_j) C_j + f(t_0)\left(I - \sum_{j=1}^{n} C_j^* C_j\right)$ *for operators* C_j *with* $\sum_{j=1}^{n} C_j^* C_j \leq I$ *and self-adjoint operator* A_j *with* $\sigma(A_j) \subseteq J$, $j=1, 2, ..., n$, *and for a fixed real number* $t_0\in J$.

(iv) $f\left(\sum_{j=1}^{n} C_j^* A_j C_j\right) \geq \sum_{j=1}^{n} C_j^* f(A_j) C_j$ *for operators* C_j *with* $\sum_{j=1}^{n} C_j^* C_j = I$, *self-adjoint operator* A_j *with* $\sigma(A_j) \subseteq J$, $j=1, 2, ..., n$, *where* $n\geq 2$.

(v) $f(PAP + t_0(I-P)) \geq P \cdot f(A) \cdot P + f(t_0)(I-P)$ for a projection P and a self-adjoint operator A with $\sigma(A) \subseteq J$ and for a fixed real number $t_0\in J$.

Definition 2.3. [8] Given a continuous $f : R^n \to R$ defined on a convex set $J\subset R^n$, consider the parametric function $\varphi : R^n \times R \to R$ defined by

$$\varphi(x,\alpha) = f(x) - \frac{1}{2}\alpha x^T x\,, \tag{2}$$

where x^T is the transposed of x. If $\varphi(x,\alpha)$ is a convex function on J for some $\alpha=\alpha^*$, then $\varphi(x,\alpha)$ is a convexification of f and α^* is its convexifier on J. Function f is convexifiable if it has a convexification.

Remark 2.4. If α is a convexifier of f, then so is every $\alpha^* \leq \alpha$.

Theorem 2.5 [9]. *If f is a continuous function $f : R^n \to R$ defined on a convex set $J\subset R^n$ then f is convex if and only if f is mid-point convex, i.e.*,

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\big(f(x)+f(y)\big),\ \forall\ x, y\in J. \tag{3}$$

Remark 2.6. Every convex function defined on a convex set from Euclidean space is mid-point convex. Over non-Euclidean space (e.g. the scalar field of rational numbers) we can construct a non-convex mid-point convex function.

With every continuous function $f : R^n \to R$ we can associate a particular function $\psi : R^n \times R^n \to R$. We denote the norm of $u\in R^n$ by $\|u\| = (u^T u)^{1/2}$.

Remark 2.7 [10]. Given a continuous function $f : R^n \to R$ and a compact convex set J in R^n the mid-point acceleration function of f on J is the function

$$\psi(x,y) = \frac{4}{\|x-y\|^2}\left[f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right],\ (\forall)\ x, y \in J,\ x\neq y. \tag{4}$$

Remark 2.8. (Justification of function's name). If we take x, y in J then their mid point is $1/2(x+y)$ and also is $x+1/2(y-x)$. Using the notation $\Delta x = 1/2(y-x)$, the mid point can be written as $x+\Delta x$, which is the same as $y-\Delta x$. Then the distance from x and $x+\Delta x$, i.e. $\|\Delta x\|$, so the average displacement of f at x in the direction of mid-point $x+\Delta x$, over distance is $\Delta f(x) = [f(x+\Delta x) - f(x)] / \|\Delta x\|$.

This is repeated at the mid-point and y, so we obtain $\Delta f(x+\Delta x) = [f(y) - f(x+\Delta x)] / \|\Delta x\|$. Hence the average "displacement of the displacement", i.e. the "acceleration" is

$$[\Delta f(x+\Delta x) - \Delta f(x)] / \|\Delta x\| = \psi(x,y).$$

Theorem 2.9 [10]. *Given a continuous function $f: \mathbb{R}^n \to \mathbb{R}$ on a compact convex set J in $\mathbb{R}^n$, function f is convexifiable on J if and only if its mid-point acceleration function is bounded on J.*

Proof. From f convexifiable we have $\varphi(x, \alpha) = f(x) - 1/2\alpha x^T x$ convex for some α. But for φ we have

$$\varphi((x+y,\alpha)/2) \le 1/2\cdot(\varphi(x,\alpha) + \varphi(y,\alpha)),\ x, y \in J.$$

After substitution, this is

$$2\cdot f((x+y)/2) - [f(x)+f(y)] \le \alpha\cdot\{1/4\cdot[\|x\|^2+2\cdot(x, y)+\|y\|^2] - 1/2\cdot[\|x\|^2+\|y\|^2]\} = -\alpha/4\cdot \|x - y\|^2.$$

So, finally

$$\alpha \le \psi(x, y), \text{ for every } x, y \in J, x \ne y.$$

Theorem 2.10 [11]. [*Jensen's inequality for convexifiable functions*]. *If f is a convexifiable function on a bounded nontrivial convex set $J \subset \mathbb{R}^n$, and α is its convexifier, then*

$$f\left(\sum_{i=1}^{p}\lambda_i x_i\right) \le \sum_{i=1}^{p}\lambda_i f(x_i) - \frac{\alpha}{2}\left(\sum_{\substack{i,j=1\\ i<j}}^{p}\lambda_i\lambda_j \| x_i - x_j \|^2\right) \tag{5}$$

for every set of points $\{x_i\}_{i=1,\dots,p}$ from J and all real scalar $\lambda_l \ge 0$ with $i = 1, 2, \dots, p$ and $\sum_{i=1}^{p}\lambda_i = 1$.

Definition 2.11. Let F be an operator over Hilbert space H. F is convex (concave) operator over $J \subset H$ if

$$F(\alpha x+\beta y) \le (\ge)\ \alpha F(x)+\beta F(y) \tag{6}$$

for any real α, β with $\alpha+\beta=1$, $\alpha, \beta \ge 0$ and $x, y \in J$.

Definition 2.12. Let F be an operator over $J \subset H$, a Hilbert space. We say that F is an convexifiable operator if exists some real number α so that for $A \in B(H)$ the new operator

$$\varphi(A,\alpha) = F(A) - \frac{1}{2}\alpha A^T A \tag{7}$$

is convex over J.

Remark 2.13. (generalization of Jensen inequality). Let A, B be self-adjoint operators with $\sigma(A) \subseteq J$ and $\sigma(B) \subseteq J$. If f is an convexifiable operator on an interval J then for $s, t \ge 0$, $s+t = 1$ we have

$$f(s\cdot A + t\cdot B) \le s\cdot f(A) + t\cdot f(B) - \frac{\alpha}{2}\left[s\cdot t\cdot \| A - B \|^2\right]. \tag{8}$$

Proof. If f is convexifiable with α its convexifier then there is a convex operator φ such that

$$\varphi(C, \alpha) = f(C) - \alpha/2C^*C.$$

If we apply Jensen's inequality for convex function to φ, for A, B, s, t with $s+t = 1$ we have

$$\varphi(sA + tB) \le s\varphi(A) + t\varphi(B). \tag{9}$$

After substitutions, the inequality is

$$f(s\cdot A+t\cdot B) \le s\cdot f(A) + t\cdot f(B) - \frac{\alpha}{2}\left[sA^2 + tB^2 - (sA+tB)^2\right].$$

Finally, from $s + t = 1$ the conclusion follows.

3. OPERATOR INEQUALITIES FOR CONVEXIFIABLE CASE

Theorem 2.14. *The following conditions are equivalent for an operator $F: J \to \mathbb{R}$, $J \subset \mathbb{R}$.*

i1. *F is a convexifiable operator with* α *its convexifier*.

i2. *For an operator C with* $\|C\|\le 1$ *and a self-adjoint operator A with* $\sigma(A)\subseteq J$ *and for fixed real number* $t_0\in J$, *the operator F with its convexifier* α *satisfy*

$$F(C^*AC+t_0(I-C^*C))\le C^*F(A)C+F(t_0)(I-C^*C)+\frac{\alpha}{2}D_1, \tag{10}$$

where

$$D_1=(C^*AC+t_0(I-C^*C))^2-C^*A^2C-t_0^2(I-C^*C). \tag{11}$$

i3. *For operators* C_j *with* $\sum_{j=1}^{n}C_j{}^*C_j\le I$ *and self-adjoint operators* A_j *with* $\sigma(A_j)\subset J$, $j=1,2,\dots n$, *and for fixed real number* $t_0\in J$, *F verify the inequality*

$$F\left(\sum_{j=1}^{n}C_j{}^*A_jC_j+t_0\left(I-\sum_{j=1}^{n}C_j{}^*C_j\right)\right)\le\sum_{j=1}^{n}C_j{}^*F(A_j)C_j+F(t_0)\left(I-\sum_{j=1}^{n}C_j{}^*C_j\right)+\frac{\alpha}{2}D_2, \tag{12}$$

where α *is its convexifier and*

$$D_2=\left(\sum_{j=1}^{n}C_j{}^*A_jC_j+t_0\left(I-\sum_{j=1}^{n}C_j{}^*C_j\right)\right)^2-\sum_{j=1}^{n}C_j{}^*A_j^2C_j-t_0^2\left(I-\sum_{j=1}^{n}C_j{}^*C_j\right). \tag{13}$$

i4. *If we have a particular case when operators* C_j *satisfy condition* $\sum_{j=1}^{n}C_j{}^*C_j=I$ *then for self-adjoint operators* A_j *with* $\sigma(A_j)\subset J$ *for* $j=1,2,\dots n$, *and for fixed real number* $t_0\in J$, *F verify the inequality*

$$F\left(\sum_{j=1}^{n}C_j{}^*A_jC_j\right)\le\sum_{j=1}^{n}C_j{}^*F(A_j)C_j+\frac{\alpha}{2}\left[\left(\sum_{j=1}^{n}C_j{}^*A_jC_j\right)^2-\sum_{j=1}^{n}C_j{}^*A_j^2C_j\right]. \tag{14}$$

i5. *If we consider an operator projection P then for a self-adjoint operator A with* $\sigma(A)\subseteq J$ *and for fixed real number* $t_0\in J$, *the operator F with its convexifier* α *satisfy the inequality*

$$\begin{aligned}F(PAP+t_0(I-P))\le{}&P\cdot F(A)\cdot P+F(t_0)(I-P)+\\&-\frac{\alpha}{2}\left[PA^2P-PAPAP+(t_0^2-t_0)(I-P)\right].\end{aligned} \tag{15}$$

Proof. The equivalence will be done following i1⇒i2⇒i3⇒i4⇒i1 and i2⇒i5⇒i1.

i1⇒ i2 From definition, if *F* is convexifiable then there is some real number α so that new operator

$$\varphi(A,\alpha)=F(A)-\frac{\alpha}{2}A^TA$$

is convex. For every convex function the inequality holds (Theorem 2.2)

$$\varphi(C^*AC+t_0(I-C^*C),\alpha)\le C^*\varphi(A,\alpha)C+\varphi(t_0,\alpha)(I-C^*C).$$

So, we have

$$\begin{aligned}&F(C^*AC+t_0(I-C^*C))-\frac{\alpha}{2}\left(C^*AC+t_0(I-C^*C)\right)^T\left(C^*AC+t_0(I-C^*C)\right)\\&\quad\le C^*\left(F(A)-\frac{\alpha}{2}A^TA\right)C+\left(F(t_0)-\frac{\alpha}{2}t_0^2\right)(I-C^*C).\end{aligned}$$

Since

$$\left(C^*AC+t_0(I-C^*C)\right)^T\left(C^*AC+t_0(I-C^*C)\right)=$$

$$=\left((C^*AC)^T+t_0(I-C^*C)^T\right)\left(C^*AC+t_0(I-C^*C)\right)=\left(C^*AC+t_0(I-C^*C)\right)^2$$

we obtain

$$F(C^*AC+t_0(I-C^*C))-\frac{\alpha}{2}\left(C^*AC+t_0(I-C^*C)\right)^2\le$$

$$\le C^*F(A)C-\frac{\alpha}{2}(C^*A^2C)+\left(F(t_0)-\frac{\alpha}{2}t_0^2\right)(I-C^*C).$$

So,

$$F(C^*AC+t_0(I-C^*C))\le C^*F(A)C+F(t_0)(I-C^*C)+$$

$$+\frac{\alpha}{2}\left[\left(C^*AC+t_0(I-C^*C)\right)^2-C^*A^2C-t_0^2(I-C^*C)\right].$$

i2 $\Rightarrow$ i3. For a convex function φ we can prove the inequality (Theorem 2.2)

$$\varphi\left(\sum_{j=1}^n C_j^*A_jC_j+t_0(I-\sum_{j=1}^n C_j{}^*C_j),\alpha\right)\le\sum_{j=1}^n C^*\varphi(A,\alpha)C+\varphi(t_0,\alpha)(I-\sum_{j=1}^n C_j{}^*C_j)\text{, i.e.}$$

$$F\left(\sum_{j=1}^n C_j^*A_jC_j+t_0\left(I-\sum_{j=1}^n C_j{}^*C_j\right)\right)-\frac{\alpha}{2}\left(\sum_{j=1}^n C_j^*A_jC_j+t_0\left(I-\sum_{j=1}^n C_j{}^*C_j\right)\right)^T\cdot$$

$$\cdot\left(\sum_{j=1}^n C_j^*A_jC_j+t_0\left(I-\sum_{j=1}^n C_j{}^*C_j\right)\right)\le$$

$$\le\sum_{j=1}^n C_j^*\left(F(A_j)-\frac{\alpha}{2}A_j{}^TA_j\right)C_j+\left(F(t_0)-\frac{\alpha}{2}t_0^2\right)\left(I-\sum_{j=1}^n C_j{}^*C_j\right).$$

But,

$$\left(\sum_{j=1}^n C_j^*A_jC_j+\left(I-\sum_{j=1}^n C_j^*C_j\right)\right)^T\left(\sum_{j=1}^n C_j^*A_jC_j+\left(I-\sum_{j=1}^n C_j^*C_j\right)\right)=\left(\sum_{j=1}^n C_j^*A_jC_j+\left(I-\sum_{j=1}^n C_j^*C_j\right)\right)^2.$$

So

$$F\left(\sum_{j=1}^n C_j^*A_jC_j+t_0\left(I-\sum_{j=1}^n C_j^*C_j\right)\right)\le$$

$$\le\sum_{j=1}^n C^*F(A_j)C+\frac{\alpha}{2}\left(\sum_{j=1}^n C_j^*A_jC_j+\left(I-\sum_{j=1}^n C_j^*C_j\right)\right)^2-\frac{\alpha}{2}\sum_{j=1}^n C_j^*A_j^2C_j+\left(F(t_0)-\frac{\alpha}{2}t_0^2\right)\left(I-\sum_{j=1}^n C_j^*C_j\right).$$

If we distribute the term, we obtain

$$F\left(\sum_{j=1}^n C_j^*A_jC_j+t_0\left(I-\sum_{j=1}^n C_j{}^*C_j\right)\right)\le\sum_{j=1}^n C_j{}^*F(A_j)C_j+f(t_0)\left(I-\sum_{j=1}^n C_j^*C_j\right)+\frac{\alpha}{2}D_2,$$

where D_2 is defined by (13).

i3$\Rightarrow$ i4. If we have $\sum_{j=1}^{n} C_j^{*} C_j = I$ then for a convexifiable operator, from (12) and (13) we have

$$F\left(\sum_{j=1}^{n} C_j^{*} A_j C_j\right) \le \sum_{j=1}^{n} C_j^{*} F(A_j) C_j + \frac{\alpha}{2}\left[\left(\sum_{j=1}^{n} C_j^{*} A_j C_j\right)^2 - \sum_{j=1}^{n} C_j^{*} A_j^2 C_j\right].$$

i4$\Rightarrow$ i1. If we take real numbers C_1, C_2 and $C_i = 0$ for I ≥ 3, we obtain $C_1^2 + C_2^2 = 1$ and (14) became:

$$F(C_1^2 A_1 + C_2^2 A_2) \le C_1^2 F(A_1) + C_2^2 F(A_2) + \alpha/2[(C_1^2 A_1 + C_2^2 A_2)^2 - C_1^2 A_1^2 - C_2^2 A_2^2],$$

the convexifiable definition of F.

i2 $\Rightarrow$ i5. If we consider C=P, a projection operator in (10) and (11), then F satisfy the relation

$$F(P^* AP + t_0(I - P^* P)) \le P^* F(A)P + F(t_0)(I - P^* P) + \frac{\alpha}{2} D,$$

with

$$D = (P^* AP + t_0(I - P^* P))^2 - P^* A^2 P - t_0^2 (I - P^* P).$$

If P is a projection then P^2=P and P^*=P so we obtain

$$D = (PAP + t_0(I-P))^2 - PA^2P - t_0^2(I-P) = PAPAP + t_0(I-P)^2 - PA^2P - t_0^2(I-P) =$$
$$= PAPAP - PA^2P - (t_0^2 - t_0)(I-P).$$

Finally we have (15).

i5 $\Rightarrow$ i1. For the self-adjoint operators C, D with $\sigma(C)$, $\sigma(D) \subseteq J$ and $\lambda \in [0,1]$ we construct some new operators over $H \oplus H$

$$X = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix},\ U = \begin{pmatrix} \lambda^{1/2} I & -(1-\lambda)^{1/2} I \\ -(1-\lambda)^{1/2} I & \lambda^{1/2} I \end{pmatrix},\ P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

We have that σ(X)= σ(C)$\cup$ σ(D) $\subseteq J$, X is an self-adjoint operator, U is unitary and P is projection. Now, relative to F, we proceed as in [1]. Since $\sigma(\lambda C+(1-\lambda)D) \subseteq J$ we get

$$F(\lambda C+(1-\lambda)D) \le \lambda F(C) + (1-\lambda) F(D) - \alpha/2\{\lambda C^2+(1-\lambda)D^2-[\lambda C+(1-\lambda)D]^2\}.$$

Remark.2.15. On the line of papers [3, 4, 5, 6] we can formulate a problem of such type.

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