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RESTRICTED OPTIMAL RETENTION IN STOP-LOSS REINSURANCE UNDER VAR AND CTE RISK MEASURES

Silvia DEDU, Roxana CIUMARA

Department of Mathematics, Academy of Economic Studies, Bucharest, Romania
E-mail: silvia.dedu@csie.ase.ro

In this paper we study the problem of existence of the restricted optimal retention in a stop-loss reinsurance. We use optimization criteria based on minimizing Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) risk measure. We prove the existence of the optimal solution of the considered problems in any conditions and we derive the optimal retention. The solutions obtained extend the results of [3] and [13]. The results obtained are illustrated using simulations. Computational results are provided.

Key words: Risk Measure; Reinsurance; Optimization; Retention; Value-at-risk.

1. INTRODUCTION

Reinsurance is a risk management tool used by an insurance company to protect itself against the risk of losses, transferring the risk to a second insurance carrier. The former part is called the cedent or the insurer and the latter is the reinsurer. Stop-loss is a special form of reinsurance. The stop-loss agreement states that reinsurer will pay the cedent's losses to the extent that losses exceed a specified amount, called retention.

Let X be the aggregate loss for an insurance portfolio. We assume that X is a nonnegative random variable with distribution function $F_X(x) = P(X \le x)$, survival function $S_X(x) = P(X > x)$ and mean $E(X) > 0$.

Let X_I and X_R be, respectively, the loss random variables of the insurer and the reinsurer in the presence of a stop-loss reinsurance. Under stop-loss agreement, the reinsurer pays part of X that exceeds the retention limit, while the insurer is effectively protected from a potential large loss by limiting the liability to the retention level.

Let $d > 0$ be the retention. The random variables X_I, X_R can be expressed as follows:

$$X_I = \begin{cases} X, & X \le d \\ d, & X > d \end{cases} \tag{1}$$

and

$$X_R = \begin{cases} 0, & X \le d \\ X - d, & X > d \end{cases} \tag{2}$$

In exchange of undertaking the risk, the reinsurer charges a reinsurance premium to the cedent. One of the most used premium principles is the expected value principle. According to [1], the net stop-loss premium is determined by:

$$\pi(d) = E(X_R) = \int_d^{\infty} S_X(x)\mathrm{d}x \tag{3}$$

and the reinsurance premium is

$$\delta(d) = (1+\rho)\pi(d) = (1+\rho)\int_d^{\infty} S_X(x)\mathrm{d}x, \tag{4}$$

where $\rho > 0$ is the relative safety loading. We note that δ is a decreasing function on d.

Let T be the total cost of the insurer in the presence of the stop-loss reinsurance. T has two components: the retained loss and the reinsurance premium

$$T = X_I + \delta(d)\,. \tag{5}$$

Risk measures such as Value-at-Risk (VaR) are used extensively in insurance. Let X be a random variable and $\alpha \in (0,1)$. The Value-at-Risk of the random variable X at the confidence level $1-\alpha$ is the number

$$\mathrm{VaR}_X(\alpha) = \inf\left\{x / P(X > x) \le \alpha\right\} = \inf\left\{x / P(X \le x) \ge \alpha\right\} \tag{6}$$

or

$$\mathrm{VaR}_X(\alpha) = \inf\left\{x / S_X(x) \le \alpha\right\} = \inf\left\{x / F_X(x) \ge \alpha\right\}. \tag{7}$$

If X has a one-to-one continuous distribution function on $(0,\infty)$, then $\mathrm{VaR}_X(\alpha)$ is the unique solution to either of the following two equations

$$P\left(X > \mathrm{VaR}_X(\alpha)\right) = \alpha \tag{8}$$

or

$$P\left(X \le \mathrm{VaR}_X(\alpha)\right) = 1-\alpha \tag{9}$$

In this case, we have

$$\mathrm{VaR}_\alpha(X) = S_X^{-1}(\alpha) = F_X^{-1}(1-\alpha). \tag{10}$$

In the same manner, one defines the *VaR* for the random variable T, defined by relation (5).

We suppose that X has a one-to-one continuous distribution function on $(0,\infty)$.

2. AN OPTIMIZATION CRITERION BASED ON VAR RISK MEASURE

We are interested in deriving a combined optimization criterion, based on minimizing VaR risk measure corresponding to the total cost of the insurer in the presence of the stop-loss reinsurance.

Let $0 < d_1 < d_2 < \infty$. We focus on determining the retention d^* that minimizes VaR risk measure provided that retention does not exceed a lower limit d_1 and an upper limit d_2, that means solving the optimization problem

$$\min_{0<d_1 \le d \le d_2} \left\{\mathrm{VaR}_T(d,\alpha)\right\}. \tag{11}$$

According to [3], for any $d > 0$ and $0 < \alpha < S_X(0)$ we have

$$\mathrm{VaR}_T(d,\alpha) = \begin{cases} d + (1+\rho)\int_d^{\infty} S_X(x)\mathrm{d}x, & 0 < d \le S_X^{-1}(\alpha) \\ S_X^{-1}(\alpha) + (1+\rho)\int_d^{\infty} S_X(x)\mathrm{d}x, & d > S_X^{-1}(\alpha) \end{cases}. \tag{12}$$

Theorem 1. *Let* $0 < d_1 < d_2 < \infty$.

1) *The optimal retention* $d^* = d^*(d_1, d_2) > 0$ *for* (11) *always exists*.

2) *The optimal retention* d^* *and the corresponding optimal value of the* VaR *risk measure are*:

a) For $0 < \alpha < \frac{1}{1+\rho} < S_X(0)$, *we have*:

*a*1*) If* $0 < d_1 < d_2 \le S_X^{-1}\left(\frac{1}{1+\rho}\right)$, *then* $d^* = d_2$ *and* $\mathrm{VaR}_T(d^*, \alpha) = d_2 + \delta(d_2)$.

*a*2*) If* $0 < d_1 \le S_X^{-1}\left(\frac{1}{1+\rho}\right) < d_2$, *then* $d^* = S_X^{-1}\left(\frac{1}{1+\rho}\right)$ *and* $\mathrm{VaR}_T(d^*, \alpha) = S_X^{-1}\left(\frac{1}{1+\rho}\right) + \delta\left(S_X^{-1}\left(\frac{1}{1+\rho}\right)\right)$.

*a*3*) If* $0 < S_X^{-1}\left(\frac{1}{1+\rho}\right) \le d_1 < d_2 \le S_X^{-1}(\alpha)$, *then* $d^* = d_1$ *and* $\mathrm{VaR}_T(d^*, \alpha) = d_1 + \delta(d_1)$.

*a*4*) If* $0 < S_X^{-1}\left(\frac{1}{1+\rho}\right) \le d_1 < S_X^{-1}(\alpha) < d_2$, *then* $d^* = \arg\min_{d \in \{d_1, d_2\}} \mathrm{VaR}_T(d, \alpha)$ *and*

$$\mathrm{VaR}_T(d^*, \alpha) = \min_{d \in \{d_1, d_2\}} \mathrm{VaR}_T(d, \alpha)).$$

*a*5*) If* $0 < S_X^{-1}(\alpha) \le d_1 < d_2$, *then* $d^* = d_2$ *and* $\mathrm{VaR}_T(d^*, \alpha) = S_X^{-1}(\alpha) + \delta(d_2)$.

b) For $\frac{1}{1+\rho} \le \alpha$ *we have* $d^* = d_2$ *and* $\mathrm{VaR}_T(d^*, \alpha) = \mathrm{VaR}_T(d_2, \alpha)$.

c) For $\frac{1}{1+\rho} \ge S_X(0)$ *we have*:

*c*1*) If* $0 < d_1 < d_2 \le S_X^{-1}(\alpha)$, *then* $d^* = d_1$ *and* $VaR_T(d^*, \alpha) = d_1 + \delta(d_1)$.

*c*2*) If* $0 < d_1 \le S_X^{-1}(\alpha) < d_2$, *then* $d^* = \arg\min_{d \in \{d_1, d_2\}} \mathrm{VaR}_T(d, \alpha)$ *and* $\mathrm{VaR}_T(d^*, \alpha) = \min_{d \in \{d_1, d_2\}} \mathrm{VaR}_T(d, \alpha))$.

*c*3*) If* $0 < S_X^{-1}(\alpha) \le d_1 < d_2$, *then* $d^* = d_2$ *and* $\mathrm{VaR}_T(d^*, \alpha) = S_X^{-1}(\alpha) + \delta(d_2)$.

Remark. For $d_1 \to 0$ and $d_2 \to \infty$ we obtain the results of [3], as special cases of the theorem. For $d_1 \to 0$ we obtain the results of [13], as special cases of the theorem.

3. AN OPTIMIZATION CRITERION BASED ON CTE RISK MEASURE

According to [1], the Conditional Tail Expectation (CTE) of the random variable X at the confidence level $1-\alpha$ is given by

$$\mathrm{CTE}_X(\alpha) = E\left[X \middle| X \ge \mathrm{VaR}_X(\alpha)\right]. \tag{13}$$

In the same manner, one defines the CTE for the random variable T defined by relation (5).

We suppose that X has a one-to-one continuous distribution function on $(0, \infty)$. The objective of the second optimization problem is determining the retention d^* that minimizes CTE risk measure provided that retention does not exceed a lower limit d_1 and an upper limit d_2, that means solving the following optimization problem:

$$\min_{0 < d_1 \le d \le d_2} \left\{\mathrm{CTE}_T(d, \alpha)\right\}. \tag{14}$$

According to [3], for any $d > 0$ and $0 < \alpha < S_X(0)$ we have

$$\mathrm{CTE}_T(d,\alpha)=\begin{cases} d+\delta(d), & \text{if } 0<d\le S_X^{-1}(\alpha) \\ S_X^{-1}(\alpha)+\delta(d)+\dfrac{1}{\alpha}\displaystyle\int_{S_X^{-1}(\alpha)}^{d} S_X(x)\mathrm{d}x, & \text{if } d>S_X^{-1}(\alpha) \end{cases}. \tag{15}$$

Remark. For $d_1 \to 0$ and $d_2 \to \infty$ we obtain the results of [3], as special cases of the theorem.

Theorem 2. *Let* $0 < d_1 < d_2 < \infty$.

1) *The optimal retention* $d^* = d^*(d_1, d_2) > 0$ *for* (14) *always exists*.

2) *The optimal retention* d^* *and the corresponding optimal value of the* CTE *risk measure are*:

a) For $0<\alpha<\dfrac{1}{1+\rho}<S_X(0)$, *we have*:

*a*1*) If* $0<d_1<d_2\le S_X^{-1}\left(\dfrac{1}{1+\rho}\right)$, *then* $d^*=d_2$ *and* $\mathrm{CTE}_T(d^*,\alpha)=d_2+\delta(d_2)$.

*a*2*) If* $0<d_1\le S_X^{-1}\left(\dfrac{1}{1+\rho}\right)<d_2$, *then* $d^*=S_X^{-1}\left(\dfrac{1}{1+\rho}\right)$ *and* $\mathrm{CTE}_T(d^*,\alpha)=S_X^{-1}\left(\dfrac{1}{1+\rho}\right)+\delta\left(S_X^{-1}\left(\dfrac{1}{1+\rho}\right)\right)$.

*a*3*) If* $0<S_X^{-1}\left(\dfrac{1}{1+\rho}\right)\le d_1<d_2$, *then* $d^*=d_1$ *and* $\mathrm{CTE}_T(d^*,\alpha)=\mathrm{CTE}_T(d_1,\alpha)$.

b) For $\dfrac{1}{1+\rho}\le\alpha$ *we have* $d^*=d_2$ *and* $\mathrm{CTE}_T(d^*,\alpha)=\mathrm{CTE}_T(d_2,\alpha)$.

c) For $\dfrac{1}{1+\rho}\ge S_X(0)$ *we have*:

*c*1*) If* $0<d_1<d_2\le S_X^{-1}(\alpha)$, *then* $d^*=d_1$ *and* $\mathrm{CTE}_T(d^*,\alpha)=d_1+\delta(d_1)$.

*c*2*) If* $0<d_1\le S_X^{-1}(\alpha)<d_2$, *then* $d^*=\arg\min\limits_{d\in\{d_1,d_2\}}\mathrm{CTE}_T(d,\alpha)$ *and* $\mathrm{CTE}_T(d^*,\alpha)=\min\limits_{d\in\{d_1,d_2\}}\mathrm{CTE}_T(d,\alpha))$.

*c*3*) If* $0<S_X^{-1}(\alpha)\le d_1<d_2$, *then* $d^*=d_2$ *and* $\mathrm{CTE}_T(d^*,\alpha)=S_X^{-1}(\alpha)+\delta(d_2)$.

4. COMPUTATIONAL RESULTS

Let X be Generalized Pareto distributed, with survival function $S_X(x)=\left(\dfrac{k+\alpha}{x+\alpha}\right)^a\cdot \mathrm{e}^{-\beta(x-k)}$, if $x\ge k$ and $S_X(x)=1$, if $x<k$. We will compute the optimal retention and the minimal value of VaR risk measure.

Using the results of the theorem, for k = 0, a = 2, α = 0.1, β = 1, ρ = 0.1, $d_1=\mathbf{0.001}$ and $d_2=\mathbf{1}$ we obtain the optimal retention $d^*(d_1,d_2)=\mathbf{0.004637946}$ and the corresponding minimal value of the VaR risk measure $\mathrm{VaR}_T(d^*,\alpha)=0.087612$.

5. CONCLUSION

In this paper we have defined and solved the problem of deriving the restricted optimal retention in a stop-loss reinsurance under Value-at-Risk and Conditional Tail Expectation risk measures. *We have proved the existence of the optimal solution of the considered problems in any conditions and we have derived the* *optimal retention,* which depends only on the assumed loss distribution and the safety loading factor. Our approach extends the results obtained in [3] and [13]**.** *On the line of papers [7,8,9,10] we can formulate some problems of such type*.

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