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EXISTENCE AND PROPERTIES OF THE OPTIMAL HUMAN CAPITAL AND CONSUMPTION EVOLUTION IN THE FRAMEWORK OF LUCAS MODEL ON FINITE HORIZON, WHEN THE NUMBER OF WORKERS IS NOT NORMALIZED

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In this paper the Lucas optimal growth of the human capital and consumption on finite horizon is analysed when the number of workers is not normalized. The main purpose is to establish the dependence of the optimal human capital and consumption evolution on the number of workers. The analysis reveals: for any initial value $h_0 > 0$ there exists an optimal evolution path of length $N+1$ for the human capital and it satisfies the Euler equation; at the moment N the human capital is given by the relation $h_N = (1-\delta)\, h_{N-1}$; the value function V_N is continuous on $\Re_+$ and there exists $A_V > 0$ such as $V_N(h_0, m_0) \le A_V \cdot h_0$; the family of functions $\{V_{N-T}\}_{T=0\ldots N-1}$ satisfies the Bellman equation and it is the unique solution of this equation which is continuous and satisfies the transversality condition.

Key words: Optimization problems; Mathematical models applied in economies.

1. INTRODUCTION

The basic problem in Lucas model on finite horizon with normalized number of workers is presented in Ref. [4] and detailed in Ref. [3].

When the number of workers is not normalized, the basic problem in the Lucas model on finite horizon is:

Find (h_t, m_t, c_t) under the constraints:

$$\forall t, \quad 0 \le c_t \le \frac{1}{m_t} f\left(\phi\left(\frac{h_{t+1}}{h_t}\right) \cdot h_t \cdot m_t\right) \tag{1}$$

$$(1-\delta)h_t \le h_{t+1} \le (1+\lambda)h_t \tag{2}$$

$$1 \le m_t \le M, \tag{3}$$

such as

$$\sum_{t=0}^{N-1} \beta^t u(c_t)$$

be maximum; $h_0 > 0$ and $m_0 \ge 1$ are given, $\beta = \frac{1+n}{1+\rho}$ is the discount factor and M is the maximum number of workers also given.

The Lucas model and the basic problem are formulated using the following quantities:

$L_t -$ the number of consumers at the moment t, where $L_t \ge 0$, $\forall\ t \ge 0$;

$N_t -$ the number of employees at the moment t, where $N_t \ge 0$, $\forall\ t \ge 0$;

h_t – the human capital at the moment t, where $h_t \geq 0$, $\forall\ t \geq 0$;

θ_t – the fraction of the working time dedicated strictly to work at the moment t, where $\theta_t \in [0,1]$, $\forall\ t \geq 0$;

c_t – the consumption at the moment t, where $L_t \geq 0$, $\forall\ t \geq 0$;

n – the growth rate of consumers, $n \geq 0$;

ρ – the time preference rate, $\rho \geq 0$;

δ – the human capital depreciation rate, $\delta \in [0,1)$;

λ – the human capital growth rate, $\lambda > 0$.

The fraction of each employee's working time used strictly to work is given by:

$$\theta_t = \phi\left(\frac{h_{t+1}}{h_t}\right), \quad 0 \leq \theta_t \leq 1, \tag{4}$$

where the function $\phi : R^1_+ \to R^1_+$ has the properties: is continuous on R_+ and is decreasing.

We keep the hypothesis H0-H3 from Ref. [3]:

H0: $\beta \in R^1$, $0 < \beta\,(1 + \lambda) < 1$

H1: The function $u : R^1_+ \to R^1_+$ is of class C^1 and satisfies:

- $u(0) = 0$
- $u'(x) > 0$, $\forall\ x > 0$
- $\exists\ A_u > 0$ such as $u(x) \leq A_u \cdot x$, $\forall\ x > 0$

H2: The function $f : R^1_+ \to R^1_+$ is of class C^2 and satisfies:

- $f(0) = 0$
- $f'(x) > 0$, $\forall\ x > 0$
- $\exists\ A_f > 0$ such as $f(x) \leq A_f \cdot x$, $\forall\ x > 0$

H3: N is a fixed natural number, $\forall\ N \geq 1$

Definition 1. A couple of finite sequences $((h_t)_{t=0\ldots N}, (m_t)_{t=0\ldots N-1})$ is called *a feasible human capital evolution* starting from (h^o, m^o) if it satisfies:

$$\forall\ t = 0 \ldots N-1,\ (1-\delta)\, h_t \leq h_{t+1} \leq (1+\lambda)\, h_t$$

$$1 \leq m_t \leq M$$

and $(h_0, m_0) = (h^o, m^o)$.

We denote by $\Pi(h^o, m^o, N)$ the set of all feasible human capital evolutions starting from (h^o, m^o).

Definition 2. A feasible human capital evolution starting from (h^o, m^o), $\left((h^o_t)_{t=0..N}, (m^o_t)_{t=0..N-1}\right)$ is called *optimal* if:

$$\max \sum_{t=0}^{N-1} \beta^t u\left(\frac{1}{m_t} f\left(\phi\left(\frac{h_{t+1}}{h_t}\right) \cdot h_t \cdot m_t\right)\right) = \sum_{t=0}^{N-1} \beta^t u\left(\frac{1}{m^o_t} \mathrm{f}\left(\phi\left(\frac{h^o_{t+1}}{h^o_t}\right) \cdot h^o_t \cdot m^o_t\right)\right),$$

where the maximum is considered on the set $\Pi(h^o, m^o, N)$.

Definition 3. A finite sequence $(c_t)_{t=0...N-1}$ of length N is called *consumption evolution* if:

$$c_t \geq 0, \ \forall \ t = 0 \ \ldots \ N-1.$$

Definition 4. A feasible human capital evolution starting from (h^o, m^o), $((h_t)_{t=0...N}, (m_t)_{t=0...N-1})$ and a consumption evolution of length N, $(c_t)_{t=0...N-1}$ is called a *feasible human capital and consumption evolution* if:

$$\forall \ t = 0 \ \ldots \ N-1, \ (1-\delta)\, h_t \leq h_{t+1} \leq (1+\lambda)\, h_t, \quad 1 \leq m_t \leq M,$$

$$0 \leq c_t \leq \frac{1}{m_t} f\left(\phi\left(\frac{h_{t+1}}{h_t}\right) \cdot h_t \cdot m_t\right).$$

Definition 5. A feasible human capital and consumption evolution, $\left((h_t^{\mathrm{o}})_{t=0..N}, (m_t^{\mathrm{o}})_{t=0..N-1}, (c_t^{\mathrm{o}})_{t=0..N-1}\right)$ is called *optimal* if:

$$\max_{\Pi(h^{\mathrm{o}}, m^{\mathrm{o}}, N)} \sum_{t=0}^{N-1} \beta^t u(c_t) = \sum_{t=0}^{N-1} \beta^t u(c_t^{\mathrm{o}}).$$

Theorem 1. *If* $((h_t)_{t=0...N}, (m_t)_{t=0...N-1})$ *is a feasible human capital evolution starting from* $(h^{\mathrm{o}}, m^{\mathrm{o}})$ *and* $c_t = \frac{1}{m_t} f\left(\phi\left(\frac{h_{t+1}}{h_t}\right) \cdot h_t \cdot m_t\right)$, $\forall \ t = 0 \ \ldots \ N-1$ *then* $\left((h_t)_{t=0..N}, (m_t)_{t=0..N-1}, (c_t)_{t=0..N-1}\right)$ *is a feasible human capital and consumption evolution and for every feasible human capital and consumption evolution of the form* $\left((h_t)_{t=0..N}, (m_t)_{t=0..N-1}, (c_t')_{t=0..N-1}\right)$ *we have*:

$$\sum_{t=0}^{N-1} \beta^t u(c_t') \leq \sum_{t=0}^{N-1} \beta^t u(c_t).$$

Proof. As $\left((h_t)_{t=0..N}, (m_t)_{t=0..N-1}, (c_t')_{t=0..N-1}\right)$ is a feasible human capital and consumption evolution, we have:

$$c_t' \leq \frac{1}{m_t} f\left(\phi\left(\frac{h_{t+1}}{h_t}\right) \cdot h_t \cdot m_t\right), \ \forall \ t = 0 \ \ldots \ N-1.$$

The function u is increasing (by the hypothesis H1) therefore:

$$u(c_t') \leq u\left(\frac{1}{m_t} f\left(\phi\left(\frac{h_{t+1}}{h_t}\right) \cdot h_t \cdot m_t\right)\right) = u(c_t)$$

$$\Rightarrow \sum_{t=0}^{N-1} \beta^t u(c_t') \leq \sum_{t=0}^{N-1} \beta^t u(c_t).$$

Consequence 1. *If* $\left((h_t^{\mathrm{o}})_{t=0..N}, (m_t^{\mathrm{o}})_{t=0..N-1}\right)$ *is an optimal human capital evolution starting from* $(h^{\mathrm{o}}, m^{\mathrm{o}})$ *and* $c_t^{\mathrm{o}} = \frac{1}{m_t^{\mathrm{o}}} f\left(\phi\left(\frac{h_{t+1}^{\mathrm{o}}}{h_t^{\mathrm{o}}}\right) \cdot h_t^{\mathrm{o}} \cdot m_t^{\mathrm{o}}\right)$ *then* $\left((h_t^{\mathrm{o}})_{t=0..N}, (m_t^{\mathrm{o}})_{t=0..N-1}, (c_t^{\mathrm{o}})_{t=0..N-1}\right)$ *is an optimal human capital and consumption evolution*.

Proof. The proof is immediate using the theorem's 1 result.

Consequence 2. *If* $\left(\left(h_t^{\text{o}}\right)_{t=0..N},\left(m_t^{\text{o}}\right)_{t=0..N-1},\left(c_t^{\text{o}}\right)_{t=0..N-1}\right)$ *is an optimal human capital and consumption evolution starting from* (h^{o}, m^{o}) *then* $c_t^{\text{o}}=\frac{1}{m_t^{o}}f\left(\phi\left(\frac{h_{t+1}^{\text{o}}}{h_t^{\text{o}}}\right)\cdot h_t^{\text{o}}\cdot m_t^{\text{o}}\right)$, $\forall\ t = 0 \ldots N-1$ and $\left(\left(h_t^{\text{o}}\right)_{t=0..N},\left(m_t^{\text{o}}\right)_{t=0..N-1}\right)$ *is an optimal human capital evolution starting from* (h^{o}, m^{o}).

Proof. We suppose that $\left(\left(h_t^{\text{o}}\right)_{t=0..N},\left(m_t^{\text{o}}\right)_{t=0..N-1}\right)$ is not an optimal evolution and we prove the contrary using the optimality of $\left(\left(h_t^{\text{o}}\right)_{t=0..N},\left(m_t^{\text{o}}\right)_{t=0..N-1},\left(c_t^{\text{o}}\right)_{t=0..N-1}\right)$.

By these consequences we observe that in order to determine an optimal human capital and consumption evolution path it's enough to determine only the part of the optimal human capital evolution of the path. Therefore, the basic problem in Lucas model is:

Find (h_t, m_t) under the constraints:

$$\forall\ t,\ \left(1-\delta\right)h_t \le h_{t+1} \le \left(1+\lambda\right)h_t, \tag{5}$$

$$1 \le m_t \le M, \tag{6}$$

such as

$$\sum_{t=0}^{N-1}\beta^t u\left(\frac{1}{m_t}f\left(\phi\left(\frac{h_{t+1}}{h_t}\right)\cdot h_t\cdot m_t\right)\right)$$

be maximum; $h_0 > 0$ and $m_0 \ge 1$ are given. The corresponding consumption c_t can be determined using the relation $c_t^{\text{o}}=\frac{1}{m_t^{\text{o}}}f\left(\phi\left(\frac{h_{t+1}^{\text{o}}}{h_t^{\text{o}}}\right)\cdot h_t^{\text{o}}\cdot m_t^{\text{o}}\right)$.

We transform this problem in an equivalent one using the multivalued application Γ and the function F defined below:

$$\Gamma : R_+^* \to R_+^* \times R_+^*,\quad \Gamma(x)=\left[(1-\delta)x,(1+\lambda)x\right],$$

$$F : R_+^* \times R_+^* \times R_+^* \to R_+^1,\quad F(x,y,z)=u\left(\frac{1}{z}f\left(x\cdot z\cdot\phi\left(\frac{y}{z}\right)\right)\right).$$

For $\left(x,y,z\right),\left(x_1,y_1,z_1\right)\in \text{graph}(\Gamma)\times R_+^1$ we define the distance

$$d((x, y, z), (x_1, y_1, z_1)) := d(x, x_1,).$$

The multivalued application Γ has the following properties:

i. $0 \in \Gamma(0)$;
ii. has compact sets as values;
iii. it is continuous on R_+^1.

The function F has the following properties:

i. is continuous on $R_+^* \times R_+^* \times R_+^*$;
ii. there exists $A_F > 0$ such as for any $\left(x,y,z\right)\in \text{graph}(\Gamma)\times R_+^1$ we have $F(x, y, z) \le A_F \cdot x$.

In terms of multivalued application Γ and function F the problem of determining the optimal human capital and consumption evolution path in Lucas model on finite horizon is:

Find (h_t, m_t) under the constraints:

$$\left(h_{t+1},m_t\right)\in\Gamma(h_t)\times[1,M],\qquad \forall t=0...N-1,$$

for which:

$$\sum_{t=0}^{N-1} \beta^t F\left(h_t, h_{t+1}, m_t\right)$$

is maximum (with $h_0 > 0$ and $m_0 \geq 1$). Therefore we have:

Definition 6. The couple of sequences $((h_t)_{t=0...N}, (m_t)_{t=0...N-1})$ is a *feasible human capital and consumption evolution* starting from (h^o, m^o) if and only if:

$$(h_{t+1}, m_t) \in \Gamma(h_t) \times [1, M],\ \forall\ t = 0 \ldots N-1 \text{ and } (h_0, m_0) = (h^o, m^o),$$

where h^o is a given strictly positive real number and m^o is a given strictly positive natural number.

Definition 7. A feasible human capital and consumption evolution starting from (h^o, m^o), $\left((h_t^o)_{t=0..N}, (m_t^o)_{t=0..N-1}\right)$ is an *optimal evolution* if and only if:

$$\max \sum_{t=0}^{N-1} \beta^t F\left(h_t, h_{t+1}, m_t\right) = \sum_{t=0}^{N-1} \beta^t F\left(h_t^o, h_{t+1}^o, m_t^o\right)$$

where the maximum is considered on the set $\Pi(h^o, m^o, N)$.

2. EXISTENCE OF THE OPTIMAL HUMAN CAPITAL AND CONSUMPTION EVOLUTION

Proposition 1. *For any fixed $h^o > 0$ and $m^o \geq 1$ there is an optimal human capital evolution (h, m) starting from (h^o, m^o).*

Proof. We proof that the function U defined by:

$$U : \Pi(h^o, m^o, N) \to R_+, \quad U(h,m) = \sum_{t=0}^{N-1} \beta^t F\left(h_t, h_{t+1}, m_t\right)$$

is continuous on the compact set $\Pi(h^o, m^o, N)$. Therefore, U is bounded thus it has a maximum on the set $\Pi(h^o, m^o, N)$.

Proposition 2. *For every fixed $h^o > 0$ and $m^o \geq 1$ there is an optimal human capital and consumption evolution path starting from (h^o, m^o), $\left((h_t)_{t=0..N}, (m_t)_{t=0..N-1}, (c_t)_{t=0..N-1}\right)$.*

3. PROPERTIES OF THE OPTIMAL HUMAN CAPITAL EVOLUTION PATH

Proposition 3. *If (h, m) is an optimal human capital evolution starting from $(h^o > 0, m^o \geq 1)$, then*

$$h_N = (1 - \delta)\, h_{N-1}.$$

Proof. We prove the statement by contradiction.

Definition 8. The function V_N defined by: $V_N : R_+^1 \times [1, M] \to R_+^1$,

$$V_N(h^o, m^o) = \max_{\Pi(h^o, m^o, N)} \sum_{t=0}^{N-1} \beta^t F\left(h_t, h_{t+1}, m_t\right),$$

where $\Pi(h^o, m^o, N)$ is the set of all feasible human capital evolution paths starting from (h^o, m^o), is called the value function.

Let $(h, m) \in \Pi(h^o, m^o, N)$. The value function V_N satisfies:

1. $V_N(0, 0) = 0$; $V_N(h^o, m^o) > 0$, for $h^o > 0$. If $\left((h_t)_{t=0..N}, (m_t)_{t=0..N-1}\right)$ is the optimal human capital evolution starting from (h^o, m^o) then $V_N(h^o, m^o) = \sum_{t=0}^{N-1} \beta^t F\left(h_t, h_{t+1}, m_t\right)$;

2. V_N is continuous on $R_+ \times [1, M]$;
3. $\exists A_V > 0$ such as $V_N(h^o, m^o) \le A_V \cdot h^o$.

Proposition 4. 1. *The functions V_{N-T} associated to h_T and V_{N-T-1} associated to h_{T+1} satisfies the Bellman equation for every $T \in \{0, \dots, N-2\}$:*

$$V_{N-T}(h_T, m_T) = \max\left\{F(h_T, h_{T+1}, m_T) + \beta V_{N-T-1}(h_{T+1}, m_{T+1}) : (h_{T+1}, m_{T+1}) \in \Gamma(h_T) \times [1, M]\right\}, \tag{7}$$

with

$$V_1(h_{N-1}, m_{N-1}) = \max\left\{F(h_{N-1}, h_N, m_{N-1}) : h_N \in \Gamma(h_{N-1})\right\}. \tag{8}$$

for any $h^o > 0$ and $m^o \ge 1$.

2. *The set of functions $\{V_{N-T}\}_{T=0\dots N-1}$* is *the unique solution of the Bellman equation which satisfies the conditions*:

i. $\forall\, t = 0 \dots N-1$, *$V_{N-T}$ is continuous on* $R_+^1 \times [1, M]$;

ii. *the transversally condition* : $\forall\, h_0 \ge 0$, $m_0 \ge 0$, $(h, m) \in \Pi(h_0, m_0, N)$ *we have*

$$V_1(h_{N-1}, m_{N-1}) = u\left(\frac{1}{m_{N-1}} f\left(h_{N-1} \cdot m_{N-1}\right)\right) \Leftrightarrow \tag{9}$$

$$\Leftrightarrow \phi\left(\frac{h_N}{h_{N-1}}\right) = 1. \tag{10}$$

3. *The sequence (h, m) is an optimal evolution path of length N+1 starting from (h_0, m_0)* if and only if:

$$\forall T = 0 \dots N-2,\ \ V_{N-T}(h_T, m_T) = F(h_T, h_{T+1}, m_T) + \beta V_{N-T-1}(h_{T+1}, m_{T+1}) \tag{11}$$

and $h_N \le (1-\delta)\, h_{N-1}$.

Proof. 1. We prove the double inequality.

2. i) The proof of the V_{N-T} function's continuity is similar to the proof of V_N continuity;

ii) Immediate by the definition of the value function. For the uniqueness we consider two solutions of the Bellman equation and we prove that they are identical.

3. Prove the double inequality.

Let G_{N-T} be a multivaluated function defined for any $T = 0, \dots, N-2$ as follows:

$$G_{N-T} : R_+^1 \times [1, M] \to 2^{R_+^1},$$

$$G_{N-T}(h_T, m_T) = \arg\max\left\{F(h_T, y, m_T) + \beta V_{N-T-1}(y, m) : (y, m) \in \Gamma(h_T) \times [1, M]\right\}$$

or equivalent

$$G_{N-T}(h_T, m_T) = \left\{(y, m) \in \Gamma(h_T) \times [1, M] : V_{N-T}(h_T, m_T) = F(h_T, y, m_T) + \beta V_{N-T-1}(y, m)\right\}.$$

Proposition 5. 1. *G_{N-T} is upper semi-continuous on* $R_+^1 \times [1, M]$, $\forall\, T \in \{0, \dots, N-2\}$.

2. *A feasible evolution path starting from (h_0, m_0) is optimal if and only if* $\forall T = 0, \dots, N-1$, $(h_T, m_T) \in G_{N-T+1}(h_{T-1}) \times [1, M]$ *and* $h_N \le (1-\delta)\, h_{N-1}$.

3. *If (h, m) is an optimal evolution starting from (h_0, m_0) then it satisfies the Euler equation*:

$$F_2(h_T, h_{T+1}, m_T) + \beta F_1(h_{T+1}, h_{T+2}, m_{T+1}) = 0,$$
$$F_3(h_{T+1}, h_{T+2}, m_{T+1}) = 0,$$

where F_1, F_2 and F_3 denotes the derivatives of F with respect to the first, the second and the third variables.

Proof. 1. We prove that $G_{N-T}(x, y)$ is a non-empty and compact set for any $(x, y) \in R_+^1 \times [1, M]$ and that for any pair of sequences $(\{x_n\}, \{y_n\})$ with $x_n \to x$, $y_n \to y$ and any pair of sequences $(\{z_n\}, \{t_n\})$, with

$(z_n, t_n) \in G_{N-T}(x_n, y_n)\ \forall n$, there is a subsequence $\{z_{n_k}\}$ and a sequence $\{t_{n_k}\}$ converging to z and t, with $(z, t) \in G(x, y)$.

2. Immediate using the Proposition's 4 results.

3. We prove that the function $\Phi(x, y) = F(h_t, x, m_t) + \beta F(x, h_{t+2}, y)$ has a local maximum in (h_{t+1}, m_{t+1}).

4. CONCLUSIONS

In the Lucas optimal growth model on finite horizon when the number of workers is not normalized for every initial value (h^o, m^o) there is an optimal human capital and consumption evolution path $\left((h_t)_{t=0..N}, (m_t)_{t=0..N-1}, (c_t)_{t=0..N-1}\right)$ and it satisfies the Euler equation. The human capital for the last period can be determined using $h_N = (1 - \delta)\, h_{N-1}$. The value function V_N is continuous and there is an $A_N > 0$ such that $V_N(h^o, m^o) \le A_V \cdot h^o$. Also, the set of value functions is the unique solution of the Bellman equation which is continuous and satisfies the transversality condition.

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