THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 11, Number x/2010, pp. 171–178

TRANSFER MATRIX METHOD FOR ORTHOTROPIC CYLINDRICAL SHELLS WITH VARIABLE WALL THICKNESS

Mihai VRABIE, Nicolae UNGUREANU, Dan DIACONU-SOTROPA

"Gheorghe Asachi" Technical University of Iasi, Faculty of Civil Engineering and Building Services,
43, Dimitrie Mangeron Blvd., Iasi 700050, Romania
Corresponding author: Mihai VRABIE, E-mail: mihai_vrabie@yahoo.com

The determination of the structural response of a cylindrical shell with variable wall thickness and loading conditions requires a large amount of computation. For isotropic shells subjected to axisymmetric loading analytic solution there is and the transfer matrix technique was developed in literature. In what follows this technique is extended for orthotropic cylindrical shells. Cylindrical tanks for fluids are used for to illustrate the analyse technique by transfer matrix method. The results are verified by FEM and a good agreement is obtained.

Key words: Orthotropic cylindrical shells; Variable wall thickness; Transfer matrix method; Finite element method.

1. INTRODUCTION

The cylindrical shells are frequently used in various domains of the technique. Often, they are made by orthotropic materials or may to possess a geometric (structural) orthotropy because of stiffening with circular and/or longitudinal ribs. The vertical cylindrical tanks for fluids storage are included in the category of closed cylindrical shells.

The vertical cylindrical tanks are frequently made from steel or reinforced/prestressed concrete [1, 2, 3, 4]. In some industries, such as chemical, food, petrochemical ones and so on, there are used also tanks or containers made from composite materials, usually reinforced with different fibers on one or two directions [5, 6, 7]. The reinforcing ratios, much different on the two directions, such as the prestressed forces, are different at the tanks made of prestressed concrete, involve also different elastic characteristics on the two principal elastic directions (material orthotropy).

In the case of relatively heigh reinforced tanks, the thickness of wall is linear variable achieved, increasingly with the pressure exerted by the liquid. At the metal tanks, but also for those from reinforced or composite materials tanks, the wall thickness is achieved frequentlly stepwise variable, being maximum at where the liquid pressure has the greates value. Some computation assumptions take into account partially filled tank, but in the case of prestressing its effect in the circumferential direction are as circular uniform forces and on the generatrix direction the effects from prestressing apear as concentrated forces at extremities. There is taking into account, simultaneously or successively, the following design parameters (Fig. 1.1):

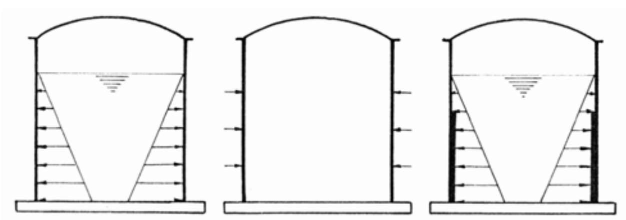

Fig. 1.1 – Different design parameters for cylindrical tanks.

– the liquid pressure in case of the partially filled tank;
– the circumferential prestressing forces;
– stepwise wall thickness variation;
– the orthotropy of the material.

In design is imposing an exhaustive analysis of the stress state corresponding to a bending theory for a more adequate reinforcing of reinforced concrete tanks or reinforced composite onse, in order to achieve joining and stability checking in the case of the metal or composite materials tanks.

The determination of the response (the stress and the strain state in the structure) of an orthotropic shell with variable geometry and loading conditions requires a large amount of computation. In mechanics of structures some variational and computational method for determination of the response was developed (finite differences, finite element methods, boundary element methods, differential quadrature and others). Only a limited number of loading and boundary conditions for the mentioned structures have exact solution. The term "exact solution" is used here to mean finding a solution that satisfies both the differential equations and boundary conditions exactly [8]. For isotropic closed cylindrical shells with axisymmetric loading and boundary conditions, the differential equation of the wall bending have exact solution [3, 4, 9, 10].

In the articles [11, 12], the transfer matrix analyze technique was developed for isotropic tanks. This method, frequently used in various domains of technique, allows to order the calculus process and facilitates the programation for the automatic computation [13, 14, 15, 16]. In what follows, this technique is extended at the orthotropic tanks.

2. DIFFERENTIAL EQUATION OF THE WALL BENDING AND SOLUTIONS

It is analyzed the general case, when the internal forces from the base and the top part of the shell interact. It is used the Cauchy and the initial parameters methods, in order to determinate particular solutions on loaded interval. By means of the transfer matrix, is determinate the state vector at different levels. The fundamental state vector, localized frequently at the base of the wall, is determinate from the boundary conditions at the top edge. There are considered various boundary conditions, both at the base and the top edge of the shell.

In order to establish the differential equation of the wall bending, it is considered orthotropic cylindrical tanks having walls with constant or variable thickness, with geometrical, elastic, supporting and loading symmetry against the vertical axis (Fig. 2.1).

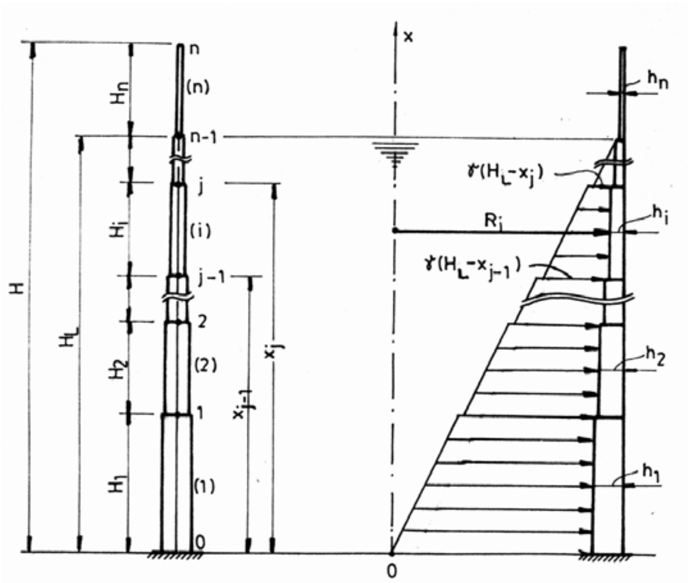

Fig. 2.1 – Geometric and loading characteristics of the cylindrical tank.

On the current section i, where the geometrical parameters R_i and h_i, also the elastic properties E_{xi}, $E_{\theta i}$, ν_{xi}, $\nu_{\theta i}$, are constant, we can use the equation [7, 9, 11]:

$$\frac{\mathrm{d}^4 w_i}{\mathrm{d}\xi^4} + 4w_i = s_i^4 \frac{q_i}{D_{xi}}, \tag{2.1}$$

where: w_i – is the radial displacement from the bending of the wall; $\xi_i = x_i/s_i$ – is the reduced length; q_i – is the distributed radial loading; s_i – is the decreasing coefficient; D_{xi} – is the bending rigidity in the generatrix direction:

$$D_{xi} = \frac{E_{xi} h_i^3}{12\left(1 - \nu_{xi}\nu_{\theta i}\right)} \tag{2.2}$$

The s_i diminution coefficient is the reversal of the damping coefficient β_i:

$$s_i = \frac{1}{\beta_i} = \frac{1}{\sqrt[4]{\dfrac{3\left(1-\nu_{xi}\nu_{\theta i}\right)}{R_i^2 h_i^2}\dfrac{E_{\theta i}}{E_{xi}}}}. \tag{2.3}$$

The displacements w_i, ϕ_i and the internal forces M_{xi}, V_{xi}, $N_{\theta i}$, in the points of the „i" segment, are expressed with the next relations:

$$\begin{aligned}
w_i &= w_{j-1} f_{1i} + s_i \phi_{j-1} f_{2i} - \frac{s_i^2 M_{x,j-1}}{D_{xi}} f_{3i} - \frac{s_i^3 V_{x,j-1}}{D_{xi}} f_{4i} + w_{pi}; \; N_{\theta i} = -\frac{E_{\theta i} h_i}{R_i} w_i, \\
\phi_i &= -\frac{4}{s_i} w_{j-1} f_{4i} + \phi_{j-1} f_{1i} - \frac{s_i M_{x,j-1}}{D_{xi}} f_{2i} - \frac{s_i^2 V_{x,j-1}}{D_{xi}} f_{3i} + w'_{pi}, \\
M_{xi} &= \frac{4}{s_i^2} D_{xi} w_{j-1} f_{3i} + \frac{4}{s_i} D_{xi} \phi_{j-1} f_{4i} + M_{x,j-1} f_{1i} + s_i V_{x,j-1} f_{2i} - D_{xi} w''_{pi}, \\
V_{xi} &= \frac{4}{s_i^3} D_{xi} w_{j-1} f_{2i} + \frac{4}{s_i^2} D_{xi} \phi_{j-1} f_{3i} - \frac{4}{s_i} M_{x,j-1} f_{4i} + V_{x,j-1} f_{1i} - D_{xi} w'''_{pi},
\end{aligned} \tag{2.4}$$

where $w_{j\text{-}1}$, $\phi_{j\text{-}1}$, $M_{x,j\text{-}1}$, $Q_{x,j\text{-}1}$ are the displacements and the internal forces in the extremity j–1 of the i tronson, considered of the origin (Fig. 2.1), and $f_{1i}, f_{2i}, f_{3i}, f_{4i}$ are the Krilov functions:

$$\begin{aligned}
f_{1i} &= \mathrm{ch}\xi_i \cos\xi_i; \; f_{2i} = \frac{1}{2}\left(\mathrm{ch}\xi_i \sin\xi_i + \mathrm{sh}\xi_i \cos\xi_i\right) \\
f_{3i} &= \frac{1}{2}\mathrm{sh}\xi_i \sin\xi_i; \; f_{4i} = \frac{1}{4}\left(\mathrm{ch}\xi_i \sin\xi_i - \mathrm{sh}\xi_i \cos\xi_i\right).
\end{aligned} \tag{2.5}$$

w_{pi} is the particular solution of the differential equation (2.1) and it can be determined using the Cauchy method:

$$w_{pi} = \frac{s_i^4}{D_{xi}} \int_0^{\xi_i} f_{4i}\left(\xi_i - \eta\right) q_i\left(\eta\right) \mathrm{d}\eta, \tag{2.6}$$

where η is the current variable on the interval [0, ξ_i].

The hydrostatic pressure on the interval i, whose extremity are j–1 and j, varies trapezoidally and it can be considered as the sum of a loading with constant intensity $q_j = \gamma(H_L–x_j)$, and a triangular one with maximum intensity $\gamma(x_j–x_{j\text{-}1})$. From the relation (2.6), we can obtain:

$$w_{pi} = \frac{s_i^4\left(H_L - x_j\right)}{4D_{xi}}\left[1 - f_{1i}\left(\xi_i\right)\right] + \frac{s_i^5\gamma}{4D_{xi}}\left[\lambda_i - \xi_i - \lambda_i f_{1i}\left(\xi_i\right) + f_{2i}\left(\xi_i\right)\right], \tag{2.7}$$

where $\lambda_i = H_i/s_i$ (H_i is the length of the interval).

The first three derivatives of the particular solution for the loading with trapezoidal variation are:

$$
\begin{aligned}
w'_{pi} &= \frac{s_i^3\left(H_L - x_j\right)\gamma}{D_{xi}} f_{4i}\left(\xi_i\right) + \frac{s_i^4\gamma}{4D_{xi}}\left[f_{1i}\left(\xi_i\right) + 4\lambda_i f_{4i}\left(\xi_i\right) - 1\right],\\
w''_{pi} &= \frac{s_i^2\left(H_L - x_j\right)\gamma}{D_{xi}} f_{3i}\left(\xi_i\right) + \frac{s_i^3\gamma}{D_{xi}}\left[\lambda_i f_{3i}\left(\xi_i\right) - f_{4i}\left(\xi_i\right)\right],\\
w'''_{pi} &= \frac{s_i\left(H_L - x_j\right)\gamma}{D_{xi}} f_{2i}\left(\xi_i\right) + \frac{s_i^2\gamma}{D_{xi}}\left[\lambda_i f_{2i}\left(\xi_i\right) - f_{3i}\left(\xi_i\right)\right].
\end{aligned}
\tag{2.8}
$$

The effects of a concentrated couple, respectively of a concentrated force in the direction of the generatrix or a radially uniform distributed force on the circumference, are introduced by analogy with those from the origin. The concentrated forces can proceed particularly from the circumferential or longitudinal prestressing in the case of the tank from prestressed concrete. The particular solutions in the case of loading with a couple M and a force P are:

$$
w_{pi} = -\frac{s_i^2 M}{D_{xi}} f_{3i}\left(\xi_i - \lambda_m\right) - \frac{s_i^3 P}{D_{xi}} f_{4i}\left(\xi_i - \lambda_p\right),
\tag{2.9}
$$

with $\lambda_m = l_m/s_i$, $\lambda_p = l_p/s_i$, where l_m and l_p are lenghts wich are distributed M and P.

3. THE USE OF THE TRANSFER MATRIX METHOD

The parameters of the static response w, ϕ, M_x, V_x and the unit are considered as components of the vector $\{S\}$, named *state vector*, for wich a current section from the interval „i”, are written in line:

$$
\{S\}_i = \{w; \phi; M_x; V_x;\ \ 1\}^{\mathrm{T}}
\tag{3.1}
$$

The state vector from the extremities j–1 and j of the interval „i” are written:

$$
\left\{S_{j-1}\right\}_i = \left\{w_{j-1}; \phi_{j-1}; M_{x,j-1}; V_{x,j-1}; 1\right\}^{\mathrm{T}},\ \left\{S_j\right\}_i = \left\{w_j; \phi_j; M_{x,j}; V_{x,j}; 1\right\}^{\mathrm{T}}.
\tag{3.2}
$$

For express the state vector $\{S_j\}_i$ depending on $\{S_{j\text{-}1}\}_i$ the relations (2.4), in which $\xi_i = H_i/s_i = \lambda_i$ are written in the matrix shape:

$$
\left\{S_j\right\}_i = \left[T\right]_{j-1}^{j}\left\{S_{j-1}\right\}_i,
\tag{3.3}
$$

where $\left[T\right]_{j-1}^{j}$ is the *segment matrix* or the *interval matrix* for the continuity interval (j–1, j) [11, 12, 13]:

$$
\left[T\right]_{j-1}^{j} = \begin{bmatrix}
f_{1i}\left(\lambda_i\right) & s_i f_{2i}\left(\lambda_i\right) & -\frac{s_i^2}{D_{xi}} f_{3i}\left(\lambda_i\right) & -\frac{s_i^3}{D_{xi}} f_{4i}\left(\lambda_i\right) & w_{pi}\left(\lambda_i\right)\\
-\frac{4}{s_i} f_{4i}\left(\lambda_i\right) & f_{1i}\left(\lambda_i\right) & -\frac{s_i}{D_{xi}} f_{2i}\left(\lambda_i\right) & -\frac{s_i^2}{D_{xi}} f_{3i}\left(\lambda_i\right) & w'_{pi}\left(\lambda_i\right)\\
\frac{4D_{xi}}{s_i^2} f_{3i}\left(\lambda_i\right) & \frac{4D_{xi}}{s_i} f_{4i}\left(\lambda_i\right) & f_{1i}\left(\lambda_i\right) & s_i f_{2i}\left(\lambda_i\right) & -D_{xi} w''_{pi}\\
\frac{4D_{xi}}{s_i^3} f_{2i}\left(\lambda_i\right) & \frac{4D_{xi}}{s_i^2} f_{3i}\left(\lambda_i\right) & -\frac{4}{s_i} f_{4i}\left(\lambda_i\right) & f_{1i}\left(\lambda_i\right) & -D_{xi} w'''_{pi}\\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\tag{3.4}
$$

The axial force $N_{\theta j}$ is determined knowing w_j.

The last column of the matrix $[T]_{j-1}^{j}$ depends on the loading and for the hydrostatic pressure is determinated from the relations (2.7), (2.8), in which $\xi_i = \lambda_i$. If the interval (j–1, j) is unloaded, than the last column is zero, excepting the last term, which remains 1.

At the passing over a concentrated couple or a radial concentrated force, which are uniform on the circumference direction and, possible, over an axial force, which is distributed also uniformly on the circumference, are used *crossing matrix*. Such a relation, which connect the state vectors from the adjacent sections k and k–1, is:

$$\{S_k\}_i = [T]_{k-1}^{k}\{S_{k-1}\}_i \tag{3.5}$$

or detailed, when the crossing matrix has all three loading cases:

$$\begin{Bmatrix} w_k \\ \phi_k \\ M_{x,k} \\ V_{x,k} \\ N_{x,k} \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & M \\ 0 & 0 & 0 & 1 & 0 & -P \\ 0 & 0 & 0 & 0 & 1 & N \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w_{k-1} \\ \phi_{k-1} \\ M_{x,k-1} \\ V_{x,k-1} \\ N_{x,k-1} \\ 1 \end{Bmatrix}. \tag{3.6}$$

The couple M has been considered clockwise, the force P having the radial direction on the towards centre and the axial force N in the direction of x axis, therefore of the tension at the wall. Obviously, some of these actions can to be absent.

The change of stepwise thickness impose an adequate segmentation, which put into evidence differents stiffnesses. In a series of relations between the succesive state vectors it is achivied requrance relationships. The state vector from the section „j" is:

$$\{S_j\}_i = [T]_{j-1}^{j}[T]_{j-2}^{j-1}\cdots[T]_1^2[T]_0^1\{S_0\}_1 \tag{3.7}$$

If the index j and i have certain values, the state vector is expressed in any section. In „n" extremity the vector is:

$$\{S_n\}_n = \leftarrow\prod_{j=0}^{n}[T]_{j-1}^{j}\{S_0\}_1 = [T]_0^n\{S_0\}_1 \tag{3.8}$$

where $\leftarrow\prod_{j=0}^{n}$ is an orderly product from the right to the left of the transfer matrix; $[T]_0^n$ is the transfer matrix between the two extremities of the tank.

In order to determinate the state vector $\{S_0\}_1$, means to set the boundary (supporting) conditions at the top part of the tank. The relation (3.8) express the connection between the state vector from the section situated on the base and the superior part of the tank, the last term has the boundary conditions. The relation (3.8) becomes:

$$\begin{aligned} &w_n = t_{11}w_0 + t_{12}\phi_0 + t_{13}M_{x0} + t_{14}V_{x0} + t_{10};\ \phi_n = t_{21}w_0 + t_{22}\phi_0 + t_{23}M_{x0} + t_{24}V_{x0} + t_{20}, \\ &M_{xn} = t_{31}w_0 + t_{32}\phi_0 + t_{33}M_{x0} + t_{34}V_{x0} + t_{30};\ V_{xn} = t_{41}w_0 + t_{42}\phi_0 + t_{43}M_{x0} + t_{44}V_{x0} + t_{40}. \end{aligned} \tag{3.9}$$

The „t" coefficients are the terms of the multiplication matrix $[T]_0^n$. For the tanks which has the walls fixed at the base, $w_0 = 0$, $\phi_0 = 0$, and for the wall which is hinged at the base $w_0 = 0$, $M_{x0} = 0$. The equations (3.9) for the two cases are:

$$\begin{aligned} &w_n = t_{13}M_{x0} + t_{14}V_{x0} + t_{10};\ \phi_n = t_{23}M_{x0} + t_{24}V_{x0} + t_{20}, \\ &M_{xn} = t_{33}M_{x0} + t_{34}V_{x0} + t_{30};\ V_{xn} = t_{43}M_{x0} + t_{44}V_{x0} + t_{40}. \end{aligned} \tag{3.10}$$

$$w_n = t_{12}\phi_0 + t_{14}V_{x0} + t_{10};\ \phi_n = t_{22}\phi_0 + t_{24}V_{x0} + t_{20},$$
$$M_{xn} = t_{32}\phi_0 + t_{34}V_{x0} + t_{30};\ V_{xn} = t_{42}\phi_0 + t_{44}V_{x0} + t_{40}. \tag{3.11}$$

In what follows, if the wall is free at the top edge, we have $M_{xn} = 0$, $V_{xn} = 0$ and, from the Eq. (3.10) we determine M_{x0}, V_{x0} and, from Eq. (3.11) we can calculate ϕ_0, V_{x0}, therefore the vector $\{S_0\}_1$ is completely determined.

In the case of an elastically supporting at the base of the wall, which has spring ties for displacement and for rotation, M_{x0} and V_{x0} can be expressed in terms of ϕ_0 and respectively w_0:

$$M_{x0} = k_\phi \phi_0,\ V_{x0} = k_w w_0, \tag{3.12}$$

where k_ϕ and k_w are the rotation, respectively displacement stiffneses of the springs, simulating the elastical connection wall-foundation. In the case of the free tank at the superior part, we can obtain the following relations in w_0 and ϕ_0:

$$\left(t_{31} + t_{34}k_w\right)w_0 + \left(t_{32} + t_{33}k_\phi\right)\phi_0 + t_{30} = 0,$$
$$\left(t_{41} + t_{44}k_w\right)w_0 + \left(t_{42} + t_{43}k_\phi\right)\phi_0 + t_{40} = 0. \tag{3.13}$$

After the determination of the fundamental state vector from the boundary conditions at the superior part, we can express with the help of the transfer matrix, the state vectors at the different levels (3.7). The developed analysis technique arranges the computation process and facilitates the programation for the automatic calculus.

4. NUMERICAL RESULTS

For the proposed of this procedure it is considered a cylindrical tank with the radius $R_i = R$ =15m and the height H = 8m, made of prestressed concrete that is considered:

a) isotropic material with $E_x = E_\theta = E$ = 300 000 daN/cm^2, $\nu_x = \nu_\theta = \nu = 0.2$;

b) orthotropic material with E_x = 240,000 daN/cm^2, E_θ = 300,000 daN/cm^2, $\nu_x = 0.16$, $\nu_\theta = 0.2$.

The tank is partially filled with water (γ = 10 kN/m^3); the liquid has the height from the base H_L = 6m. The wall of the tank has a thickness h_1 = 25cm, for a height H_1 = 3m from the base, untill the superior part the thickness is 15 cm. The wall of the tank, fixed at the base and free at the superior part, is divided in three segments for the calculus (Fig. 4.1).

It is calculated the segment matrix $[T]_0^1, [T]_1^2, [T]_2^3$ and than, the multiplication matrix $[T]_0^3 = [T]_2^3[T]_1^2[T]_0^1$, by which is written the state vector from the free end $\{S_3\}_3 = [T]_0^3\{S_0\}_1$. Using the boundary conditions $M_{x3} = 0$, $V_{x3} = 0$, is determinated the components of the fundamental state vector. The state vectors from the sections 0, 1, 2 in the two cases of material are presented in the Table 4.1:

Table 4.1

The state vectors in three sections of the tank wall

State vector	Section	0		1		2	
	Material	*a*	*b*	*a*	*b*	*a*	*b*
w [cm]		0	0	0.113	0.117	0.018	0.017
ϕ [rad]		0	0	2.0358E-04	1.8086E-04	–3.1177E-04	–2.9735E-04
M_x [daN]		–4965.712	–4481.515	628.69	516.54	–196.97	–169.88
V_x [daN/cm]		75.112	71.664	–1.451	–1.289	–0.167	0.046
N_x [daN/cm]		0	0	6780	5834	54	51

The plotting of the internal forces M_x and V_x on the interval (1) (where they have significant values), is presented in Fig. 4.2 (for the case of the isotropic material).

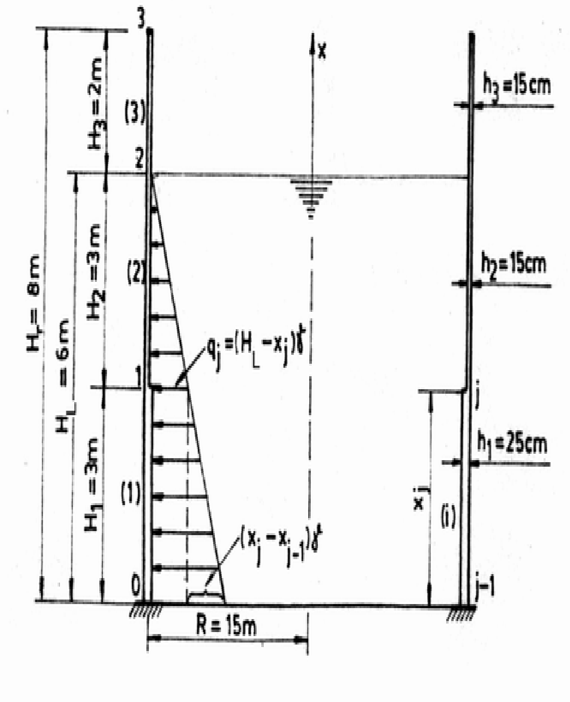

Fig. 4.1 – Geometric and loading conditions of the tank.

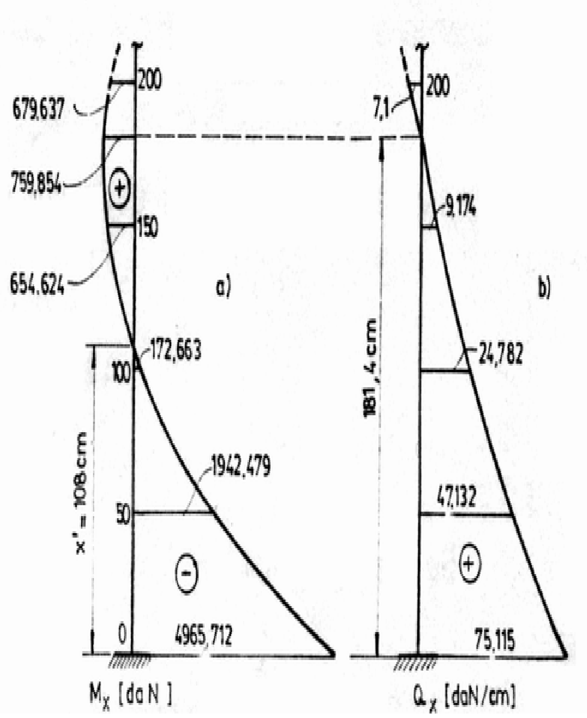

Fig. 4.2 – Bending moment and shear force variation.

Tests have been made with some automatic calculus programs, which are based on FEM. The results obtained for the displacements are close to those obtained with the transfer matrix method (1.1585 mm with ANSYS, 1.188 mm with ROBOT and 1.163 mm with AXIS VM software towards 1,17mm by the transfer matrix method).

5. CONCLUSIONS

1. The determination of the stress state in the structure of an isotropic shell with variable geometry and loading conditions implies a large number of design parameters. This number of parameters is bigger for the orthotropic shells, and this fact lead to the complications in the calculus.

2. Analyze technique with transfer matrix has some advantages:

- allow to take into account the various loading conditions, variable geometry of the shell walls and various boundary conditions;
- allow to order the calculus process and facilitates the programation for the automatic computation.

3. The results obtained from the calculus example show that, in the case “b” (the tank made from orthotropic material), the values of the internal forces is smaller than in the case “a” (isotropic tank) – with 10% for M_x and 5% for V_x in section 0 at the base of the tank wall.

4. The values of the radial displacements w is bigger in the case “b” that in the case “a” (with 3,5% in section 1, at 3m from the base of the tank).

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the financial support of the CNCSIS – UEFISCSU, through the research project ID_619/1.10.2007.

REFERENCES

1. CIOCLOV Dragoş, *Pressure Vessels. Stress and Strain Analysis* (in Romanian), Publishing House of the Romanian Academy, 1983.
2. PAVEL A., *Pressure vessels stability* (in Romanian), Publishing House of the Romanian Academy, 1985.

3. MÎRŞU Ovidiu, FRIEDRICH Richard, *Special Industrial Constructions Made of Reinforced Concrete* (in Romanian), Edit. Didactică şi Pedagogică, Bucharest, 1975.
4. HOBJILĂ Vasile, LUCA Mihail, MITROI Amedeo, *Cylindrical Tank Made of Prestressed Concrete with Posttensioned Cables* (in Romanian), Publishing House CERMI, Iasi, 1999.
5. ALIPERIN V.I., KOROLIKOV N.V., MOTAVKIN A.V., ROGHINSKII S.L., TELESHOV V.A., *Glass-Plastics Constructions* (in Russian), Publishing House „Himia", Moscow, 1979.
6. VASILIEV Valery, MOROZOV Evgeny, *Mechanics and Analysis of Composite Materials,* Elsevier, 2001.
7. VRABIE Mihai, *Orthotropic Cylindrical Shells Applied on Tanks. Theory and Computation* (in Romanian)*,* Publishing House of Academic Society "Matei-Teiu Botez", Iasi, 2004.
8. QUATU S. Mohamad, *Vibrations of Laminated Shells and Plates,* Elsevier, 2004.
9. NEGOIȚĂ Alexandru (co-ordinator), UNGUREANU Nicolae *et. al., Applications for Earthquake Engineering II* (in Romanian), Chapter 21 – "Tanks", Technical Publishing House, Bucharest, 1990.
10. IEREMIA Mircea, *Elasticity, Plasticity, Nonlinearity* (in Romanian), Publishing House Printech, Bucharest, 1998.
11. UNGUREANU Nicolae, STRAT Lucian, VRABIE Mihai, *Bending effects in liquid storage tanks with stepwise variable wall thickness,* Proc. of the 17-th Congress Committee Yugoslav Society of Mechanics, Belgrad, 1986.
12. VRABIE Mihai, VLAD Ioana, GORBANESCU Dumitru, UNGUREANU Nicolae, *Cylindrical tanks of variable geometry and loading conditions,* Bulletin of the Polytechnic Institute of Jassy, Construction Architecture Section, XXXIX(XLIII), Fasc.*1–4*, 1993.
13. MASSONET Ch., DEPREZ G., MAQUOI R., MULLER R., FONDER G., *Calcules des structures sur ordinateur* (in Romanian), Technical Publishing House, Bucharest, 1974.
14. BAXTER J. Rodney, *Exactly solved models in statistical mechanics,* Academic Press Inc., London, 1982.
15. HUANG Yuying, *Transfer Matrix Method on Dynamic Analysis of Circular Cylindrical Tanks with Variable Wall-Thickness,* CNIKI Journal*,* Huazhong University, 1989.
16. WUNDERLICH Walter, PILKEY D. Walter, *Mechanics of structures: variational and computational methods,* 2-nd edition, CRC Press, 2003.

Received November 12, 2009