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EXISTENCE AND STABILITY OF PERIODIC MOTIONS IN SOME ROLL-COUPLING DYNAMICS OF AN AIRCRAFT

Andrei HALANAY[1], Achim IONIȚĂ[2]

[1] Department of Mathematics I,
University "Politehnica" of Bucharest, 313 Splaiul Independenței
RO-060042 Bucharest, ROMÂNIA
E-mail: halanay@mathem.pub.ro
[2] Institute for Theoretical and Experimental
Analysis of Aeronautical Structures, 220 Bd. Iuliu Maniu
RO-061126 Bucharest, ROMÂNIA
E-mail: achim.ionita@straero.ro

Existence of periodic motions in a roll-coupling model of an aircraft under the action of a periodic force due to gravitational terms is investigated. Some general sufficient conditions for exponential asymptotic stability of such a periodic solution are deduced and applied to a particular case.

Key words: Periodic solution; Liapunov stability; Initial growth rate; Roll-coupling dynamics.

1. INTRODUCTION

The paper addresses the problem of existence and Lyapunov stability of periodic motions during the flight of an aircraft. It can be seen as a contribution to understanding undesirable events that occur during aircraft-pilot coupling in a generic model that includes the interaction between aircraft dynamics, flight control, pilot's characteristics and gravitational forces. For details on the model, we refer to [1, 2, 3, 4]. The state variables are α, the angle of attack, β, the angle of side slip, p, q, r the angular velocity components in body axes. The flight control consists in a feedback component and a component due to pilot's action that will be taken as a constant vector. With g the gravity acceleration, V the velocity of the aircraft mass centre and $x = (\alpha, \beta, p, q, r)$ the system that will be investigated is:

$$\begin{aligned}
\dot{\alpha} &= f_1(x) + \frac{g}{V}\cos\varphi_0 t,\\
\dot{\beta} &= f_2(x) + \frac{g}{V}\sin\varphi_0 t,\\
\dot{p} &= f_3(x),\\
\dot{q} &= f_4(x) + \frac{g}{V} m\cos\varphi_0 t,\\
\dot{r} &= f_5(x).
\end{aligned} \tag{1.1}$$

Here φ_0 and m are nonzero constants and $f_1, \ldots, f_5$, are polynomials. Denote $f = (f_1, \ldots, f_5)$. We analyze the situation when the system described by:

$$\dot{x} = f(x) \tag{1.2}$$

has an asymptotically stable equilibrium point and shows how a stable periodic solution of (1.1) can appear as an effect of the action of gravitational terms given by:

$$G(t)=\left(\frac{g}{V}\cos\varphi_0 t,\quad \frac{g}{V}\sin\varphi_0 t,\quad 0,\quad \frac{g}{V}m\cos\varphi_0 t,\quad 0\right). \tag{1.3}$$

Stability of equilibrium in (1.2) is achieved by the feedback flight control system and is preserved under pilot's action. Pilot's actions influence the value of these equilibrium thus the spectrum of the Jacobian matrix calculated in these equilibrium and this, together with the higher order terms of f around the equilibrium point, will play an important role in the existence and stability of periodic solutions.

The paper is organized as follows. In Section 2, a criterion of existence of periodic solutions for (1.1) will be proved as well as a condition that implies its stability. In Section 3, a case study will be presented. A final section is dedicated to concluding remarks.

2. EXISTENCE AND STABILITY OF PERIODIC SOLUTIONS

System (1.1) will be written as:

$$\dot{x}=f(x)+G(t), \tag{2.1}$$

where $G(t+\omega)=G(t)\quad \forall t,\ \omega=\dfrac{2\pi}{\varphi_0}$. We take x_0 with

$$f(x_0)=0 \tag{2.2}$$

and we define

$$A=f'(x_0). \tag{2.3}$$

In what follows by $\|\ \|$ will be denoted the Euclidean norm on $\mathbf{R}^5$ and for $B\in\mathcal{M}(\mathbf{R})$, the space of quadratic matrices (5×5 in our case) with real entries, $\|B\|$ will denote the operatorial norm of B when the Euclidean norm is considered on $\mathbf{R}^5$. Recall from [6, 7] the following definition.

Definition. For $A\in\mathcal{M}(\mathbf{R})$, the *initial growth rate* $\mu(A)$ (called $\mu_2(A)$ in [6]) is defined by:

$$\mu(A)=\min\left\{\mu\in\mathbf{R}\,\middle|\,\|\mathrm{e}^{At}\|\le \mathrm{e}^{\mu t}\quad \forall t\ge 0\right\},$$

with the norms chosen as mentioned above,

$$\mu(A)=\frac{1}{2}\max\left\{\lambda\in\sigma(A+A^*)\right\} \tag{2.4}$$

(see [6]). Suppose that $\mu(A)<0$. Then A is Hurwitz (the spectrum of A, $\sigma(A)$, is contained in $\boldsymbol{C}\_=\{z\in\mathbf{C}\,|\,\mathrm{Re}\,z<0\}$. One defines

$$c_0=-\mu(A). \tag{2.5}$$

Then $\|e^{At}\|\le \mathrm{e}^{-c_0 t}$ and $\|(I-\mathrm{e}^{At})^{-1}\|\le \dfrac{1}{1-\mathrm{e}^{-c_0 t}}\quad \forall t>0$. Perform next a translation to zero through $y=x-x_0$ and rewrite (2.1) as

$$\dot{y}=Ay+R(y)+G(t), \tag{2.6}$$

where $R(y)$ contains the terms of order greater of equal to two in the Taylor expansion of f around x_0, thus $R(0)=0$. Recall (see, e.g. [5]) that a fundamental matrix of solutions for the linear system $\dot{\eta}=A\eta$ is given by $C(t,s)=\mathrm{e}^{A(t-s)}$. To find conditions for existence of ω-periodic solutions of (2.6) we follow [5], §3.3: the ω-periodic solution must be a fixed point of the operator $\Omega: C([0,\infty),\mathbf{R}^5\to C([0,\infty),\mathbf{R}^5$ defined through

$$\Omega(y)(t)=\mathrm{e}^{A(t+\omega)}(I-\mathrm{e}^{A\omega})^{-1}\int_0^{\omega}\mathrm{e}^{-As}\left\{R[y(s)]+G(s)\right\}\mathrm{d}s+\int_0^{t}\mathrm{e}^{A(t-s)}\left\{R[y(s)]+G(s)\right\}\mathrm{d}s. \tag{2.7}$$

Since A is Hurwitz, $(I-\mathrm{e}^{A\omega})^{-1}$ is well defined and Ω is a compact operator (called also completely continuous).

Theorem 2.1. *Define, for R given in (2.6),*

$$\delta=\sup\{\|R(y)\|\,|\,\|y\|\le 1\}. \tag{2.8}$$

For c_0 defined in (2.5) suppose (see (1.1)) that

$$c_0\ge\delta+\frac{g}{V}\sqrt{1+m^2}:=\delta_1. \tag{2.9}$$

Then (2.6) has an ω-periodic solution.

Proof. We show that Ω defined in (2.7) invariates the unit ball in $C([0,\infty), \mathbf{R}^5)$ so the result in the theorem follows by applying Schauder's fixed point theorem. Since $\|G(t)\|\le\frac{g}{V}\sqrt{1+m^2}$, if $\|y(t)\|\le 1\quad\forall t\ge 0$ we infer from (2.7) and (2.9) that, for every $t\ge 0$,

$$\|\Omega(y)(t)\|\le\|\mathrm{e}^{At}\|\|(I-\mathrm{e}^{A\omega})^{-1}\|\int_0^{\omega}\|\mathrm{e}^{A(\omega-s)}\|\delta_1\mathrm{d}s+\int_0^{\omega}\|\mathrm{e}^{A(t-s)}\|\delta_1\mathrm{d}s\le$$

$$\le\mathrm{e}^{-c_0t}\frac{\delta_1}{1-\mathrm{e}^{-c_0\omega}}\int_0^{\omega}\mathrm{e}^{-c_0(\omega-s)}\mathrm{d}s+\delta_1\mathrm{e}^{-c_0t}\int_0^{t}\mathrm{e}^{c_0s}\mathrm{d}s=$$

$$=\frac{\mathrm{e}^{-c_0t}\delta_1}{1-\mathrm{e}^{-c_0\omega}}e^{-c_0\omega}\frac{\mathrm{e}^{c_0\omega}-1}{c_0}+\delta_1\mathrm{e}^{-c_0t}\frac{\mathrm{e}^{c_0t}-1}{c_0}=\frac{\delta_1}{c_0}\le 1.$$

Schauder's fixed point theorem implies the existence of a fixed point of Ω in the unit ball of $C([0,\infty), \mathbf{R}^5)$ so (2.6) has an ω-periodic solution φ and $\|\varphi\|\le 1$ in $C([0,\infty), \mathbf{R}^5)$.

Corollary 2.2. *Under the assumption in Theorem* 2.1*, equation* (2.1) ((1.1)) *has the periodic solution* $\psi(t)=x_0+\varphi(t)$ *with* φ *the periodic solution of* (2.6) *given by Theorem* 2.1 *and* x_0 *given by* (2.2).

Proof. Since $f(x)=A(x-x_0)+R(x-x_0)$,

$$\dot\psi=\dot\varphi=A\varphi+R(\varphi)+G(t)=A(\psi-x_0)+R(\psi-x_0)+G(t)=f(\psi)+G(t).$$

Suppose again that $\mu(A)<0$ and that (2.9) holds. Let φ be a periodic solution of (2.6) given by Theorem 2.1 so $\|\varphi(t)\|\le 1\ \forall t\ge 0$. To study its stability performs a translation to zero through $\xi=y-\varphi$. Then

$$\dot\xi=A\xi+R(\xi+\varphi)-R(\varphi):=A\xi+F(t,\xi). \tag{2.10}$$

Theorem 2.3 *Suppose that, for F defined in* (2.10), *the following holds*

$$\|F(t,\xi)\|\le K\|\xi\|\forall t\ge 0,\ \forall\xi\in\mathbf{R}^5,\ \|\xi\|\le 1 \tag{2.11}$$

and suppose also that, with c_0 defined in (2.5),

$$K<c_0. \tag{2.12}$$

Then the zero solution of (2.10) *is exponentially asymptotically stable thus the solution* φ *is exponentially asymptotically stable for* (2.6) *and the same is true for the solution* $\psi=x_0+\varphi$ of (2.1).

Proof. The proof consists in an adaptation of the proof of Theorem 1.7 in [5].
The formula of variations of constants applied to (2.10) gives for $\xi(t;t_0,\xi_0)$, the solution of (2.10) that verifies $\xi(t_0)=\xi_0$, that

$$\xi(t;t_0,\xi_0)=\mathrm{e}^{A(t-t_0)}\xi_0+\int_{t_0}^{t}\mathrm{e}^{A(t-s)}F[s,\xi(s;t_0,\xi_0)]\,\mathrm{d}s.$$

Since $\left\|\mathrm{e}^{At}\right\|\le \mathrm{e}^{-c_0 t}\quad \forall t\ge 0$ by (2.5), it follows that, for $t\ge t_0$,

$$\left\|\xi(t;t_0,\xi_0)\right\|\le \mathrm{e}^{-c_0(t-t_0)}\left\|\xi_0\right\|+\int_{t_0}^{t}\mathrm{e}^{-c_0(t-s)}\left\|F[s,\xi(s;t_0,\xi_0)]\right\|\mathrm{d}s.$$

If $\left\|\xi(t;t_0,\xi_0)\right\|\le 1$ for $t_0\le s\le t$, (2.11) implies that $\left\|F[s,\xi(s;t_0,\xi_0)]\right\|\le K\,\|\,\xi(s;t_0,\xi_0)\,\|$ and then

$$\left\|\xi(t;t_0,\xi_0)\right\|\le \mathrm{e}^{-c_0(t-t_0)}\left\|\xi_0\right\|+\mathrm{e}^{-c_0 t}K\int_{t_0}^{t}\mathrm{e}^{c_0 s}\left\|\xi(s;t_0,\xi_0)\right\|\mathrm{d}s,$$

so

$$\mathrm{e}^{c_0 t}\left\|\xi(t;t_0,\xi_0)\right\|\le \mathrm{e}^{c_0 t_0}\left\|\xi_0\right\|+K\int_{t_0}^{t}\mathrm{e}^{c_0 s}\left\|\xi(s;t_0,\xi_0)\right\|\mathrm{d}s. \tag{2.13}$$

With $u(t)=\mathrm{e}^{c_0 t}\left\|\xi(s;t_0,\xi_0)\right\|$, (2.13) is rewritten as $u(t)\le u(t_0)+K\int_{t_0}^{t}u(s)\,\mathrm{d}s$ and the well-known Gronwall inequality (see, e.g. [5], Lemma I.6) gives $u(t)\le u(t_0)\mathrm{e}^{K(t-t_0)}$ hence $\mathrm{e}^{c_0 t}\left\|\xi(t;t_0,\xi_0)\right\|\le \mathrm{e}^{c_0 t_0}\mathrm{e}^{K(t-t_0)}$ and finally $\left\|\xi(t;t_0,\xi_0)\right\|\le \mathrm{e}^{(K-c_0)(t-t_0)}\left\|\xi_0\right\|\le 1,$ if $\left\|\xi_0\right\|\le 1,$ due to (2.12). It follows that if $\left\|\xi_0\right\|\le 1$ then

$$\left\|\xi(t;0,\xi_0)\right\|\le \mathrm{e}^{(K-c_0)t}\left\|\xi_0\right\|\quad \forall t\ge 0$$

and since $K-c_0<0$ the theorem is proved.

1. CASE STUDY

Consider in (1.1) $\alpha=\beta=0$ and

$$\begin{aligned}
\dot p &= a_{11}p+a_{13}r-i_1 rq+b_{11}\delta_1+b_{13}\delta_3,\\
\dot q &= a_{22}q+i_3 pr+b_{22}\delta_2+\frac{g}{V}m\cos\varphi_0 t,\\
\dot r &= a_{31}p+a_{33}r-i_2 pq+b_{31}\delta_1+b_{33}\delta_3,
\end{aligned}\tag{3.1}$$

where, in the framework of the model in [1], $a_{11}=-15.9709$, $a_{13}=18.1171$, $i_1=0.952$, $b_{11}=17.0387$, $b_{13}=2.12983,$ $a_{22}=-14.4124,$ $i_3=0.594$, $b_{22}=-6.38593,$ $a_{31}=-6.1768,$ $a_{33}=-15.4735,$ $i_2=0.247,$ $b_{31}=1.23357,$ $b_{33}=-2.00459,$ $g=9.81$, $V=84.5$, $m=-5.26416$. With $\delta_1=-\frac{\pi}{10}$, $\delta_2=0$, $\delta_3=\frac{\pi}{15}$ one finds

$$x_0=(p_0,q_0,r_0)=(-0.252214,\ -0.00050415,\ 0.0485).$$

The initials growth rate of A calculated by (2.4) is $\mu(A) = -14.4106$ so $c_0 = 14.4106$. Since

$$R(y) = -(-i_1 y_2 y_3,\ i_3 y_1 y3,\ -i_2 y_1 y_2)$$

it follows that $\delta = \frac{1}{2}\sqrt{i_1^2 + i_2^2 + i_3^2} = 0.574489$ and that $\delta_1 = \delta + \frac{g}{V}\sqrt{1+m^2} = 1.19656 < c_0$.

Then (3.1) has a periodic solution φ. For it, the function F in (2.10) is

$$F(t,\xi) = (-i_1(\xi_2\xi_3 + \varphi_2\xi_3 + \varphi_3\xi_2), i_2(\xi_1\xi_3 + \varphi_1\xi_3 + \varphi_3\xi_1), -i_3(\xi_1\xi_2 + \varphi_1\xi_2 + \varphi_2\xi_1)) .$$

Since

$$\|\varphi(t)\| \le 1,\ \forall t \ge 0,\ \|F(t,\xi)\| \le \frac{3}{2}\sqrt{i_1^2 + i_2^2 + i_3^2}\,\|\xi\|,\ \|\xi\| \le 1$$

so $K = 3\delta < c_0$ and by Theorem 2.3 the solution φ is exponentially asymptotically stable.

2. CONCLUDING REMARKS

The emergence of periodic motions during an aircraft flight might be extremely dangerous. A situation when this becomes possible was presented in this paper. The criterion involves aerodynamic data and pilot action thus can be seen as a contribution to the research on Pilot Induced Oscillations. The conclusion is that, in certain particular situations, a combination of automatic flight control and pilot action can enable the gravitational forces to produce undesired stable periodic motions.

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