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ON ITERATED INTEGRATED TAIL

Gheorghiță ZBĂGANU

Bucharest University, Academiei 14, Bucharest, E-mail:zbagang@fmi.unibuc.ro and
"Gheorghe Mihoc – Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy
Casa Academiei Române, Calea 13 Septembrie no. 13, 050711 Bucharest, Romania

Let F be a distribution on $[0,\infty)$, with $\underline{F}(x):=F((x,\infty))$ its right tail. Suppose that F has a finite first moment $\mu = \int_0^\infty \underline{F}(x)\mathrm{d}x$. The ***Lorenz curve*** of F is the graph of $L(F):[0,1] \to [0,1]$ defined by $L(F)(y) = \frac{1}{\mu}\int_0^y F^{-1}(t)\mathrm{d}t$, where $F^{-1}(y) = \sup\{x : F(x) \le y\}$ is the pseudoinverse of F. As $L(F)(0)=0$, $L(F)(1)=1$ and L_F is increasing, it is the distribution function of some other measure F_1. Tzvetan Ignatov proved [5] that if we construct the sequence defined by the recurrence $F_{n+1}=L(F_n)$, this sequence has always a limit which does not depend on F. From a geometric point of view $L(F)(y)$ is the ratio $A(y)/A(0)$ where $A(y)$ is the area of the set $\{(x,z) : y \le z \le \underline{F}(x)\}$. If we replace this ratio by $B(x)/B(0)$, $B(x)$ being the area of the set $\{(t,z)\colon 0 \le z \le \underline{F}(t),\ t \ge x\}$ we obtain the tail of another distribution which is denoted by F_I and it is called ***the integrated tail*** of F [1, 3, 8, 13, 14]. Ignatov conjectured that if we construct the sequence defined by the recurrence $F_{n+1}=(F_n)_I$, this sequence has always a limit which is an exponential distribution. We prove that this is true in some cases if we agree to add Dirac's measure δ_0 and the null measure δ_∞ to the family of exponential distributions under the name $\mathrm{Exp}(\infty)$ and $\mathrm{Exp}(0)$.

Key words: Weak convergence; Integrated tail; Hazard rate.

1. DEFINITIONS AND STATEMENT OF THE PROBLEM

Let (Ω,K,P) be a probability space and $\boldsymbol{L} = \bigcap L_+^p(\Omega,\boldsymbol{K},P)$. So, $X \in \boldsymbol{L}$ iff $X \ge 0$ (a.s.) and $\mathrm{E}X^p < \infty$ for every $1 \le p < \infty$. Let $\boldsymbol{M}$ be the set of the distribution of the random variables $X \in \boldsymbol{L}$. Thus $F \in \boldsymbol{M}$ iff $F([0,\infty))$ $= 1$ and $\int x^p \mathrm{d}F(x) < \infty\ \forall\ 1 \le p < \infty$. The integral $\int x^p \mathrm{d}F(x) := \mathrm{E}X^p$ will be denoted by $\mu_p(F)$. For $p = 1$ it will be simply denoted $\mu(F)$. $\mu_p(F)$ is called the pth moment of X. We shall denote by $F(x)$ the distribution function of F and by $\underline{F}(x)$ its right tail. Precisely, $F(x)$ will stand for $F([0,x])$ and $\underline{F}(x)$ for $F((x,\infty))$.

Let $g:[0,\infty) \to \Re$ be a continuous function. Suppose that it is differentiable at almost all its points, with the possible exception of a discrete set. We shall often use the formula (integration by parts).

$$F \in \boldsymbol{M} \Rightarrow \int g\mathrm{d}F = g(0) + \int_0^\infty g'(x)\underline{F}(x)\mathrm{d}x. \tag{1.1}$$

For instance, if $g(x) = (x-a)_+$ we get $X \ge 0 \Rightarrow E(X-a)_+ = \int_a^\infty \underline{F}(x)\mathrm{d}x$ $\mathrm{E}(X\text{-}a)_+ = \int_a^\infty \underline{F}(x)\mathrm{d}x$.

In renewal and ruin theories the following distribution is of interest: it is called *the integrated tail*. Its tail is defined by

$$\underline{F}_I(x)=\int_x^{\infty}\underline{F}(y)\mathrm{d}y\Big/\int_0^{\infty}\underline{F}(y)\mathrm{d}y. \tag{1.2}$$

We shall to study the mapping T: $\boldsymbol{M}\rightarrow\boldsymbol{M}$ defined by $T(F)=F_I$ and the sequence defined by

$$F_0=F,\ F_{n+1}=(F_n)_I. \tag{1.3}$$

For the history of the operator "integrated tail" the reader can consult [1] or [3].

2. STRAIGHTFORWARD PROPERTIES

Let $\boldsymbol{M}_{\text{ac}}$ be the family of absolutely continuous (with respect to Lebesgue measure) distribution from $\boldsymbol{M}$. If $F\in\boldsymbol{M}_{\text{ac}}$ we denote by f its density.

The first remark is that no matter $F\in\boldsymbol{M}$, $T(F)$ is absolutely continuous. Its density is

$$f_I(x)=\frac{\underline{F}(x)}{\mu_1(F)}. \tag{2.1}$$

Here are some simple properties of the operator T.

Proposition 2.1.

(i) $F_I(x)=\dfrac{\mathrm{E}(X-x)_+}{\mathrm{E}X}=\dfrac{\mathrm{E}(X-x)_+}{\mu_1}$, *where* $X\sim F$.

(ii) $T(F)\in\boldsymbol{M}_{\text{ac}}$ *for every* $F\in\boldsymbol{M}$ *and* Range $(T)=\{G\in\boldsymbol{M}_{\text{ac}}\mid g$ *is non-increasing*$\}$; *the range consists of all absolutely continuous distribution with non-increasing densities.*

(iii) *Let* $\varphi(t)=\int \mathrm{e}^{\mathrm{i}tx}\mathrm{d}F(x)$ *be the characteristic function of* F *and* $\varphi_I(t)=\int \mathrm{e}^{\mathrm{i}tx}\mathrm{d}F_I(x)$ *the characteristic function of* F_I. *Suppose that* $\mu_1>0$. *Then* $\varphi_I(t)=\dfrac{\varphi(t)-1}{it\mu_1}=\dfrac{\varphi(t)-1}{t\varphi'(0)}$.

(iv) *Let* $\boldsymbol{m}(t)=\int \mathrm{e}^{tx}\mathrm{d}F(x)$ *be the m.g.f. of* F *and* $\boldsymbol{m}_I(t)=\int \mathrm{e}^{tx}\mathrm{d}F_I(x)$ *the m.g.f. of* F_I. *Suppose that* $\mu_1>0$. *Then* $\boldsymbol{m}_I(t)=\dfrac{\boldsymbol{m}(t)-1}{t\mu_1}=\dfrac{\boldsymbol{m}(t)-1}{t\boldsymbol{m}'(0)}$.

(v) Unicity up to mixtures with δ_0: $F,G\in\boldsymbol{M}\Rightarrow F_I=G_I$ *iff* $G=(1-p)F+p\delta_0$ *for some* $p\in[0,1]$.

(vi) *Let* $\boldsymbol{M}_0=\{F\in\boldsymbol{M}\mid F((0,\infty))=1\}=\{F\in\boldsymbol{M}\mid F(0)=0\}$. *The mapping* $\boldsymbol{T}:\boldsymbol{M}_0\rightarrow\boldsymbol{M}_{\text{ac}}$ *is one to one.*

Proof. **(i)**. Apply (1.1) for $g(t)=(t\text{-}x)_+$. **(iii)** and **(iv)** are consequences of (1.1) for $g_t(x)=e^{itx}$ and for $g(x)=\mathrm{e}^{tx}$: for instance $\varphi_I(t)=\int_0^{\infty}\mathrm{e}^{\mathrm{i}tx}\mathrm{d}F_I(x)=\int_0^{\infty}\mathrm{e}^{\mathrm{i}tx}\dfrac{\underline{F}(x)}{\mu_1}\mathrm{d}x=\dfrac{1}{\mu_1}\int_0^{\infty}\mathrm{e}^{\mathrm{i}tx}\underline{F}(x)\mathrm{d}x$.

By (1.1), $\varphi(t)=1+it\int_0^{\infty}\mathrm{e}^{\mathrm{i}tx}\underline{F}(x)\mathrm{d}x$, hence $\int_0^{\infty}\mathrm{e}^{\mathrm{i}tx}\underline{F}(x)\mathrm{d}x=\dfrac{\varphi(t)-1}{it}$ and (iii) follows. Equality (iv) has the same proof.

(ii) For the second assertion, let G be an absolutely continuous distribution on $[0,\infty)$ such that its density g is non-increasing and right-continuous. Then $\underline{F}(x)=\dfrac{g(x)}{g(0)}$ is the tail of some distribution F and $G=F_I$. To prove the first assertion, we have to check that F_I has finite moments or, which is the same thing, that φ_I is indefinitely differentiable at 0; or, which is the same thing, that $\lim\limits_{t\to 0}t^{-n}\left(\varphi_I(t)-1\right)$ does exist and is finite. But that is obvious, by Hospital's rule $\lim\limits_{t\to 0}\dfrac{\varphi_I(t)-1}{t^n}=\lim\limits_{t\to 0}\dfrac{\varphi(t)-1-t\mu_1}{it^{n+1}\mu_1}=\dfrac{\varphi^{(n+1)}(0)}{i(n+1)!\mu_1}$, and the last quantity does exist since φ is indefinitely differentiable.

As a byproduct we have

Corollary 2. 2. *The moments of* $\boldsymbol{F}_{\mathbf{I}}$ *are given by*

$$\int x^k \mathrm{d}F_I(x) := \mu_k\left(F_I\right) = \frac{\mu_{k+1}(F)}{(k+1)\mu_1(F)} \quad \forall\, k \geq 0. \tag{2.2}$$

Proof. $\mu_k(F_I) = \dfrac{(\varphi_I)^{(k)}(0)}{i^k} = \lim\limits_{t\to 0}\dfrac{\varphi_I(t)-1}{i^k t^k / k!} = \lim\limits_{t\to 0}\dfrac{k!(\varphi(t)-1-t\mu_1)}{i^{k+1}t^{k+1}\mu_1} = \dfrac{\mu_{k+1}(F)}{(k+1)\mu_1(F)}$.

Now, we study ***continuity properties*** of T. To begin with, notice that T is **not** continuous in the weak topology since for any distribution F the sequence $F_n = \left(1-n^{-1}\right)F + n^{-1}\delta_0$ obviously converges to F but the tails $(\underline{F_n})_I(x) = \dfrac{\left(1-\frac{1}{n}\right)\int_x^\infty \underline{F}(t)\mathrm{d}t + \frac{(n-x)_+}{n}}{\mu_1(F)+1}$ converge to $\dfrac{\int_x^\infty \underline{F}(t)\mathrm{d}t + 1}{\mu_1(F)+1}$ as $n \to \infty$. The limit function does not vanish at infinity, hence it is not the tail of a distribution from $\boldsymbol{M}$.

However, T is ***monotonously continuous***.

Definition. Let F and G be two distribution on the real line. We say that F is stochastically dominated by G, and write $F \prec_{\mathrm{st}} G$ iff $\underline{F} \leq \underline{G}$. (For a survey on stochastic orderings the reader may see [8, 12, 13] and the references therein). We write $F_n \uparrow F$ (respectively $F_n \downarrow F$) iff $F_n \Rightarrow F$ and $n \leq n+1 \Rightarrow F_n \prec_{\mathrm{st}} F_{n+1}$ (respectively $F_{n+1} \prec_{\mathrm{st}} F_n$).

Proposition 2.3. *If* $F_n, F \in \boldsymbol{M}$ *and* $F_n \uparrow F$ (*respectively* $F_n \downarrow F$), *then* $T(F_n) \Rightarrow T(F)$.

Proof. Apply Beppo-Levi's theorem: if $\underline{F_n} \uparrow \underline{F}$, then $\int_x^\infty \underline{F}_n(y)\mathrm{d}y \uparrow \int_x^\infty \underline{F}(y)\mathrm{d}y$ for any x. For $x = 0$ we see that $\mu_1(F_n) \to \mu_1(F)$. Thus, $(F_n)_I$ converges weakly to F_I.

If we are interested in the possible limits of the sequence $(F_n)_n$ defined by (1.3), we should study the ***fixed points*** of T.

Proposition 2.4. *Let* $F \in \boldsymbol{M}$. *Then* $T(F) = F \Leftrightarrow F = \delta_0$ *or if* $F = \mathrm{Exp}(\lambda)$ *for some* $\lambda > 0$.

(By $\mathrm{Exp}(\lambda)$ we denote the distribution F with tail $\underline{F}(x) = \mathrm{e}^{-\lambda x}$).

Proof. Let $F \in \boldsymbol{M}$ be such that $T(F) = F$ and let φ be its characteristic function. Let μ be its expectation. If $\mu = 0$, then $F = \delta_0$ and of course, $T(F) = F$. Suppose that $\mu > 0$. According to Proposition 2.1(iii), φ should satisfy the equation $\varphi(t) = \dfrac{\varphi(t)-1}{it\mu_1} \Leftrightarrow \varphi(t) = \dfrac{1}{1-it\mu_1}$. But this is the characteristic function of $\mathrm{Exp}(1/\mu)$. The uniqueness theorem says that $F = \mathrm{Exp}(1/\mu)$.

After that, we should study the ***monotonicity*** of T. Say that T is ***increasing*** if $F \prec_{\mathrm{st}} G \Rightarrow F \prec_{\mathrm{st}} G$ and ***decreasing*** if $F \prec_{\mathrm{st}} G \Rightarrow G \prec_{\mathrm{st}} F$. The fact is that T is **not** monotonous. Indeed, if $\underline{F}(x) = (1 - x/2)_+$ and $\underline{G}(x) = 1_{[0,1)}(x) + (1 - x/2)_+$ then $\underline{F} \leq \underline{G}$, but there is no domination between F_I and G_I : indeed, $0 < x < 4/5 \Rightarrow \underline{F_I}(x) < \underline{G_I}(x)$ and $x \geq 4/5 \Rightarrow \underline{F_I}(x) \geq \underline{G_I}(x)$.

However, T has a weaker monotonicity: it is **HR - *increasing.***

Definitions. Suppose that $F \in \boldsymbol{M}$ is absolutely continuous. Then its tail can be written as

$$\underline{F}(x) = \mathrm{e}^{-\int_0^x \lambda(y)\mathrm{d}y} \tag{2.3}$$

with $\lambda(x) = -\underline{F}'(x)/\underline{F}(x)$.

The mapping $\lambda = \lambda_F : [0,\infty) \to [0,\infty]$ defined by (2.3) is called ***the hazard rate*** of F. We make the convention that if $\underline{F}(x) = 0$ then $\lambda_F(x) = \infty$ (see [2] or [12]).

Let F and $G \in \boldsymbol{M}_{\mathbf{ac}}$. If $\lambda_F \geq \lambda_G$ we say that F is **HR-*dominated*** by G and write $F \prec_{\mathrm{HR}} G$. It is obvious that $F \prec_{\mathrm{HR}} G \Rightarrow F \prec_{\mathrm{st}} G$. (Indeed, $\underline{F}(x) = \mathrm{e}^{-\int_0^x \lambda_F(y)\mathrm{d}y} \leq \mathrm{e}^{-\int_0^x \lambda_G(y)\mathrm{d}y} = \underline{G}(x)$!).

If T is an operator from $\boldsymbol{M}_{\mathbf{ac}}$ to $\boldsymbol{M}_{\mathbf{ac}}$ with the property that $F \prec_{\mathrm{HR}} G \Rightarrow T(F) \prec_{\mathrm{HR}} T(G)$, we say that T is **HR-*increasing***.

Proposition 2.5. (i) *Let $F \in \boldsymbol{M}$ be have the hazard rate λ. Then the hazard rate of F_I is*

$$\lambda_I(x) = \underline{F}(x) \Big/ \int_x^\infty \underline{F}(y)\mathrm{d}y \,. \tag{2.4}$$

(ii). *The mapping $T(F) = F_I$ is* HR-*increasing.*

Proof. **(i)** The density of F_I is $f_I(x) = \underline{F}(x)/\mu(F)$ and its tail is $\underline{F}_I(x) = \int_x^\infty \underline{F}(y)\mathrm{d}y \Big/ \mu(F)$. Thus, its hazard rate is their ratio, hence $\lambda_I(x) = \underline{F}(x) \Big/ \int_x^\infty \underline{F}(y)\mathrm{d}y$.

(ii) Notice that if $F, G \in \boldsymbol{M}$ are absolutely continuous, then

$$F \prec_{\mathrm{HR}} G \Leftrightarrow x \mapsto \frac{\underline{F}(x)}{\underline{G}(x)} \text{ is non-increasing.} \tag{2.5}$$

Indeed, let λ_F and λ_G be the hazard rates of F and G. $F \prec_{\mathrm{HR}} G \Leftrightarrow \lambda_F \geq \lambda_G \Rightarrow \dfrac{\underline{F}(x)}{\underline{G}(x)} = \mathrm{e}^{\int_0^x (\lambda_G - \lambda_F)(y)\mathrm{d}y}$ is obviously non-increasing. Conversely, if the mapping $x \mapsto \dfrac{\underline{F}(x)}{\underline{G}(x)}$ is non-increasing, then the mapping $h(x) = \int_0^x (\lambda_F - \lambda_G)(y)\mathrm{d}y$ is non-decreasing, hence its derivative should be non-negative: $\lambda_F - \lambda_G \geq 0$. So, (2.5) is true. To prove that T is HR-monotonous, Let $F \prec_{\mathrm{HR}} G$. According to (2.5) we can write $\underline{F} = \Lambda \underline{G}$ with Λ non-increasing. Then $\lambda_{F_I}(x) = \underline{F}(x) \Big/ \int_x^\infty \underline{F}(y)\mathrm{d}y = \Lambda(x)\underline{G}(x) \Big/ \int_x^\infty \Lambda(y)\underline{G}(y)\mathrm{d}y$.

We claim that $\lambda_{F_I}(x) \geq \lambda_{G_I}(x)$. Indeed, the inequality $\Lambda(x)\underline{G}(x) \Big/ \int_x^\infty \Lambda(y)\underline{G}(y)\mathrm{d}y \geq \underline{G}(x) \Big/ \int_x^\infty \underline{G}(y)\mathrm{d}y$ is equivalent to $\int_x^\infty \Lambda(x)\underline{G}(y)\mathrm{d}y \geq \int_x^\infty \Lambda(y)\underline{G}(y)\mathrm{d}y$ and the last inequality is obvious.

From the point of view of the hazard rates, there are two interesting classes of distributions from $\boldsymbol{M}_{\mathbf{ac}}$: the IFRs and the DFRs.

Definition. Let $F \in \boldsymbol{M}_{\mathbf{ac}}$. We write $F \in$ IFR (= Increasing Failure Rate) iff λ_{F} is non-decreasing. If λ_F is non-increasing, then we write $F \in$ DFR (= Decreasing Failure rate) [2, 6, 7, 9, 10 or 13].
It happens that the operator $T = (\cdot)_I$ preserves these two classes.

Proposition 2.6. $F \in$ IFR $\Rightarrow F_I \in$ IFR *while* $F \in$ DFR $\Rightarrow F_I \in$ DFR.
Proof. Suppose that $F \in$ IFR. Let λ be its hazard rate. By our assumption, λ is non-decreasing. We want to show that the mapping $\lambda_I(x) = \underline{F}(x) / \int_x^\infty \underline{F}(y)\mathrm{d}y$ is increasing, too.

Write $\frac{1}{\lambda_I(x)} = \int_x^\infty \frac{\underline{F}(y)}{\underline{F}(x)}\mathrm{d}y = \int_0^\infty \frac{\underline{F}(x+t)}{\underline{F}(x)}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\int_0^t \lambda(x+u)\mathrm{d}u}\mathrm{d}t$. Let $a < b$ be positive. Then $\lambda(a+u) \le \lambda(b+u)\ \forall\ u$ implies the inequality $\int_0^t \lambda(a+u)\mathrm{d}u \le \int_0^t \lambda(b+u)\mathrm{d}u \quad \forall t > 0 \Rightarrow \frac{1}{\lambda_I(a)} \ge \frac{1}{\lambda_I(b)}$. Thus, λ_I is non-decreasing hence $F_I \in$ IFR. The proof for the DFRs is similar.

3. ITERATED INTEGRATED TAIL

Our problem is: when the sequence of measures defined by the recurrence

$$F_0 = F,\ F_{n+1} = T(F_n) \tag{3.1}$$

does have a weak limit? The main help will come from the HR-monotonicity of $T = (\cdot)_I$.

Proposition 3.1. *If* $F \in$ IFR *then* $T(F) \prec_{\mathrm{HR}} F$ *while if* $F \in$ DFR *then* $F \prec_{\mathrm{HR}} T(F)$.

Proof **(i).** Let $\lambda:[0,\infty) \to [0,\infty)$ be the hazard rate of F and λ_I be the hazard rate of $T(F) := F_I$. By the definition, we have to prove that if λ is non-decreasing then $\lambda(x)\int_x^\infty \underline{F}(y)\mathrm{d}y \le \underline{F}(x)$ and if λ **is** non-increasing, then $\lambda(x)\int_x^\infty \underline{F}(y)\mathrm{d}y \ge \underline{F}(x)$. As $\underline{F}(x) = \mathrm{e}^{-\int_0^x \lambda(t)\mathrm{d}t}$, the claimed inequalities become $\lambda(x)\int_x^\infty \mathrm{e}^{-\int_x^y \lambda(t)\mathrm{d}t}\mathrm{d}y \le 1$ (if λ is non-decreasing) and $\lambda(x)\int_x^\infty \mathrm{e}^{-\int_x^y \lambda(t)\mathrm{d}t}\mathrm{d}y \ge 1$ (if λ is non-increasing).

In the first case $\int_x^y \lambda(t)\mathrm{d}t \ge \lambda(x)(y-x)$ and in the second one $\int_x^y \lambda(t)\mathrm{d}t \ge \lambda(x)(y-x)$. Thus, in the first case $\lambda(x)\mathrm{e}^{-\int_x^y \lambda(t)\mathrm{d}t} \le \lambda(x)\mathrm{e}^{-\lambda(x)(y-x)} \Rightarrow \lambda(x)\int_x^\infty \mathrm{e}^{-\int_x^y \lambda(t)\mathrm{d}t}\mathrm{d}y \le \int_x^\infty \lambda(x)\mathrm{e}^{-\lambda(x)(y-x)}\mathrm{d}y = \int_0^\infty \lambda(x)\mathrm{e}^{-\lambda(x)t}\mathrm{d}t = 1$ while in the second one the converse inequality holds.

Corollary 3.2. *If* $F \in$ IFR *then the sequence* $F_n = T^n(F)$ *is HR-decreasing while if* $F \in$ DFR *the sequence is* HR-*increasing*.

Proof. Obvious from Proposition 3.1.

Corollary 3.3. *If* $F \in$ IFR, *then* $T^n(F)$ *has a limit, G. If* $F \in$ DFR *then the sequence of non-increasing right continuous functions* $(\underline{T^n(F)})_n$ *has a limit* $\underline{G}$ *too. If* $\underline{G}(\infty) = 0$, *then* $T^n(F)$ *weakly converges to G.*

Proof. Obvious. If $F_n = T^n(F)$ then the sequence of the tails $(\underline{F_n})_n$ is monotonic – either increasing, or decreasing.

Proposition 3.4.

I. (The IFR case). *Let* $F \in$ IFR *have the hazard rate* λ.

(i) *If* $\lambda(\infty) < \infty$ *then* $\lim_{n\to\infty} T^n(F) = \mathrm{Exp}(\lambda(\infty))$.

(ii) *If* $\lambda(\infty) = \infty$ *then* $\lim_{n\to\infty} T^n(F) = \delta_0$.

II. (The DFR case). *Let* $F \in$ DFR *have the hazard rate* λ.

(i) *If* $\lambda(\infty) > 0$ *then* $\lim_{n\to\infty} T^n(F) = \mathrm{Exp}(\lambda(\infty))$.

(ii) *If* $\lambda(\infty) = 0$ *then the limit does not exist anymore.*

Remark. In case **II**(ii) we could say that the limit is δ_∞, but that makes little sense for distribution.

Proof.

I. Let λ_n be the hazard rate of $F_n = T^n(F)$. Then

$$\lambda_{n+1}(x) = \underline{F_n}(x) / \int_x^\infty \underline{F_n}(y)\mathrm{d}y\,. \tag{3.2}$$

(i). The sequence $(\lambda_n)_n$ is non-decreasing. Therefore, it has a limit, λ^*. Let F^* be the distribution with tail $\underline{F^*}(x) = \mathrm{e}^{-\int_0^x \lambda^*(t)\mathrm{d}t}$. As $\lambda_n \to \lambda^*$ monotonously, $\underline{F_n}(x)$ converges to the tail $\underline{F^*}(x)$. We claim that $F^* = \mathrm{Exp}(\lambda(\infty))$. The first step is to check that λ^* **must be a constant**. Anyway, the fact that $\lambda(0) \le \lambda(x) \le \lambda(\infty)$ means that $\mathrm{Exp}(\lambda(\infty)) \prec_{\mathrm{HR}} F \prec_{\mathrm{HR}} \mathrm{Exp}(\lambda(0))$. By Proposition 2.5, T is HR-monotonic hence $\mathrm{T}(\mathrm{Exp}(\lambda(\infty))) \prec_{\mathrm{HR}} T(F) \prec_{\mathrm{HR}} \mathrm{T}(\mathrm{Exp}(\lambda(0))$. By Proposition 2.4, the exponential distributions are fixed points for T. Therefore, $\mathrm{Exp}(\lambda(\infty)) \prec_{\mathrm{HR}} F_1 \prec_{\mathrm{HR}} \mathrm{Exp}(\lambda(0)$. By induction, we see that

$$\mathrm{Exp}(\lambda(\infty)) \prec_{\mathrm{HR}} F_n \prec_{\mathrm{HR}} \mathrm{Exp}(\lambda(0). \tag{3.3}$$

Letting $n \to \infty$, we get $\mathrm{Exp}\,(\lambda(\infty)) \prec_{\mathrm{HR}} F^* \prec_{\mathrm{HR}} \mathrm{Exp}\,(\lambda(0)$. Or, in terms of hazard rates,

$$\lambda(0) \le \lambda^*(x) \le \lambda(\infty)\ \forall\, x \ge 0. \tag{3.4}$$

Proposition 2.3 says that T is monotonically continuous: if $F_n \Rightarrow F^*$ monotonically , then $T(F_n) \Rightarrow T(F^*)$. On the other hand, $T(F_n) = F_{n+1}$ converges to F^* *hence* $F^* = T(F^*)$. According to Proposition 3(iii), the only fixed points of T are the exponential distributions. Thus $\lambda^* = \mathrm{const}$. As $\lambda^* \ge \lambda$ (recall that the sequence (λ_n) is increasing !) , $\lambda^* \ge \lambda(x)$ for any $x \ge 0$. Letting $x \to \infty$, $\lambda^* \ge \lambda(\infty)$. On the other hand, inequality (3.4) points out that $\lambda^* \le \lambda(\infty)$.

(ii). We reason as before: $\lambda^* \ge \lambda(x)\ \forall\, x \ge 0 \Rightarrow \lambda^* \ge \lambda(\infty)$ i.e. $\lambda^* = \infty$. The limit is δ_0.

Proof for the **DFR case.** Now, the sequence $(\lambda_n)_n$ is ***decreasing***. As in the proof of **I,** the limit λ^* must be a constant such that $\lambda(\infty) \le \lambda^* \le \lambda(0)$ and $\lambda^* \le \lambda(x)\ \forall\, x$. If this constant is equal to 0, there is no limit among distributions from $\boldsymbol{M}$ since all the mass vanishes. The limit, G, provided that it does exist, should dominate all the distributions $\mathrm{Exp}(\lambda)$, meaning that $\underline{G}(x) \ge e^{-\lambda x}$ for any $\lambda \Rightarrow \underline{G}(x) = 1$ for any x , or $G = \delta_\infty$.

Now we prove our main result.

Theorem 3.5. *Let* $F \in \boldsymbol{M}_{\mathbf{ac}}$ *be a distribution such that the limit* $\lambda := \lambda_F(\infty)$ *does exist. Then*
– *if* $\lambda \in (0,\infty)$ *then* $T^n(F) \Rightarrow \mathrm{Exp}(\lambda)$;
– *if* $\lambda = \infty$ *then* $T^n(F) \Rightarrow \delta_0$;
– *if* $\lambda = 0$ *then* $T^n(F)$ *diverges. Precisely,* $\underline{T^n(F)}\,(x) \to 1$ *as* $n \to \infty$.

Proof. Let us define $\lambda_*(x) = \inf_{y\ge 0}\lambda_F(x+y)$ and $\lambda^*(x) = \sup_{y\ge 0}\lambda_F(x+y)$. Consider the first case, $\lambda \in (0,\infty)$.Then λ_* is non-decreasing and λ^* is non-decreasing. Moreover, $\lambda_*(\infty) = \lambda^*(\infty) = \lambda$ and

$$\lambda_*(x) \le \lambda_F(x) \le \lambda^*(x)\ \forall\, x \ge 0. \tag{3.5}$$

Let F^*, F_* be distributions from $\boldsymbol{M}_{\mathbf{ac}}$ such that $\lambda_{F^*} = \lambda^*$ and $\lambda_{F_*} = \lambda_*$. Then

$$F_* \in \mathrm{IFR},\ F^* \in \mathrm{DFR} \text{ and } F^* \prec_{\mathrm{HR}} F \prec_{\mathrm{HR}} F_*. \tag{3.6}$$

As T is HR-increasing, we infer from (3.6) that

$$T^n(F^*) \prec_{\mathrm{HR}} T^n(F) \prec_{\mathrm{HR}} T^n(F_*). \tag{3.7}$$

According to Proposition (3.4), $T^n(F^*) \Rightarrow \mathrm{Exp}(\lambda^*(\infty)) = \mathrm{Exp}(\lambda)$. In the same way, $T^n(F_*) \Rightarrow \mathrm{Exp}(\lambda_*(\infty)) = \mathrm{Exp}(\lambda)$. Thus, both sequences $(T^n(F^*))_n$ and $(T^n(F_*))$ have the same limit. But clearly

$$G_n \prec_{\mathrm{st}} F_n \prec_{\mathrm{st}} H_n\,,\ G_n \Rightarrow F,\ H_n \Rightarrow F \tag{3.8}$$

implies that $(F_n)_n$ is convergent and $F_n \Rightarrow F$. Indeed, (3.8) means $\underline{G_n} \le \underline{F_n} \le \underline{H_n}$, $\underline{G_n} \to \underline{F}$, $\underline{H_n} \to \underline{F}$ as $n \to \infty$.

Therefore, $(T^n(F))_n$ must converge to the same limit, namely Exp(λ).

Consider now the case $\lambda = \infty$. Then $T^n(F_*) \Rightarrow \delta_0 \Leftrightarrow$ the tails of $T^n(F_*)$ converge to 0. The fact that $T^n(F) \prec_{st} T^n(F_*)$ implies that the tails of $T^n(F)$ converge to 0, too, hence $T^n(F) \Rightarrow \delta_0$.

The last case is $\lambda = 0$. Then the distribution $T^n(F)$ dominates the DFR distributions $T^n(F^*)$. Their mass vanishes to infinity, so the same must happen with the mass of $T^n(F)$.

Do we have a clue to find the limit when we are not able to compute the hazard rate λ_F? The answer is YES, we have. If we are able to prove somehow that $\lambda_F(\infty)$ does exist. We should look at the mgf of F.

Proposition 3.6. *Let* $F \in \boldsymbol{M}_{\mathbf{ac}}$ *and* $\boldsymbol{m}(t) = \int \mathrm{e}^{tx}\mathrm{d}F(x) = 1 + t\int_0^\infty \mathrm{e}^{tx}\underline{F}(x)\mathrm{d}x$ *be its mgf.*

Let $t^* = \sup\{t \in \Re: \boldsymbol{m}(t) < \infty\}$. *If* $\lambda_F(\infty)$ *does exist, then* $\lambda_F(\infty) = t^*$.

Proof. If $t^* > 0$, then $\boldsymbol{m}(t) < \infty \Leftrightarrow \int_0^\infty \mathrm{e}^{tx}\underline{F}(x)\mathrm{d}x < \infty$. Let $\lambda = \lambda_F$. Write $\underline{F}(x) = \mathrm{e}^{-\int_0^x \lambda(y)\mathrm{d}y}$. Then

$$\int_0^\infty \mathrm{e}^{tx}\underline{F}(x)\mathrm{d}x = \int_0^\infty \mathrm{e}^{tx - \int_0^x \lambda(y)\mathrm{d}y}\mathrm{d}x = \int_0^\infty \mathrm{e}^{\int_0^x (t-\lambda(y))\mathrm{d}y}\mathrm{d}x .$$

Suppose that $t < \lambda(\infty)$. There exists some $\varepsilon > 0$ and some $a > 0$ such that $y > a \Rightarrow \lambda(y) > t - \varepsilon \Leftrightarrow t - \lambda(y) \le -\varepsilon$. For $x > a$ we have

$$\int_0^x (t-\lambda(y))\mathrm{d}y = \int_0^a (t-\lambda(y))\mathrm{d}y + \int_a^x (t-\lambda(y))\mathrm{d}y = C(a) + \int_a^x (t-\lambda(y)) \le C(a) - \varepsilon(x-a) = K - \varepsilon x .$$

It follows that $\underline{F}(x) \le A\mathrm{e}^{-\varepsilon x}$ for some $A > 0$ hence $\int_0^\infty \mathrm{e}^{tx}\underline{F}(x)\mathrm{d}x < \infty$.

We thus proved that $t < \lambda(\infty) \Rightarrow t \le t^*$ hence $\lambda(\infty) \le t^*$.

On the other hand, if $t > \lambda(\infty)$, we can find some $a > 0$ and $\varepsilon > 0$ such that $y > a \Rightarrow t - \lambda(y) \ge \varepsilon$ and the same reasoning as before yields $\mathrm{e}^{tx}\underline{F}(x) \ge B\mathrm{e}^{\varepsilon x}$ for some constant B. Obviously, this means that

$$\int_0^\infty e^{tx}\underline{F}(x)\mathrm{d}x = \infty \Leftrightarrow t \ge t^*.$$

Thus, $t^* = \lambda(\infty)$.

In the same way one can proves the exception cases. If $\lambda(\infty) = \infty$ then $\int_0^\infty \mathrm{e}^{tx}\underline{F}(x)\mathrm{d}x < \infty$ for every t, hence $t^* = \infty$ and if $\lambda(\infty) = 0$ then $t^* = 0$.

If we agree to denote the measure δ_∞ by Exp(0) (the sense is that the tail of this measure is equal to 1), then we can restate Theorem 3.5. as

Corollary 3.7. *Let Let* $F \in \boldsymbol{M}_{\mathbf{ac}}$ *and* $\boldsymbol{m}(t)$ *its mgf. Let* t^* *defined as in Proposition* 3.6. *Suppose that the limit* $\lambda(\infty)$ *does exist. Then* $T^n(F)$ *converges to* Exp(t^*).

Remark. We could call a distribution F ***short tailed*** if $t^* = \infty$, ***medium tailed*** if $t^* \in (0,\infty)$ and ***long tailed*** if $t^* = 0$. This agrees with the various definitions for long tailed distributions from [1, 4, 7].

Example 3.8. The Poisson distribution F = Poisson(λ) has the mgf $\boldsymbol{m}(t) = \mathrm{e}^{\lambda(\mathrm{e}^t - 1)}$. As $t^* = \infty$, $T^n(F)$ should converge to δ_0. F is not absolutely continuous, hence we cannot speak about its hazard rate. But F_1 has the density $f = \sum_{k=0}^\infty q_k 1_{[k,k+1)}$ with $q_k = \underline{F}(k)/\lambda = \sum_{j=k+1}^\infty \frac{\lambda^{j-1}}{j!}\mathrm{e}^{-\lambda}$ and the tail $\underline{F_1}(x) = \int_x^\infty f(y)\mathrm{d}y$.

The ratio $\lambda(x) = f(x)/\underline{F_1}(x)$ is increasing on $[k, k+1)$. We claim that $\lambda(\infty) = \infty$. Indeed, it is enough to prove that $\lambda(k) \to \infty$ as $k \to \infty$. We have

$$\lambda(k) = q_k / (q_{k+1} + q_{k+2} + \ldots) = \left(\frac{\lambda^k}{(k+1)!} + \frac{\lambda^{k+1}}{(k+2)!} + \ldots \right) / \left(\frac{\lambda^{k+1}}{(k+2)!} + 2\frac{\lambda^{k+2}}{(k+3)!} + 3\frac{\lambda^{k+3}}{(k+4)!} + \ldots \right) =$$

$$= \frac{k+2}{\lambda}\left(1 + \frac{\lambda}{k+2} + \frac{\lambda^2}{(k+2)(k+3)} + \ldots \right) / \left(1 + \frac{2\lambda}{(k+3)} + \frac{3\lambda^2}{(k+3)(k+4)} + \ldots \right)$$

which converges to ∞ as $k \to \infty$. Thus if F = Poisson(λ), then $T^n(F) \to \delta_0$.

Example 3.9. The lognormal distribution belongs to the class DFR and $t^* = 0$. This means that the limit does not exist. The distribution Gamma(ν,λ) are IFR distributions (see, for instance [5, 6, 10]) hence the limit is Exp(λ). An interesting example is the inverse Gaussian distribution IG(μ,λ) which naturally arises from first passage problems for Brownian motion (see for instance[11]). This time the hazard rate λ is not monotonous: these distributions are neither IFR nor DFR. Its density is $f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{-\frac{\lambda(x-\mu)^2}{2x\mu^2}}$ and its tail is

$\underline{F}(x) = 1 - \left(\Phi\left(-\frac{\sqrt{\lambda}}{\sqrt{x}} + \frac{\sqrt{\lambda}}{\mu}\sqrt{x} \right) + e^{2\frac{\lambda}{\mu}} \Phi\left(-\frac{\sqrt{\lambda}}{\sqrt{x}} - \frac{\sqrt{\lambda}}{\mu}\sqrt{x} \right) \right)$. Then $\lambda(\infty) = \lambda(\infty) = \frac{\lambda}{2\mu^2}$ (use twice L'Hospital rule).

Open problem. We still do not know at this stage if it is possible that $(T^n(F))_n$ have no limit at all in other cases than the one stated in Proposition 3.4 II(ii). In the stated case it is true that the sequence of distributions $(T^n(F))_n$ has no limit because all the mass vanishes; however the sequence of tails $(\underline{T^n(F)})_n$ converges to 1. Is it possible that this sequence of tails have no limit?

REFERENCES

1. S. ASSMUSSEN, S., *Applied probabilities and queues,* Wiley, New York, 1987.
2. BARLOW, R.E., PROSCHAN, F., *Mathematical theory of reliability*, John Wiley & Sons, Inc., New York, 1965.
3. CRAMER, H., *Collective Risk Theory*, Skandia, Jubilee Volume, Stockholm, 1955.
4. EMBRECHTS, P., KLUPPELBERG, C., MIKOSCH, T., *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin 1997.
5. IGNATOV, T., *The use of the Lorenz curves and one of its dynamic limit to measure the inequality of incomes,* Paper presented at the 2007 conference of the Society for Probability and Statistics of Romania, Abstract at http://www.csm.ro/spsr/rezumat2007.pdf; Full text: *Utilisation des courbes de Lorenz et une de leur limite dynamique pour mesurer l'inegalite de revenus,* "RILA"-project 2/9 2005–2006, pp.149–166, 2006 (in Bulgarian).
6. KARLIN, S., *Total Positivity*, Vol I, Standford University Press, Stanford C A., 1968.
7. KARLIN, S., TAYLOR, H. M., *A Second Course in Stochastic Processes*, Academic Press, New York 1981.
8. KLUGMAN, S., PANJER, H. WILLMONT, G., *Loss Models. From Data to Decisions*, Wiley, New York, 2008.
9. PATEL, J. K., *Hazard rate and other classifications of distributions*, in *Encyclopedia of Statistical Sciences* 3, Wiley, New York, pp. 590–594. (1983).
10. PROCHAN, F., *Theoretical explanation of observed decreasing failure rate*. *Technometrics*, **5**, pp. 375–383. (1963).
11. SESHADRI, V., *The Inverse Gaussian Distribution,* Oxford Univ. Press, 1993
12. SHAKED, M. and SHANTIKUMAR, J. G., *Stochastic Orders and Their Applications*, Academic Press, New York, 1994.
13. ZBĂGANU, G., *Mathematical Methods in Risk Theory and Actuaries* (in Romanian), Bucharest Univ. Press, 2004.
14. ZBĂGANU, G ., *Elements of Ruin Theory* (in Romanian), Balkanpress, Bucharest, 2007.

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