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EXACT SOLUTIONS FOR THE FLOW OF A GENERALIZED OLDROYD-B FLUID INDUCED BY A SUDDENLY MOVED PLATE BETWEEN TWO SIDE WALLS PERPENDICULAR TO THE PLATE

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The unsteady flow of an incompressible generalized Oldroyd-B fluid induced by a suddenly moved plate between two side walls perpendicular to the plate is studied by means of Fourier sine and Laplace transforms. The velocity field v(y,z,t), written in terms of the generalized $G_{a,b,c}(\cdot,\cdot)$ functions, is presented as a sum between the Newtonian solution and the corresponding non-Newtonian contribution. The solutions corresponding to the generalized Maxwell fluids, as well as the solutions for ordinary Maxwell and Oldroyd-B fluids, performing the same motion, are obtained as limiting cases of the general solutions. In the absence of side walls, namely when the distance between walls tends to infinity, the solutions corresponding to the motion over an infinite suddenly moved plate are recovered. Finally, the effect of fractional parameters on the velocity field, as well as the influence of the side walls on the fluid motion, is spotlighted by graphical illustrations.

Key words: Generalized Oldroyd-B fluid; Side walls; Suddenly moved plate; Exact solutions.

1. INTRODUCTION

The laminar flow of a great number of fluids such as polymeric liquids, food products, paints, foams, slurries, biological fluids and so forth cannot be adequately described by means of the classical linearly viscous Newtonian model. The departure from the Newtonian behavior manifests itself in a variety of ways: non-Newtonian viscosity (shear thinning or shear thickening), stress-relaxation, non-linear creeping, development of normal stress differences and yield stress. Numerous models have been proposed to describe the response characteristics of these fluids, they being classified as fluids of differential type, rate type and integral type. Amongst the many rate type models that have been developed, the Oldroyd-B model is amenable to analysis and more importantly experimental corroboration. It has had some success in describing the response of some polymeric liquids being viewed as one of the most successful models for describing the response of a sub-class of such fluids.

Recently, the fractional calculus has encountered much success in the description of viscoelasticity, it proving to be a valuable tool to handle viscoelastic properties [1–5]. Especially, the rheological constitutive equations with fractional derivatives play an important role in description of the behavior of the polymer solutions and melts. In particular, it has been shown that the predictions of a fractional derivative Maxwell model are in excellent agreement with the linear viscoelastic data in the glass transition and glass state [2, 6]. The list of such applications is quite long, it including fractal media, fractional wave diffusion, fractional Hamiltonian dynamics and many other topics in physics. That motivated further work on the one-dimensional fractional derivative models. The starting point of the fractional derivative models of non-Newtonian fluids is usually a classical differential equation which is modified by replacing the time derivative of an integer order by so-called Riemann-Liouville fractional differential operator. This generalization allows us to define precisely non-integer order integrals or derivatives [7]. During the last years, a lot of papers regarding these fluids have been published. Here we shall refer only to those regarding generalized Oldroyd-B fluids (GOF) [8-16] and the references therein.

The aim of this note is to establish exact solutions for the velocity field corresponding to the unsteady flow of an incompressible GOF between two side walls perpendicular to a plate and to underline the influence of the fractional parameters and of the side walls on the motion. The motion of the fluid is produced by the plate, which at time t equal zero is impulsively set in motion with a constant velocity V. The solutions that have been obtained, presented under integral and series form in terms of generalized $G_{a,b,c}(\cdot,\cdot)$ functions, are established by means of Fourier sine and Laplace transforms. They are presented as a sum between the Newtonian solutions and the corresponding non-Newtonian contributions. The similar solutions for generalized Maxwell fluids as well as those for ordinary Maxwell and Oldroyd-B fluids can be obtained as limiting cases of general solutions. Furthermore, in the absence of the side walls, all previous solutions reduce to those corresponding to the motion over an infinite plate, the well known classical solution for the first problem of Stokes being also recovered as a limiting case.

2. STATEMENT OF THE PROBLEM

Let us consider an incompressible Oldroyd-B fluid at rest occupying the space above an infinite plate perpendicular to the y-axis and between two side walls situated in the planes $z = 0$ and $z = d$ [17]. At time $t = 0^+$ the infinite plate begins to slide in its plane with the constant velocity V. Due to the shear the fluid above the plate is gradually moved, its velocity being of the form

$$\mathbf{V} = \mathbf{V}(y, z, t) = \mathrm{v}(y, z, t)\,\mathbf{i}, \tag{1}$$

where $\mathbf{i}$ is the unit vector along the x-direction. For this flow, the constraint of incompressibility is automatically satisfied. Assuming that the extra-stress $\mathbf{S}$, as well as the velocity $\mathbf{V}$, is a function of y, z and t only, it is easy to show that in the absence of body forces and a pressure gradient in the flow direction, the governing equation for this flow is [17]

$$(1 + \lambda\partial_t)\frac{\partial \mathrm{v}(y, z, t)}{\partial t} = \nu(1 + \lambda_r\partial_t)\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathrm{v}(y, z, t);\ y, t > 0,\ z \in (0, d), \tag{2}$$

where ν is the kinematic viscosity of the fluid and λ and λ_r are the relaxation and retardation times. If the fractional calculus approach is used in the constitutive relationship of the fluid model, the governing equation (2) takes the form (cf. with [12] and [16])

$$(1 + \lambda D_t^{\alpha})\frac{\partial \mathrm{v}(y, z, t)}{\partial t} = \nu(1 + \lambda_r D_t^{\beta})\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathrm{v}(y, z, t);\ y, t > 0,\ z \in (0, d), \tag{3}$$

where the fractional differentiation operator D_t^p can be defined as [7]

$$D_t^p f(t) = \frac{1}{\Gamma(1-p)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{f(\tau)}{(t-\tau)^p}\,\mathrm{d}\tau;\ \ 0 \le p < 1, \tag{4}$$

$\Gamma(\cdot)$ being the Gamma function. It should be noted that the governing equation (3) is correct from dimensional point of view only if the new material constants λ and λ_r have the dimensions of t^{α} and t^{β}, respectively. In some recent papers the authors have used λ^{α} and λ_r^{β} instead of λ and λ_r into the constitutive equations of a GOF. However, for simplicity, we are keeping the same notations although these material constants have different significations into Eqs. (2) and (3). The appropriate initial and boundary conditions corresponding to this problem are [17]

$$\mathrm{v}(y, z, 0) = \frac{\partial \mathrm{v}(y, z, 0)}{\partial z} = 0;\ y > 0,\ 0 \le z \le d, \tag{5}$$

$$\mathrm{v}(0,z,t)=V;\ t>0,\ 0<z<d \ \text{ and } \ \mathrm{v}(y,0,t)=\mathrm{v}(y,d,t)=0;\ y,t>0. \tag{6}$$

Moreover, the natural conditions at infinity

$$\mathrm{v}(y,z,t),\ \frac{\partial \mathrm{v}(y,z,t)}{\partial y}\to 0 \ \text{ as } \ y\to\infty \ \text{ and } \ t>0, \tag{7}$$

have to be also satisfied. In the following, the fractional differential equation (3), with the initial and boundary conditions (5)-(7), will be solved by means of Fourier sine and Laplace transforms.

3. THE SOLUTION OF THE PROBLEM

Multiplying Eq. (3) by $\sqrt{2/\pi}\,\sin(y\xi)\sin(\lambda_n z)$, where $\lambda_n=n\pi/d$, integrating the result with respect to y and z from 0 to infinity, respectively, 0 to d and taking into account the initial and boundary conditions (5)-(7), we find that (see also Eq. (A1) from Appendix)

$$(1+\lambda D_t^{\alpha})\frac{\partial \mathrm{v}_{sn}(\xi,t)}{\partial t}+\nu(\xi^2+\lambda_n^2)(1+\lambda_r D_t^{\beta})\mathrm{v}_{sn}(\xi,t)=\nu V\xi\sqrt{\frac{2}{\pi}}\frac{1-(-1)^n}{\lambda_n}\left[1+\beta_r\frac{t^{-\beta}}{\Gamma(1-\beta)}\right];\xi,t>0, \tag{8}$$

where $\beta_r=\lambda_r\operatorname{sign}(1-\beta)$ and the double Fourier sine transforms $\mathrm{v}_{sn}(\xi,t)$ of $\mathrm{v}(y,z,t)$ [18] have to satisfy the initial conditions

$$\mathrm{v}_{sn}(\xi,0)=\frac{\partial \mathrm{v}_{sn}(\xi,0)}{\partial t}=0 \ \text{ for } \ \xi>0 \ \text{ and } \ n=1,2,3,4... \tag{9}$$

Applying the Laplace transform to Eq. (8) and using the Laplace transform formula for sequential fractional derivatives [7], we find that the image function $\bar{\mathrm{v}}_{sn}(\xi,q)$ of $\mathrm{v}_{sn}(\xi,t)$ is given by (see (A2))

$$\bar{\mathrm{v}}_{sn}(\xi,q)=\nu V\xi\sqrt{\frac{2}{\pi}}\frac{(1-(-1)^n)}{\lambda_n}\frac{1+\beta_r q^{\beta}}{q[\lambda q^{\alpha+1}+q+\nu(\xi^2+\lambda_n^2)+\alpha_r(\xi^2+\lambda_n^2)q^{\beta}]};\alpha,\beta\in(0,1), \tag{10}$$

where $\alpha_r=\nu\lambda_r$. In order to obtain $\mathrm{v}_{sn}(\xi,t)$ and to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method [8-16]. However, for a suitable presentation of the final results, we firstly rewrite the last factor from Eq. (10) in the equivalent form

$$\begin{aligned}\frac{1+\beta_r q^{\beta}}{q[\lambda q^{\alpha+1}+q+\nu(\xi^2+\lambda_n^2)+\alpha_r(\xi^2+\lambda_n^2)q^{\beta}]}&=\frac{1}{\nu(\xi^2+\lambda_n^2)}\left[\frac{1}{q}-\frac{1}{q+\nu(\xi^2+\lambda_n^2)}\right]+\\&+\left(\frac{1}{q+\nu(\xi^2+\lambda_n^2)}\right)\left(\frac{\beta_r q^{\beta}-\lambda q^{\alpha}}{\lambda q^{\alpha+1}+q+\nu(\xi^2+\lambda_n^2)+\alpha_r(\xi^2+\lambda_n^2)q^{\beta}}\right).\end{aligned} \tag{11}$$

In view of the equalities (A3) from appendix, the second factor of the last term from (11) can be written as a double series, under the form

$$\begin{aligned}\frac{\beta_r q^{\beta}-\lambda q^{\alpha}}{\lambda q^{\alpha+1}+q+\nu(\xi^2+\lambda_n^2)+\alpha_r(\xi^2+\lambda_n^2)q^{\beta}}&=\sum_{k=0}^{\infty}\sum_{m,l\ge 0}^{m+l=k}\left(\frac{-1}{\lambda}\right)^k\frac{k![\alpha_r(\xi^2+\lambda_n^2)]^m}{m!l!}\times\\&\times\left[\frac{\beta_r}{\lambda}\frac{q^{\beta(m+1)+l}}{[q^{\alpha+1}+(\nu/\lambda)(\xi^2+\lambda_n^2)]^{k+1}}-\frac{q^{\beta m+\alpha+l}}{[q^{\alpha+1}+(\nu/\lambda)(\xi^2+\lambda_n^2)]^{k+1}}\right].\end{aligned} \tag{12}$$

Introducing (11) and (12) into (10), inverting the result by means of the Fourier sine formulae [18], applying then the discrete inverse Laplace transform to the obtained result and using Eqs. (A4), (A6) and the property (A7) from Appendix, we find the velocity $\mathrm{v}(y,z,t)$ under the form

$$
\begin{aligned}
\mathrm{v}(y,z,t) &= \mathrm{v}_N(y,z,t) + \frac{8\nu V}{\pi d}\sum_{n=1}^{\infty}\frac{\sin(\lambda_N z)}{\lambda_N}\sum_{k=0}^{\infty}\sum_{m,l\geq 0}^{m+l=k}\left(\frac{-1}{\lambda}\right)^k \frac{k!\,\alpha_r^m}{m!\,l!}\int_0^{\infty}\xi(\xi^2+\lambda_N^2)^m \sin(y\xi) \\
&\times \int_0^t \mathrm{e}^{-\nu(\xi^2+\lambda_N^2)(t-s)}\left\{\frac{\beta_r}{\lambda}G_{\alpha+1,a_m,b}\left[\frac{-\nu}{\lambda}(\xi^2+\lambda_N^2),s\right] - G_{\alpha+1,c_m,b}\left[\frac{-\nu}{\lambda}(\xi^2+\lambda_N^2),s\right]\right\}\mathrm{d}s\mathrm{d}\xi,
\end{aligned} \tag{13}
$$

where $N = 2n-1$, $a_m = \beta(m+1)+l$, $c_m = \beta m + \alpha + l$, $b = k+1$, and (see [19, pp. 14 and 15])

$$
G_{\alpha,a,b}(d,t) = \sum_{j=0}^{\infty}\frac{(b)_j\, t^{\alpha(j+b)-a-1}}{\Gamma(j+1)\Gamma(\alpha(j+b)-a)}(d)^j. \tag{14}
$$

Into above relations $(b)_j = b(b+1)(b+2)...(b+j-1)$ is the Pochhammer polynomial and

$$
\begin{aligned}
\mathrm{v}_N(y,z,t) &= \frac{4V}{d}\sum_{n=1}^{\infty}\frac{\sin(\lambda_N z)}{\lambda_N}\mathrm{e}^{-\lambda_N y} - \frac{2V}{d}\sum_{n=1}^{\infty}\frac{\sin(\lambda_N z)}{\lambda_N}\times \\
&\times\left[\mathrm{e}^{-\lambda_N y}\operatorname{erfc}\left(\lambda_N\sqrt{\nu t}-\frac{y}{2\sqrt{\nu t}}\right) - \mathrm{e}^{\lambda_N y}\operatorname{erfc}\left(\lambda_N\sqrt{\nu t}+\frac{y}{2\sqrt{\nu t}}\right)\right],
\end{aligned} \tag{15}
$$

represents the velocity field corresponding to a Newtonian fluid performing the same motion. In the following, for later use, we take $d = 2h$ and change the origin of the coordinate system at the middle of the channel. Consequently, putting $z = z^* + h$ and dropping out the star notation, Eqs. (13) and (15) take the more suitable forms (see for instance [17, Eq. (32)] for the Newtonian velocity)

$$
\begin{aligned}
\mathrm{v}(y,z,t) &= \mathrm{v}_N(y,z,t) + \frac{4\nu V}{\pi h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\mu_N z)}{\mu_N}\sum_{k=0}^{\infty}\sum_{m,l\geq 0}^{m+l=k}\left(\frac{-1}{\lambda}\right)^k \frac{k!\alpha_r^m}{m!l!}\int_0^{\infty}\xi(\xi^2+\mu_N^2)^m \sin(y\xi)\times \\
&\times\int_0^t \mathrm{e}^{-\nu(\xi^2+\mu_N^2)(t-s)}\left\{\frac{\beta_r}{\lambda}G_{\alpha+1,a_m,b}\left[\frac{-\nu}{\lambda}(\xi^2+\mu_N^2),s\right] - G_{\alpha+1,c_m,b}\left[\frac{-\nu}{\lambda}(\xi^2+\mu_N^2),s\right]\right\}\mathrm{d}s\mathrm{d}\xi,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\mathrm{v}_N(y,z,t) &= \frac{2V}{h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\mu_N z)}{\mu_N}\mathrm{e}^{-\mu_N y} - \frac{V}{h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\mu_N z)}{\mu_N}\times \\
&\times\left[\mathrm{e}^{-\mu_N y}\operatorname{erfc}\left(\mu_N\sqrt{\nu t}-\frac{y}{2\sqrt{\nu t}}\right) - \mathrm{e}^{\mu_N y}\operatorname{erfc}\left(\mu_N\sqrt{\nu t}+\frac{y}{2\sqrt{\nu t}}\right)\right],
\end{aligned} \tag{17}
$$

where $\mu_N = (2n-1)\pi/(2h)$ and $\operatorname{erfc}(\cdot)$ is the complementary error function of Gauss.

4. SPECIAL CASES

1. Taking $\lambda_r = 0$ into Eq. (16), we find the velocity field

$$
\begin{aligned}
\mathrm{v}(y,z,t) &= \mathrm{v}_N(y,z,t) - \frac{4\nu V}{\pi h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\mu_N z)}{\mu_N}\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^k\int_0^{\infty}\xi\sin(y\xi)\times \\
&\times\int_0^t \mathrm{e}^{-\nu(\xi^2+\mu_N^2)(t-s)}G_{\alpha+1,\alpha+k,k+1}\left[\frac{-\nu}{\lambda}(\xi^2+\mu_N^2),s\right]\mathrm{d}s\mathrm{d}\xi,
\end{aligned} \tag{18}
$$

corresponding to a generalized Maxwell fluid performing the same motion.

2. By now letting $\alpha \to 1$ into (18), the similar solution

$$\mathrm{v}(y,z,t)=\mathrm{v}_N(y,z,t)-\frac{4\nu V}{\pi h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\mu_N z)}{\mu_N}\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^k\int_0^{\infty}\xi\sin(y\xi)\times$$
$$\times\int_0^t \mathrm{e}^{-\nu(\xi^2+\mu_N^2)(t-s)}G_{2,k+1,k+1}\left[\frac{-\nu}{\lambda}(\xi^2+\mu_N^2),s\right]\mathrm{d}s\mathrm{d}\xi, \tag{19}$$

corresponding to the classical Maxwell model is obtained.

3. Taking the limit of the general solution (16) for α and $\beta\to 1$, the velocity field

$$\mathrm{v}(y,z,t)=\mathrm{v}_N(y,z,t)-\frac{4\nu V}{\pi h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\mu_N z)}{\mu_N}\sum_{k=0}^{\infty}\int_0^{\infty}\xi\left[-\frac{1+\alpha_r(\xi^2+\mu_N^2)}{\lambda}\right]^k\sin(y\xi)\times$$
$$\times\int_0^t \mathrm{e}^{-\nu(\xi^2+\mu_N^2)(t-s)}G_{2,k+1,k+1}\left[\frac{-\nu}{\lambda}(\xi^2+\mu_N^2),s\right]\mathrm{d}s\mathrm{d}\xi, \tag{20}$$

corresponding to an ordinary Oldroyd-B fluid is recovered.

4. Finally, in view of Eq. (A7), it is worthy pointing out that the Newtonian solution $\mathrm{v}_N(y,z,t)$ can be obtained from anyone of Eqs. (18), (19) or (20) for $\lambda\to 0$. Furthermore, it can be also obtained as a limiting case of Eqs. (13) and (16) for λ_r and $\lambda\to 0$.

5. LIMITING CASE $h\to\infty$ (FLOW OVER AN INFINITE PLATE)

In the absence of the side walls, namely when $h\to\infty$ into above equalities, the velocity field corresponding to the motion over an infinite plate (Stokes' first problem) is recovered [12]. The solutions corresponding to (16) and (20), for instance, become

$$\mathrm{v}(y,t)=\mathrm{v}_N(y,t)+\frac{2\nu V}{\pi}\sum_{k=0}^{\infty}\sum_{m,l\geq 0}^{m+l=k}\left(\frac{-1}{\lambda}\right)^k\frac{k!\alpha_r^m}{m!l!}\int_0^{\infty}\xi^{2m+1}\sin(y\xi)\int_0^t \mathrm{e}^{-\nu\xi^2(t-s)}\times$$
$$\times\left\{\frac{\beta_r}{\lambda}G_{\alpha+1,a_m,b}\left(\frac{-\nu\xi^2}{\lambda},s\right)-G_{\alpha+1,c_m,b}\left(\frac{-\nu\xi^2}{\lambda},s\right)\right\}\mathrm{d}s\mathrm{d}\xi, \tag{21}$$

$$\mathrm{v}(y,t)=\mathrm{v}_N(y,t)-\frac{2\nu V}{\pi}\sum_{k=0}^{\infty}\int_0^{\infty}\xi\left(-\frac{1+\alpha_r\xi^2}{\lambda}\right)^k\sin(y\xi)\int_0^t \mathrm{e}^{-\nu\xi^2(t-s)}G_{2,k+1,k+1}\left(\frac{-\nu\xi^2}{\lambda},s\right)\mathrm{d}s\mathrm{d}\xi, \tag{22}$$

where $\mathrm{v}_N(y,t)=V\mathrm{erfc}(y/2\sqrt{\nu t})$ is the classical solution corresponding to the first problem of Stokes. As a check of our calculi, we showed that the diagrams of $\mathrm{v}(y,t)$ given by Eq. (22) are identical to those corresponding to the similar solution obtained in [20] by a different technique.

Passing to the limit as $\lambda_r\to 0$ into (21), we obtain the similar solution for a Maxwell fluid with fractional derivatives, namely

$$\mathrm{v}(y,t)=\mathrm{v}_N(y,t)-\frac{2\nu V}{\pi}\sum_{k=0}^{\infty}\left(-\frac{1}{\lambda}\right)^k\int_0^{\infty}\xi\sin(y\xi)\int_0^t \mathrm{e}^{-\nu\xi^2(t-s)}G_{\alpha+1,\alpha+k,k+1}\left(\frac{-\nu\xi^2}{\lambda},s\right)\mathrm{d}s\mathrm{d}\xi. \tag{23}$$

Furthermore, by now letting $\alpha\to 1$ into (23), the solution for an ordinary Maxwell fluid

$$\mathrm{v}(y,t)=\mathrm{v}_N(y,t)-\frac{2\nu V}{\pi}\sum_{k=0}^{\infty}\left(-\frac{1}{\lambda}\right)^k\int_0^{\infty}\xi\sin(y\xi)\int_0^t \mathrm{e}^{-\nu\xi^2(t-s)}G_{2,k+1,k+1}\left(\frac{-\nu\xi^2}{\lambda},s\right)\mathrm{d}s\mathrm{d}\xi, \tag{24}$$

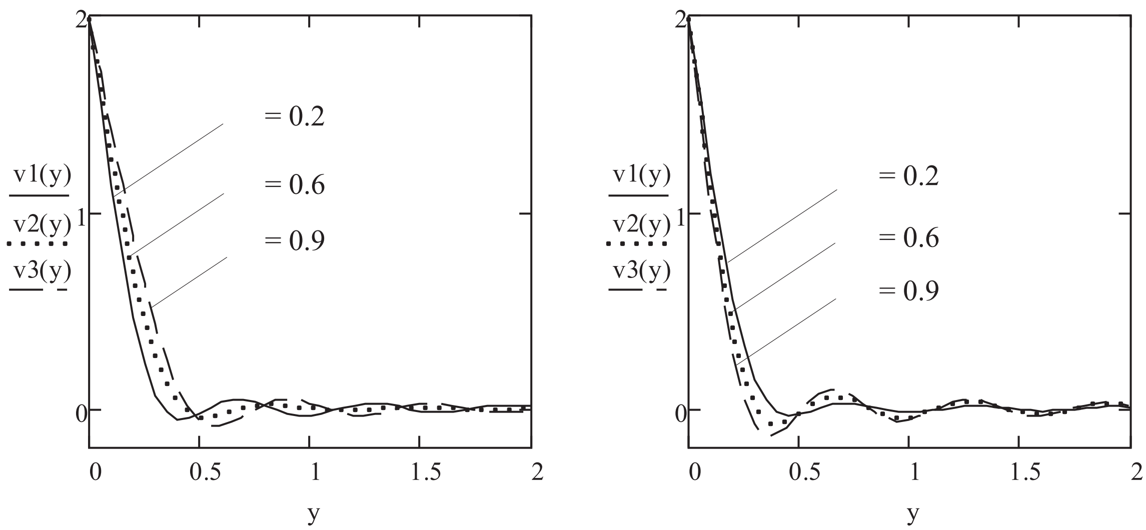
is obtained. As form, this last solution is completely different of the solutions obtained in [20] and [21]. However, by means of graphical illustrations, we showed that our solution (24) is equivalent to the similar solution obtained in [20] by a different technique.

6. CONCLUSIONS AND NUMERICAL RESULTS

The main purpose of this note is to provide the velocity field for the unsteady flow of an incompressible generalized Oldroyd-B fluid induced by a suddenly moved plate between two side walls perpendicular to the plate and to spotlight the influence of the fractional parameters on the fluid motion. The exact solutions, obtained using Fourier sine and Laplace transforms, are presented under integral and series form in terms of the generalized $G_{a,b,c}(\cdot,\cdot)$ functions. The corresponding solutions for fractional Maxwell fluids as well as those for ordinary Maxwell and Oldroyd -B fluids are also obtained as limiting cases of our general solutions. Furthermore, unlike the previous solutions from the literature, the present solutions are presented as a sum of the Newtonian solution and the corresponding non-Newtonian contributions. In the absence of side walls, namely when $h \to \infty$, all solutions that have been obtained reduce to the solutions corresponding to the motion over an infinite suddenly moved plate.

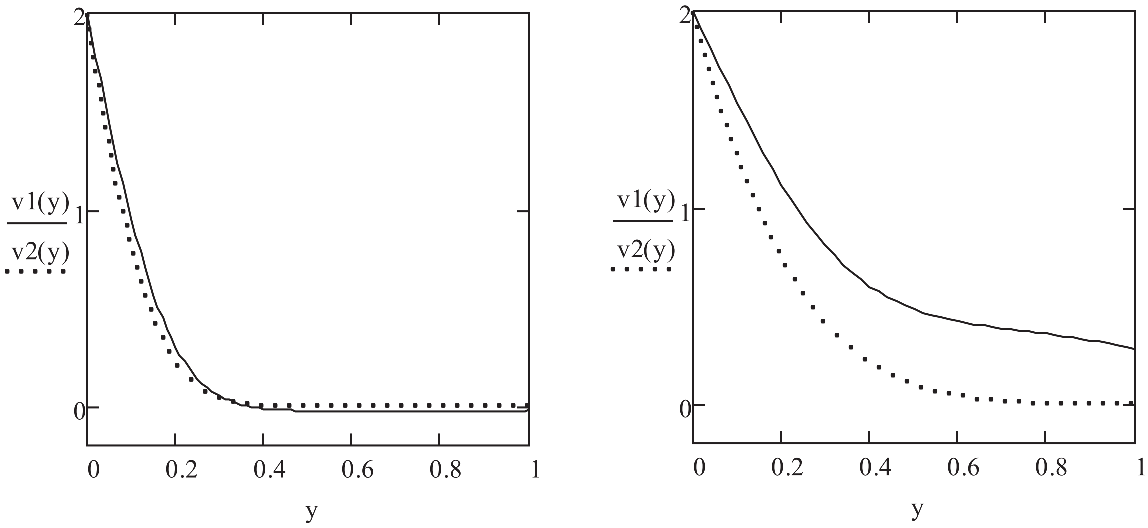
In order to reveal some relevant physical aspects regarding the obtained results, the diagrams of $\mathrm{v}(y,0,t)$, giving the velocity profiles at the middle of the channel, have been drawn against y for different values of the fractional parameters α and β. The effects of the two parameters, as it results from Figs. 1a and 1b, are opposite. In the region near the plate, for instance, the velocity of the fluid is an increasing function with respect to α and a decreasing one with respect to β. The influence of the side walls on the fluid motion is underlined in Figs. 2a and 2b. The velocity of the fluid is smaller in the presence of the side walls and the difference $\mathrm{v}(y,t) - \mathrm{v}(y,0,t)$, as it was to be expected, increases in time.

Finally, it is worthy pointing out that the general solutions (13), (16) and (21), corresponding to an incompressible generalized Oldroyd-B fluid, tend to the Newtonian solutions $\mathrm{v}_N(y,z,t)$ and $\mathrm{v}_N(y,t)$ if $\lambda_r \to \lambda$ and $\alpha \to \beta$. This result can be looked as an extension of Joseph's remark [22, Section 2.2] concerning Oldroyd-B fluids.



a) $\beta = 0.3$ b) $\alpha = 0.3$

Fig. 1 – Profiles of the velocity field $\mathrm{v}(y,0,t)$ given by Eq. (16) – curves $\mathrm{v1}(y)$, $\mathrm{v2}(y)$, $\mathrm{v3}(y)$ for $V = 2$, $h = 0.6$, $\lambda = 5$, $\lambda_r = 4$, $t = 5\mathrm{s}$ and different values of α and β.



a) $t = 5s$ b) $t = 20s$

Fig. 2 – Profiles of the velocity field $\mathrm{v}(y,0,t)$ given by Eq. (20) – curves $\mathrm{v1}(y)$ and Eq. (22) – curves $\mathrm{v2}(y)$ for $V = 2$, $\nu = 0.002$, $h = 0.6$, $\lambda = 5$, $\lambda_r = 4$.

APPENDIX

$$D_t^{\beta} V = \operatorname{sign}(1-\beta)\frac{Vt^{-\beta}}{\Gamma(1-\beta)} = \begin{cases} 0, & \beta = 1 \\ Vt^{-\beta}/\Gamma(1-\beta), & \beta \in (0,1) \end{cases} \tag{A1}$$

$$L\left\{\frac{t^{-\beta}}{\Gamma(1-\beta)}\right\} = \frac{1}{q^{1-\beta}};\, L\{D_t^p \mathrm{v}_{sn}(\xi,t)\} = q^p \overline{\mathrm{v}}_{sn}(\xi,q);\, L\left\{D_t^p\, \frac{\partial \mathrm{v}_{sn}(\xi,t)}{\partial t}\right\} = q^{p+1}\overline{\mathrm{v}}_{sn}(\xi,q), \tag{A2}$$

$$\frac{1}{y+a} = \sum_{k=0}^{\infty}(-1)^k\, \frac{y^k}{a^{k+1}};\, (a+b)^k = \sum_{m=0}^{\infty}\binom{m}{k} a^m b^{k-m} = \sum_{m,l\ge 0}^{m+l=k} \frac{k!}{m!\,l!}\, a^m b^l, \tag{A3}$$

$$\int_0^{\infty} \frac{\xi \sin(y\xi)}{\xi^2 + b^2}\, \mathrm{e}^{-a\xi^2} d\xi = \frac{\pi}{4}\, \mathrm{e}^{ab^2} \begin{bmatrix} \mathrm{e}^{-by}\operatorname{erfc}\left(b\sqrt{a} - \dfrac{y}{2\sqrt{a}}\right) \\ -\,\mathrm{e}^{by}\operatorname{erfc}\left(b\sqrt{a} + \dfrac{y}{2\sqrt{a}}\right) \end{bmatrix};\, \operatorname{Re}(a) > 0, \operatorname{Re}(b) \ge 0. \tag{A4}$$

$$G_{a,b,c}(d,t) = L^{-1}\left\{\frac{q^b}{(q^a - d)^c}\right\};\, \operatorname{Re}(ac-b) > 0, \operatorname{Re}(q) > 0, \left|\frac{d}{q^a}\right| < 1, \tag{A5}$$

$$(u_1 * u_2)(t) = \int_0^t u_1(t-s)u_2(s)\mathrm{d}s = \int_0^t u_1(s)u_2(t-s)\mathrm{d}s = L^{-1}\{\overline{u}_1(q)\overline{u}_2(q)\} \text{ if}$$
$$u_1(t) = L^{-1}\{u_1(q)\} \text{ and } u_2(t) = L^{-1}\{u_2(q)\}, \tag{A6}$$

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^c}\, G_{a,b,c}\left(-\frac{d}{\lambda}, t\right) = \frac{1}{d^c}\, \frac{t^{-b-1}}{\Gamma(-b)};\; b < 0. \tag{A7}$$

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