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ON EQUILIBRIUM PROBLEMS AND EQUILIBRIUM PROBLEM SYSTEMS

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We consider sufficient regularity and coercivity conditions for Minty and Stampacchia type equilibrium systems. Thus, under suitable assumptions, if the independent equilibrium problems are solvable, then the system of equilibrium problems also has a solution.

Key words: Minty and Stampacchia types equilibrium problems, Generalized monotonicity, Coercitivity, Hemicontinuity.

1. MINTY AND STAMPACCHIA TYPE EQUILIBRIUM PROBLEMS - THE STATEMENT OF THE PROBLEM

Let X and Y be topological spaces. Let C be a nonempty subset of Y, $\Psi : X\times C\times C \to \mathbb{R}$ and $F : C \to 2^X$ be a set valued mapping with nonempty values.

The equilibrium problem of Minty type is

$M\left(\Psi, F; C\right)$: Find $u\in C$ such that for all $v\in C$ and for all $x\in F\left(v\right)$,

$$\Psi\left(x,v,u\right)\geq 0 .$$

We note that $u\in C$ is a solution of $M\left(\Psi, F; C\right)$ if and only if

$$\inf \Psi\left(F\left(v\right),v,u\right) := \inf_{x\in F(v)} \Psi\left(x,v,u\right)\geq 0 \quad \text{for all } v\in C .$$

The equilibrium problem of Stampacchia type is

$S\left(\Psi, F; C\right)$: Find $u\in C$ such that for all $v\in C$ there exists $x\in F\left(v\right)$ for which

$$\Psi\left(x,v,u\right)\geq 0 .$$

If the set valued function F has compact values, then the solution u of $S\left(\Psi, F; C\right)$ is such that

$$\sup \Psi\left(F(u),v,u\right) := \sup_{x\in F(u)} \Psi\left(x,v,u\right)\geq 0 \quad \text{for all } v\in C .$$

Let us define the set valued maps

$$M_{\Psi F}(v) := \left\{u\in C \mid \ \forall\ x\in F\left(v\right),\ \Psi\left(x,v,u\right)\geq 0\right\},\ v\in C ,$$

and

$$S_{\Psi F}(v) := \left\{ u \in C \mid \exists\, x \in F(u),\ \Psi(x, v, u) \geq 0 \right\},\ v \in C .$$

We now see that *u* is a solution of $M(\Psi, F; C)$ if and only if $u \in \bigcap_{v \in C} M_{\Psi F}(v)$. Similarly, *u* is a solution of $S(\Psi, F; C)$ if and only if $u \in \bigcap_{v \in C} S_{\Psi F}(v)$.

Definition 1.1. *A set valued map F is said to be* Ψ*-properly quasimonotone if for all* $m \in \mathbb{N}$*, all vectors* $v_1, \ldots, v_m \in C$ *and scalars* $\lambda_1, \ldots, \lambda_m > 0$ *with* $\sum_{i=1}^{m} \lambda_i$ *we have*

$$\max_{1 \leq j \leq m} \inf(F(v_j), v_j, \sum_{i=1}^{m} \lambda_i v_i) = 0.$$

Remark 1.1. In our case, the proper quasimonotonicity of *F* w.r.t. Ψ is equivalent to the KKM property of the map $M_{\Psi F}$ (see [2]).

The connection between the Ψ - properly quasimonotonicity of *F* and the maps $M_{\Psi F}(v)$ and $S_{\Psi F}(v)$ is given by the results below.

Proposition 1.1.

(i) *F is* Ψ *- properly quasimonotone if and only if* $M_{\Psi F}$ *is a KKM map.*

(ii) *If C is a convex set,* $\Psi(F(x), \cdot, u)$ *is convex in the second argument for any fixed* $u \in C$ *and* $x \in F(u)$*, and* $(F(x), u, u) = 0$ *for any* $u \in C$ *and* $x \in F(u)$*, then* $S_{\Psi F}$ *is a KKM map.*

Definition 1.2. F is Ψ - pseudomonotone on C if and only if $S_{\Psi F}(v) \subseteq M_{\Psi F}(v)$ *for any* $v \in C$.

Proposition 1.2. F is said to be pseudomonotone w.r.t . Ψ if and only if $S_{\Psi F}(v) \subseteq M_{\Psi F}(v)$ *for any* $v \in C$.

2. MINTY AND STAMPACCHIA TYPE EQUILIBRIUM PROBLEM SYSTEMS - THE STATEMENT OF THE PROBLEM

Now we take up the case of the equilibrium problem systems.

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be real Hausdorff topological vector spaces. For $i = 1, \ldots, n$, consider the functions $\Psi_i^* : X_i \times C_i \times C_i \to \mathbb{R}$, where $C_i \subset Y_i$, $i = 1, \ldots, n$, are nonempty sets. Further, for $i = 1, \ldots, n$, we consider set valued functions $F_i : C_1 \times \ldots \times C_n \to 2^{X_i}$ with nonempty values.

Let us denote $I = \{1, \ldots, n\}$, $C = C_1 \times \ldots \times C_n$, $u = (u_1, \ldots, u_n)$ and $(u^i, v) = (u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_n)$. Also, we use the notation *F* for the family $F_1, \ldots, F_n$.

The equilibrium problems system of Minty type is

$M(\Psi^*, F; C)$: Find $u \in C$ such that for all $i \in I$ and $v \in C_i$, and for all $x \in F_i(u^i, v)$, we have

$$\Psi^*(x, v, u_i) \geq 0 .$$

The equilibrium problem system of Stampacchia type is

$S(\Psi^*, F; C)$: Find $u \in C$ such that for all $i \in I$ and $v \in C_i$ there exists $x \in F_i(u^i, v)$ for which

$$\Psi^*(x, v, u_i) \geq 0 .$$

Remark 2.1. For the case of classical types of variational inequalities, see [1] and [3]. The implicit case was studied in [6].

3. MINTY AND STAMPACCHIA TYPE EQUILIBRIUM PROBLEMS - MAIN RESULTS

In the following we consider problems $M(\Psi, F; C)$ and $S(\Psi, F; C)$. We first note that in the case when the set C is not a compact convex set, it is necessary to consider a coercivity condition of Minty type, namely,

$(\Psi - \mathrm{CM})$: There exist a compact set $K \subset C$ and $v_1, ..., v_m \in C$ such that

$$\min_{1 \le j \le m} \inf \Psi\left(F(v_j), v_j, u\right) < 0 \text{ for all } u \in C \setminus K.$$

Remark 3.1. For $\Psi(x,v,u) = \langle x, v-u \rangle$ we get the coercivity condition used in [4]. Also, in this case, for $m=1$ we obtain a coercivity condition from [5].

We now state a first result for solvability of problem $M(\Psi, F; C)$.

Theorem 3.1. *Assume that*
(i) *C is a closed set*;
(ii) *F is a Ψ - properly quasimonotone mapping* ;
(iii) *the coercivity condition* $(\Psi - \mathrm{CM})$ *is satisfied*;
(iv) *the mapping $u \mapsto \Psi(x,v,u)$ is lower semicontinuous on K for any fixed $x \in F(v)$, $v \in K$* .

Then there exists a solution of problem $M(\Psi, F; C)$ in K.

We can also consider a Stampacchia type coercivity condition
$(\Psi - \mathrm{CS})$: There exist a compact set $K \subset C$ and $v_0 \in C$ such that

$$\Psi(x, v_0, u) < 0 \text{ for all } u \in C \setminus K \text{ and } x \in F(u).$$

Theorem 3.2. If the conditions (i) and (iv) together with
(v) *C is a convex set,*
(vi) *F is a pseudomonotone mapping w.r.t . Ψ , and*
(vii) *the coercivity condition* $(\Psi - \mathrm{CS})$
are fulfilled, then the problem $M(\Psi, F; C)$ has a solution in K.

We consider now the case of problem $S(\Psi, F; C)$.

Definition 3.1. *Let C be a convex set. We say that the set valued map F is Ψ - upper hemicontinuous on C if for all $u, v \in C$ the mappig $t \mapsto F(v + t(u-v)), t \in [0,1]$, is upper semicontinuous on* [0,1].

Theorem 3.3 *We suppose that the conditions* (i), (v) *are fulfilled together with*
(viii) *F is Ψ - upper hemicontinuous on C w.r.t Ψ* ,
(ix) *$\Psi(\cdot, v, u)$ is a semicontinuous mapping for all $v, u \in C$* ,
(x) $\Psi(x, v_t, u) = t^\tau \Psi(x, v, u),\ t \in [0,1],\ x \in F(v_t),\ \tau > 0$.

If problem $M(\Psi, F; C)$ is solvable, then problem $S(\Psi, F; C)$ also is solvable. Further, every solution of $M(\Psi, F; C)$ also is a solution of $S(\Psi, F; C)$.

Remark 3.2. The next results are based on Theorems 3.1 and 3.3.

4. MINTY AND STAMPACCHIA TYPE EQUILIBRIUM PROBLEM SYSTEMS - MAIN RESULTS

Relative to the Minty problem $M(\Psi^*, F; C)$ we consider the *i*-th subproblem. For $i \in I$ and fixed elements $u_j \in C_j$, $j \neq i$, the Minty equilibrium problem concerning the *i*-th inequality of $M(\Psi^*, F; C)$ is

$M(\Psi_i^*, F_i; (u^i, C_i))$: Find $u \in C$ such that for all $v \in C$ and for all $x \in F_i(u^i, v)$,

$$\Psi_i^*\left(x,v,u_i\right)\geq 0 .$$

Now, the solvability of problem $M\left(\Psi^*,F;C\right)$ is obtained from the independent solvability of the components of the equilibrium problems.

Theorem 4.1. *For each* $i\in\{1,2,\dots,n\}$ *let* $\mathrm{K}_i\subset\mathrm{C}_i$ *be a compact convex set. Assume that*

(xi) *for each fixed* $u_j\in K_j$, $j\neq i$, *problem* $M\left(\Psi_i^*,F_i;(u^i,C_i)\right)$ *is solvable in* K_i;

(xii) *for each fixed* $\mathrm{v}_i\in\mathrm{K}_i$, *the set valued function* $u^i\to F_i(u^i,v_i)$ *is lower semicontinuous on* $K^i=K_1\times\ldots\times K_{i-1}\times K_{i+1}\times\ldots\times K_n$;

(xiii) *for any fixed x,v,* $\Psi_i^*(x,v,\cdot)$ *is quasiconcave for all* $i\in I$;

(xiv) *for any fixed v,* $\Psi_i^*(\cdot,v,\cdot)$ *is semicontinuous on* $X_i\times C_i$ *for all* $i\in I$.

Then problem $M\left(\Psi^*,F;C\right)$ *has a solution in* $K_1\times\ldots\times K_n$.

Theorem 4.2. For each $i\in\{1,2,\dots,n\}$ let $\mathrm{K}_i\subset\mathrm{C}_i$ be a compact convex set. Assume that (xi)-(xiv) from Theorem 4.1 are fulfilled toghether with

(xv) $\Psi_i^*(x,v_t,u_i)=t^{\tau_i}\Psi_i^*(x,v,u_i)$ *for all* $t\in[0,1]$, $x\in F_i(u^i,v)$, $v_t=tv+\left(1-t\right)u_i,\tau_i>0$;

(xvi) *for any fixed* $u_j\in K_j$, *the set valued function* $v\to F_i(u^i,v)$ *is upper semicontinuous on* C_i.

Then problem $S\left(\Psi,F;C\right)$ *has a solution in* $K_1\times\ldots\times K_n$.

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