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# DUALITY FOR A CLASS OF CONTINUOUS-TIME PROGRAMMING PROBLEMS

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We consider a class of nonlinear continuous-time programming problems and a general Mond-Weir dual model for this class. We get that WD-invexity property is a necessary and sufficient condition for weak duality.

*Key words*: WD-invexity, continuous-time nonlinear programming, Mond-Weir duality, weak duality.

## 1. INTRODUCTION

In 1953, Bellman [1] introduced a certain class of continuous-time optimization problems. Since then, many new classes of continuous-time nonlinear problems were considered. See De Oliveira and Rojas-Medar [4], Rojas-Medar *et al*. [5], Zalmai [6] and the references therein.

De Oliveira and Rojas-Medar [4] gave a generalization of the notions of KKT-invexity and WD-invexity for the continuous-time nonlinear programming problem, introduced by Martin [2] for the mathematical programming case. Also, they proved two very interesting results: KKT-invexity is a necessary and sufficient condition for global optimality of a Karush-Kuhn-Tucker point and WD-invexity is a necessary and sufficient condition for weak duality, where a Lagrangian dual (a continuous-time analogue to Wolfe's duality) is considered.

We prove that WD-invexity property of a general Mond-Weir dual for the continuous-time nonlinear programming problem also is a necessary and sufficient condition for weak duality. In this respect, we consider a general WD-invexity concept and a general qualification constraint, generalizations of the notions introduced by Oliveira and Rojas-Medar [4].

## 2. PRELIMINARIES

We consider the following continuous-time nonlinear programming problem

(CNP)
$$\text{minimize } \phi(x) = \int_0^T f(x(t),t)\,dt$$

subject to

$$g\left(x\left(t\right),t\right) \le 0 \text{ a.e. in } \left[0,T\right], x \in X,$$

where $X$ is a nonempty open convex subset of the Banach space $L_\infty^n\left[0,T\right], \varphi : X \to \mathbb{R}, f\left(x\left(t\right),t\right) = \xi\left(x\right)\left(t\right)$ and $g\left(x\left(t\right),t\right) = \gamma\left(x\right)\left(t\right)$, with the mappings $\xi$ and $\gamma$ from $X$ into $\Lambda_1^1[0,T]$ and $\Lambda_1^m[0,T]$, respectively. Here, $L_\infty^n\left[0,T\right]$ is the space of all $n$-dimensional vector-valued Lebesgue measurable functions defined on the compact interval $\left[0,T\right]\subset\mathbb{R}$ that are essentially bounded, with the norm $\|\cdot\|_{\infty}$ defined by

$$\| x\|_{\infty}=\max_{1\le j\le n} ess\sup\left\{\left|x_j\left(t\right)\right|,0\le t\le T\right\},$$

where $\left(x_j\left(t\right)\right)_{1\le j\le n}=x\left(t\right)\in\mathbb{R}^n$; the space $\Lambda_1^m[0,T]$ is the space of all $m$-dimensional vector-valued functions defined on $\left[0,T\right]$ that are essentially bounded and Lebesgue measurable, with the norm $\|\cdot\|_1$ defined by

$$\| y\|_1=\max_{1\le j\le n}\int_0^T |y_j(t)|\,dt$$

for $y\left(t\right)=\left(y_j\left(t\right)\right)_{1\le j\le n}\in\mathbb{R}^n$.

Let $F=\{x\in X: g\left(x\left(t\right),t\right)\le 0$ a.e. in $[0,T]\}$ be the set of all feasible solutions of (CNP). We suppose that $F$ is a nonempty set and all vectors are column vectors. For $w\in\mathbb{R}^p, w\le 0$ means that $w_i\le 0$ for all $i=1,2,...,p$; $w<0$ means that $w_i<0$ for $i=1,2,...,p$ and $w'$ stands for the transposed of $w$.

Now, for (CNP) problem, we consider a general Mond-Weir dual. We suppose that the functions $t\mapsto\nabla f\left(x\left(t\right),t\right)$ and $t\mapsto\nabla g_i'\left(x\left(t\right),t\right)z\left(t\right), i\in I=\left\{1,2,...,p\right\}$, are Lebesgue integrable in $[0,T]$ for all $x\in X$ and for all $z\in L_{\infty}^n\left[0,T\right]$. The general Mond-Weir type dual is

(MWDP) $$\text{maximize}\,\psi(x,\lambda)=\int_0^T\left[f(x(t),t)+\lambda'_{I0}(t)g_{I0}(x(t),t)\right]dt$$

subject to

$$\int_0^T\left[\nabla f'(x(t),t)+\sum_{i\in I}\lambda_i(t)\nabla g_i'(x(t),t)\right]z(t)dt=0,$$

$$\lambda_{I_k}'\left(t\right)g_{I_k}\left(x\left(t\right),t\right)\ge 0 \text{ a.e. in } \left[0,T\right],\ k=\overline{1,\nu},$$

$$\lambda_i\left(t\right)\ge 0 \text{ a.e. in } \left[0,T\right],\ i\in I,$$

$$z\in L_{\infty}^n[0,T],\ x\in X,\ \lambda\in L_{\infty}^m[0,T],$$

where $\nu\ge 0, I_{\alpha}\cap I_{\beta}=\Phi$ for $\alpha\ne\beta$ and $\bigcup_{\alpha=0}^{\nu} I_{\alpha}=\{1,...,m\}$, $\lambda_{I_k}=\left(\lambda_i\right)_{i\in I_k}$ and $g_{I_k}=\left(g_i\right)_{i\in I_k}$ with

$$\lambda_{I_k}'\left(t\right)g_{I_k}\left(x\left(t\right),t\right)=\sum_{i\in I_k}\lambda_i'\left(t\right)g_i\left(x\left(t\right),t\right).$$

This dual problem (MWDP) may be considered as the continuous-time analogue of a general Mond-Weir duality formulation [3].

Let **FD** denote the set of all feasible solutions of (MWDP).

## 3. INVEXITY AND WEAK DUALITY

**Definition 3.1.** ([4]) There is weak duality between the problems (CNP) and (MWDP) if

$$\varphi\left(\mathrm{x}\right)\ge\psi\left(\mathrm{y},\lambda\right)$$

*for all* $x\in F$ *and all* $\left(y,\lambda\right)\in$ **FD**.

**Definition 3.2.** ([4]) The (CNP) problem is said to be invex if there exists a function $\eta : V \times V \times [0,T] \to \mathbb{R}^n$ such that $t \mapsto \eta(x(t), y(t), t) \in L_\infty^n[0,T]$ and

$$\varphi(x) - \varphi(y) \geq \int_0^T \nabla f'(y(t),t)\eta(x(t), y(t),t)dt,$$

$$g(x(t),t) - g(y(t),t) \geq \nabla g_i'(y(t),t)\eta(x(t),y(t),t) \text{ a.e. } in\ [0,T],\ i \in I,$$

*for all* $x \in F$ *and* $y \in X$.

**Theorem 3.1.** The invexity of (CNP) implies the weak duality between (CNP) and (MWDP).

We note that the omission of the terms $g_i(x(t),t)$, $i \in I$ and $g_i(y(t),t)$, $i \notin I_0$ in the last inequality from Definition 3.2 does not affect the conclusion of Theorem 3.1. Thus, this makes it possible to use a generalized WD-invexity for (CNP).

## 4. GENERALIZED WD- INVEXITY AND WEAK DUALITY

In this section we introduce a generalized WD-invexity and a generalized constraint qualification. Then we prove the equivalence of weak duality for (CNP) and generalized WD-invexity defined below.

**Definition 4.1.** *We say that the* (CNP) *problem is generalized weak duality invex* (*generalized* WD-*invex*) *if there exists a function* $\eta : V \times V \times [0,T] \to \mathbb{R}^n$ *such that* $t \mapsto \eta(x(t), y(t), t) \in L_\infty^n[0,T]$ *and*

$$\varphi(x) - \varphi(y) \geq \int_0^T \nabla f'(y(t),t)\eta(x(t), y(t),t)dt,$$

$$-g_i(y(t),t) \geq \nabla g_i'(y(t),t)\eta(x(t),y(t),t) \text{ a.e. in } [0,T],\ i \in I_0,$$

$$0 \geq \nabla g_i'(y(t),t)\eta(x(t),y(t),t) \text{ a.e. in } [0,T],\ i \notin I_0,$$

*for all* $x \in F$ *and* $y \in X$.

**Remark 4.1**. For $\nu = 0$ generalized WD-invexity reduces to WD-invexity defined in [4].

**Definition 4.2**. *We say that g satisfies the generalized constraint qualification* (GCQ) *if there is no* $v_i \in L_\infty[0,T]$, $v_i(t) \geq 0$ a.e. *in* $[0,T]$, $\mathrm{i} \in \mathrm{I}_0$, *not all zero, such that*

$$\int_0^T \sum_{i \in I_0} v_i(t) g_i(x(t),t)dt \geq 0 \ \ for\ all\ x \in X.$$

**Remark 4.2**. For $\mathrm{I}_k = \Phi, \mathrm{k} = \overline{1,\nu}$, (GCQ) becomes (CQ2) introduced in [4].

**Theorem 4.1**. *Under* (GCQ), *weak duality holds between* (CNP) *and* (GMWDP) *if and only if* (CNP) *is generalized* WD-*invex*.

## 5. CONCLUSION

In this note we proved that in the context of continuous-time nonlinear programming problem, weak duality is attained if the general Mond-Weir dual of the problem previously mentioned has the WD-invexity property. On account of the importance and accuracy of the results from [4], we think that it is interesting and useful to establish corresponding formulations for both the multiobjective continuous case and the case where invexity is replaced by, for example, ρ-invexity.

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