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SOME RESULTS ON E-P-ALGEBRAS

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Abstract: In this paper we present some results about the finite dimensional Lie p-algebras L and some properties of the Frattini p-subalgebra of L. In addition, some properties of E-algebra and E-p-algebra are pointed out.

Key words: Lie algebra, Frattini subalgebra, p-algebra

1. INTRODUCTION

The Lie algebras have become an interesting subject in both mathematics and physics. This theory has undergone a remarkable evolution during the last years. The notion of of Lie p-algebra was introduced by Nathan Jacobson, [5], without devoted(awarded) a special study. It is well known that restricted Lie algebras have played an important role in the classification of the finite-dimensional modular simple Lie algebras.

The theory of the Frattini subgroup of a group is well advanced and has proved useful in the study of certain types of problems in the group theory. Analogous problems for algebras can be posed, and these are of independent interest. It therefore seems desirable to investigate the possibility of establishing a parallel theory for algebras. The many close connections which Lie algebras have with groups render them the obvious choice for a first attempt at an analogous theory, and such investigations have been successfully carried out by Barnes and Gastineau-Hills, Hochschild, Reutenauer, Roggenkamp, Schue, Winter, et al. ([1], [4], [9], [10], [11], [15]).

Although the corresponding concept of the Frattini subalgebra of a Lie algebra has been widely recognized, so far there has not been developed a theory of the Frattini subalgebra analogous to that of the Frattini subgroup.

In this section, we recall some notions and properties necessary in the paper. Throughout L will denote a finite dimensional Lie algebra over a field K.

Definition 1.1. A *Lie p-algebra* is a Lie algebra L, over a field K of characteristic p>0, with a p-map $a \rightarrow a^p$, such that:

$(\alpha x)^p = \alpha^p x^p$, for all $\alpha \in K, x \in L$

$x(ady^p)=x(ady)^p$, for all, $x,y \in L$, and

$(x+y)^p=x^p+y^p+\sum_{i=1}^{p-1} s_i(x,y)$ for all $x,y \in L$,

where $is_i(x,y)$ is the coefficient of λ^{i-1} in the expansion of $x(ad(\lambda x+y))^{p-1}$.

A Lie subalgebra (respectively, *Lie ideal*) of L is a Lie p-subalgebra (respectively, a Lie p-ideal) if it is closed under the p-map.

The notions of maximal Lie p-subalgebra respectively maximal Lie p-ideal of L are defined as usual.The intersection of Lie p-subalgebras (respectively Lie p-ideals) is a Lie p-subalgebra (respectively a Lie p-ideal) of L.

We denote by $\Phi_p(L)$ the Lie p-subalgebra of L obtained by intersecting all maximal Lie p-subalgebras of L and we call it the *Frattini p-subalgebra* of L.

The largest Lie p-ideal of L included into $\Phi_p(L)$ is called the *Frattini p-ideal* and is denoted by $F_p(L)$.

These are the corresponding notions to the Frattini subalgebra $\Phi(L)$ and the Frattini ideal F(L) for a Lie algebra.

We shall use the following notations:

[x,y] is the product of x,y in L;

$L^{(1)}$ the derived algebra of L;

$L^{(n)} = (L^{(n-1)})^{(1)}$, for all $n\geq 2$;

(A) is the subalgebra generated by the subset A of L;

$(A)_p = (\{x^{p^n} \mid x\in (A), n\in N\})$, where $x^{p^n} = (x^{p^{n-1}})^p$;

$A^p = (\{x^p \mid x\in A\})$, where A is a subalgebra of L;

$A^{p^n} = (A^{p^{n-1}})^p$;

Z(L) is the center of L;

N(L) is the nilradical of L;

Note that, if L is a p-algebra (finit dimensional), then Z(L) is closed as p-ideal of L.

The reader can consult for instance [2], [3], [5]- [8], [12]-[14]

2. LIE P -ALGEBRAS WHICH ARE F_P-FREE

A particular case of Lemma 3.1 [6] is the following result

Proposition 2.1. *If L is a finite dimensional Lie algebra over a field K, then*

$$L^{(1)}\cap Z(L)\subseteq F(L).$$

In [8], Lincoln and Towers have proved the following

Lemma 2.2. *If L is a finite dimensional Lie p-algebra over a field K, then we have*

$$(L^{(1)})_p\cap Z(L) \subseteq F_p(L).$$

The *abelian socle* S(L) is the sum of all minimal abelian Lie ideals of L.

We may define the *abelian p-socle* of the finite dimensional Lie p-algebra L as being the sum of all minimal abelian Lie p-ideals of L and we denote it by Sp(L).

The abelian socle (respectively, the abelian p-socle) of a finite dimensional Lie (p-) algebra is a Lie ideal (a Lie p-ideal) of L, as one can show easily.

Definition 2.3. Let L be a finite dimensional Lie p-algebra and I be a Lie p-ideal of L. We say that *L p-splits* over I if there exist a Lie p-subalgebra A of L such that $L=I\dot{+}A$. A is called a *p-complement* of the p-ideal I.

In these hypothesis the following statements are true.

Theorem 2.4.[8] *Let L be a finite dimensional Lie p-algebra such that $L^{(1)}\neq 0$ and $L^{(1)}$ is nilpotent. Then the following statements are equivalent:*

(i) $F_p(L)=0$.

(ii) $Sp(L)=N(L)$, and L p-splits over N(L).

(iii) $L^{(1)}$ is abelian, $(L^{(1)})^p=0$, L p-splits over $L^{(1)}\oplus Z(L)$, and

$$Sp(L)=L^{(1)}\oplus Z(L).$$

In the same hypotheses, the Cartan subalgebra of L is exactly those subalgebras which have $L^{(1)}$ as a p-complement.

It can be inferred that:

Corollary 2.5. *If L is a finite dimensional Lie p-algebra over K with $L^{(1)}$ nilpotent and nonzero, $F_p(L)=0$ and K is perfect, then the maximal toral subalgebras are precisely those having as p-complement $L^{(1)}\oplus Z(L)$.*

Examples 2.6. We know which are the Lie algebras of dimension 2 over K, and we take L=I+V, where $I = Kx+Ky, V = Ku_1+Ku_2$, $u_1^{\,p} = u_2^{\,p} = y^p = 0$, $x^p = x$,

$[V,V] = 0, [x,y] = 0, [x,u_1] = u_1, [x,u_2] = u_2, [y,u_1] = u_2, [y, u_2] = 0$.

Then $L^{(1)}=V$ is abelian, $(L^{(1)})^p = 0$, $Z(L) = 0$. Now, $N(L) = Ky + Ku_1 + Ku_2$. Also Ku_2 is a minimal p-ideal. Let J be a minimal p-ideal contained in N(L). Since $[N(L),N(L)]=Ku_2$, either $J=K u_2$ or $[N(L),J]=0$. Suppose that $J\neq K u_2$. Then $[y,J]=0$ so $J\subseteq Ky+K u_2$, and $[u_1,J]=0$ so $J \subseteq Ku_1 + Ku_2$. Thus $J \subseteq Ku_2$, a contradiction. Hence $N(L)\neq Sp(L)$.

In [12], E. L. Stitzinger has shown that, for any Lie algebra L over the arbitrary field K, such that $L^{(1)}$ is nilpotent, L is F-free (that is F(L)=0) if and only if each subalgebra of L is F-free.

The complete analogue of this result does not hold if F(L) is replaced by $F_p(L)$. In this respect we give an example.

Example 2.7. Taking $L=Kx+Ky+Ku_1+Ku_2$ with $K=Z_2$,
$x^2=x$, $y^2=x+y$, $[x, u_1] = u_1, [x, u_2]= u_2, [b, u_1]= u_2, [b, u_2]= u_1 + u_2$
$[x,y]=[u_1, u_2]=0$, $u_1^{\,2} = u_2^{\,2} = 0$, and $I = Kx + Ky$ we get $F_0(L) = 0$
where as $F_p(I)=Kx$.

However some partial results can be obtained in this direction as we can deduce from the following result.

Theorem 2.8.[8] *Let L be a finite-dimensional Lie p-algebra. Then the following statements are equivalent:*

i) $L^{(1)}$ is nilpotent and $F_p(L)=0$;

ii) $L = I \dot{+} A$ where A is an abelian Lie subalgebra, I is an abelian Lie p-ideal, the (adjoint) action of A on I is faithful and completely reducible, Z(L) is completely reducible under the p-map, and the p-map is trivial on [A,I].

Corollary 2.9. *Let L be a finite dimensional Lie p-algebra with $L^{(1)}$ nilpotent and $F_p(L) = 0$. We claim that the following assertion are true:*

i) If A is a Lie p-subalgebra of L containing Sp(L), then $F_p(A)=0$.

ii) If J is a Lie ideal of L, then $F_p(J)=0$.

3.THE RELATIONSHIP BETWEEN E-ALGEBRAS AND E- p -ALGEBRAS

Definition 3.1. A finite dimensional Lie algebra (respectively Lie p-algebra), L, is called *elementary* (respectively *p-elementary*), if F(A)=0 (respectively $F_p(A)=0$) for every Lie subalgebra (respectively Lie p-subalgebra) A of L.

Corollary 3.2.[8] *Assume L is a finite dimensional Lie p-algebra with nilpotent $L^{(1)}$ and $F_p(L)=0$. Let $L=Sp(L) \dot{+} A$ as in Theorem 2.8 (ii). Then L is p-elementary, if and only if A=Sp(A).*

Lemma 3.3. *Let L be a finite dimensional Lie p-algebra over an algebraically closed field K of characteristic p>0, and suppose that $L^{(1)}$ is nilpotent. Then L is p-elementary, if and only if $F_p(L) = 0$.*

Proof. Suppose that $F_p(L) = 0$ and write $L = Sp(L) \dot{+} A$ as in Theorem 2.8 (ii). Then A has a faithful completely reducible representation on Sp(L) which is equivalent to the fact that A has non-zero nil ideals [11]. As A is abelian this is equivalent to the injectivity of the p-map. Since K is algebraically closed, this is equivalent to Sp(A) = A. Now it follows from Corollary 3.2. that L is p-elementary. The reverse assertion is immediately.

Definition 3.4. We say that a finite dimensional Lie algebra (respectively Lie p-algebra), L, is an *E-algebra* (respectively, *E-p-algebra*) if for every Lie subalgebra (respectively, Lie p-subalgebra) U of L we have $F(U) \subseteq F(L)$ (respectively, $F_p(U) \subseteq F_p(L)$)

Now here is available one version of Theorem I.4.2 [Ci] which we presents bellow.

Theorem 3.5. *If L is a finite dimensional Lie p-algebra, then L is an E-p-algebra, if and only if $L/F_p(L)$is p-elementary.*

Proof. Suppose that L is an E-p-algebra, and let $U/F_p(L)$ be a subalgebra of $L/F_p(L)$. We choose a p-subalgebra m of L which is minimal with respect to $F_p(L) + m = U$(so m could be U because we know by [2] that $F_p(L) \cap m \subseteq F_p(m) \subseteq F_p(L)$ so $F_p(L) \subseteq m$ and m=U). Let I be a p-ideal of U such that $I/ F_p(L) = F_p(U/ F_p(L))$, and we suppose that $I \neq F_p(L)$.

Then $I = I \cap U = I \cap (F_p(L) + m) = F_p(L) + I \cap m$ and $I \cap m \not\subset F_p(L)$.

It follows that $I \cap m \not\subset F_p(m)$ since L is an E-p-algebra. But $I \cap m$ is a p-ideal of m, so $I \cap m \not\subset \Phi_p(m)$. Hence there is a maximal p-subalgebra M of m such that $I \cap m \not\subset M$, and $m = I \cap m + M$.

By the minimality of m, we have $F_p(L) + M \neq U$. We claim that $F_p(L) + M$ is a maximal p-subalgebra of U. Suppose that $F_p(L) + M \subset J \subset U$. Than $M \subseteq J \cap m \subseteq m$ and so by the maximality of M, $J \cap m = M$ or $J \cap m = m$.

The statement $J \cap m = M$ implies that $F_p(L) + M = F_p(L) + J \cap m = J \cap (F_p(L) + m) = J \cap U = J$, a contradiction.

The second statement, $J \cap m = m$ implies $m \subseteq J$ and hence $J \supseteq F_p(L) + m = U$, also a contradiction. From that results the maximality of $F_p(L) + M$ in U.

Thus $(F_p(L) + M)/ F_p(L) \supseteq F_p(U/ F_p(L)) = I/ F_p(L)$, and so $I \subseteq F_p(L) + M$.

But $I \cap m \subseteq F_p(L) + M$ and so $U = F_p(L) + m = F_p(L) + I \cap m + M = F_p(L) + M$, contradicting the minimality of m. We conclude that $I = F_p(L)$, whence $F_p(U/ F_p(L)) = 0$ and $L/F_p(L)$ is p-elementary.

Conversely, suppose that $L/F_p(L)$ is p-elementary. Let U be a p-subalgebra of L.

Then $(F_p(U) + F_p(L))/ F_p(L) \subseteq F_p((U + F_p(L))/ F_p(L)) = 0$, and so $F_p(U) \subseteq F_p(L)$.

Corollary 3.6. *Let L be a finite dimensional Lie p-algebra over an algebraically closed field K of characteristic p > 0, and suppose that $L^{(1)}$is nilpotent. Then L is a E-p-algebra.*

Proof. Because $L^{(1)}$ is nilpotent we have $F_p(L) = 0$. Now from Corollary 3.2. and Theorem 3.5 results the fact that L is a E-p-algebra.

After following Theorem we finish by presenting the relationship between elementary and p-elementary Lie p-algebras (respectively E-algebras and E-p-algebras relationship) exposed by Theorem 3.8 below.

Theorem 3.7. *Let L be a finite dimensional Lie p-algebra and A a subalgebra of L. Then:*

(i) *$F(A) \subseteq F((A)_p)$;*

(ii) *If L is nilpotent, then $F(A) \subseteq F_p(L) \Rightarrow F(A) \subseteq F(L)$.*

Proof. (i) Let M be a maximal subalgebra of $(A)_p$, and suppose that F(A) $\not\subset$ M. Then $(A)_p$ = M + F(A), and so A = M $\bigcap$ A + F(A) = M $\bigcap$ A, these from the fact that L is a Lie p-algebra, B a Lie p-subalgebra of L and B+F(L)=B we have B=L (see for these [2]. From these A $\subseteq$ M and so F(A) $\subseteq$ M, contrary to our assumption. Thus F(A) $\subseteq$ $\Phi((A)_p)$, whence F(A) $\subseteq$ $F((A)_p)$.

Suppose that F(A) $\subseteq$ $F_p(L)$ and let M be a maximal subalgebra of L such that F(A) $\not\subset$ M. We know that L is nilpotent if and only if $\Phi(L) = L^{(1)}$. On the other hand, since L is nilpotent we have F(L)=[L,L]. But now F(A)= $\Phi(A) = [A, A] \subseteq [L, L] = F(L) \subseteq M$ which is a contradiction. Hence F(A)$\subseteq$F(L).

If L is a Lie p-algebra, F(L)$\subseteq F_p(L)$ and F(A) $\subseteq$ F(L) $\Rightarrow$ F(A)$\subseteq F_p(L)$.

Theorem 3.8. *Let L be a finite dimensional Lie p-algebra. Then:*

(i) If L is p-elementary then L is elementary;

(ii) If L is an E-p-algebra, then L is an E-algebra.

Proof. (i) Let L be a Lie p-algebra, p-elementary and let U be a subalgebra of L. We have F(U) $\subseteq$ $F((U)_p)$ and $F((U)_p)\subseteq F_p((U)_p)$. Thus F(U) $\subseteq$ $F((U)_p)$ $\subseteq$ $F_p((U)_p)$ = 0 since L is p-elementary. So L is elementary.

(ii) Let L be an E-p-algebra and let U be a subalgebra of L. Then by mens of (1), we have F(U) $\subseteq$ $F((U)_p)$ $\subseteq$ $F_p((U)_p)$ $\subseteq$ $F_p(L)$ since L is an E-p-algebra, i.e., F(U) $\subseteq$ $F_p(L)$. So F(U) $\subseteq$ F(L) in the light of the Theorem 3.7. (2), that is, L is an E-algebra.

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