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AN ANALYTICAL APPROACH FOR APPROXIMATION OF EXPERIMENTAL HYSTERETIC LOOPS BY BOUC –WEN MODEL

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In this paper an analytical method is developed to identify the Bouc-Wen model parameters from the experimental data of periodic loading tests. The model parameters are determined by closed analytical relationships such as the predicted and experimental hysteresis loops to have exactly the same maximum force values and coordinates of loop- axes crossing points. Asymmetric hysteretic characteristics are modeled by sewing the solutions of two different Bouc-Wen equations, corresponding to negative and positive values of the imposed cyclic displacement. The method efficiency is illustrated by its application to portraying the asymmetric hysteretic behavior of two vibration control devices.

Keywords: hysterezis, Bouc-Wen model, fitting experimental data, analytical approach.

1. INTRODUCTION

The Bouc-Wen model, widely used in structural and mechanical engineering, gives an analytical description of a smooth hysteretic behavior. It was introduced by Bouc [1] and extended by Wen [2], who demonstrated its versatility by producing a variety of hysteretic characteristics. The hysteretic behavior of materials, structural elements or vibration isolators is treated in a unified manner by a single nonlinear differential equation with no need to distinguish different phases of the applied loading pattern. In practice, the Bouc-Wen model is mostly used within the following inverse problem approach: given a set of experimental input–output data, how to adjust the Bouc-Wen model parameters so that the output of the model matches the experimental data. Once an identification method has been applied to tune the Bouc-Wen model parameters, the resulting model is considered as a “good” approximation of the true hysteresis when the error between the experimental data and the output of the model is small enough from practical point of view. Usually, the experimental data are obtained by imposing cyclic relative motions between the mounting ends on the testing rig of a sample material, structural element or vibration isolator and by recording the evolution of the developed force versus the imposed displacement. Once the hysteresis model was identified for a specific input, it should be validated for different types of inputs that can be applied on the testing rig, such as to simulate as close as possible the expected real inputs. Then this model can be used to study the dynamic behavior of different systems containing the tested structural elements or devices under different excitations.

Various methods were developed to identify the model parameters from the experimental data of periodic vibration tests. A frequency domain method was employed to model the hysteretic behavior of wire-cable isolators [3], iterative procedures were proposed for the parametric identification of a smoothed hysteretic model with slip [4], of a modified Bouc-Wen model to portray the dynamic behavior of magnetorhological dampers [5], etc. The Genetic Algorithms were widely used for curve fitting the Bouc-Wen model to experimentally obtained hysteresis loops for composite materials [6], nonlinear degrading structures [7] or magnetorheological fluid dampers [8-10].

In the present work, our primary focus is to give closed analytical relationships to determine the parameters of the Bouc-Wen model such as the predicted hysteresis curves and the experimental loops to have same absolute values of the maximum forces and same coordinates of the loop-axes crossing points.

The derived equations can be used for fitting the Bouc-Wen model to both symmetric and asymmetric experimental loops. The asymmetry of experimental hysteresis curves is due to the asymmetry of the mechanical properties of the tested element, of the imposed cyclic motion, or of both factors. In most cases, the identified model output turns out to be a "good" approximation of experimental output. When this approximation is not satisfactory, the obtained parameter values can be used as initial values within an iterative algorithm to improve the model accuracy.

2. ANALYTICAL APPROACH

Suppose the experimental hysteretic characteristic is a symmetric loop $-F_{\mathrm{m}} \le F(x) \le F_{\mathrm{m}}$, obtained for a periodic motion $-x_m(t) \le x(t) \le x_m(t)$, imposed between the mounting ends of the tested element. The loop-axes crossing points are: $A(0,\ F_0)$, $C(x_0,\ 0)$, $D(0,\ -F_0)$ and $E(-x_0,\ 0)$.

By introducing the dimensionless magnitudes

$$\begin{gathered}\tau = t/T,\ \xi(\tau) = x(\tau T)/x_u,\ \xi'(\tau) = \mathrm{d}\xi/\mathrm{d}\tau,\ z(\xi) = F(x_u\xi)/F_u, \\ \xi_m = \max\left|\xi(\tau)\right|,\ z_m = \max\left|z(\xi)\right|,\ \ \xi_0 = x_0/x_u,\ z_0 = F_0/F_u\end{gathered} \tag{1}$$

where T is the period of the imposed cyclic motion and x_u, F_u are displacement and force reference units such as $\xi_m \le 1$, $z_m \le 1$, a generic plot of the symmetric hysteresis loop $z(\xi)$ can be represented as shown in figure 1.

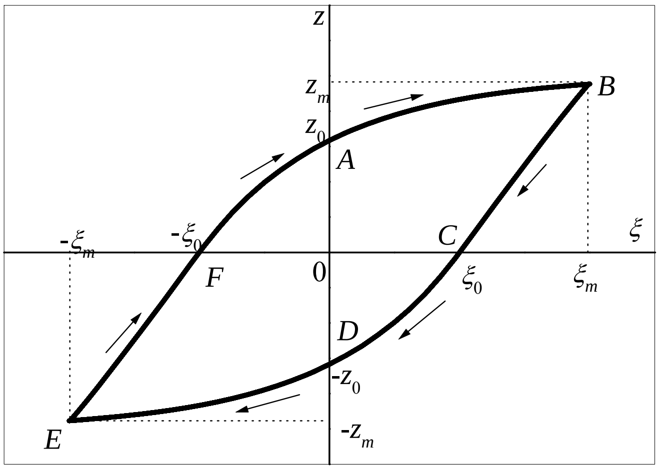

Fig.1. Generic experimental hysteretic loop

The Bouc –Wen model, chosen to fit the hysteresis loop shown in figure 1, is described by the following non-linear differential equation

$$\frac{\mathrm{d}z}{A - |z|^n\left[\beta + \gamma \mathrm{sgn}(\xi' z)\right]} = \mathrm{d}\xi \tag{2}$$

where $A,\ \beta,\ \gamma,\ n$ are loop parameters controlling the shape and magnitude of the hysteresis loop $z(\xi)$. Due to the symmetry of hysteresis curve, only the branches AB, BC and CD, corresponding to positive values of the imposed displacement $\xi(\tau)$, will be considered.

The model parameters are to be determined such as the steady-state solution of equation (2), under symmetric cyclic excitation, to satisfy the matching conditions considered in (3).

$$z(0) = z_0 \text{ at } A,\ z(\xi_m) = z_m,\ z(\xi_0) = 0,\ z(0) = -z_0 \text{ at } D \tag{3}$$

Equation (2) is solved analytically for $n = 1$ and 2. For arbitrary values of n, the equation can be solved numerically. In the present work, the proposed method for fitting the solution of equation (2) to the experimental hysteresis loop shown in figure 1, is illustrated for $n = 1$ and 2.

Fitting the Bouc-Wen model to symmetric experimental hysteresis loops for n=1

If are considered the notations

$$\alpha = \beta + \gamma,\ \delta = \beta - \gamma \tag{4}$$

the equation (2) takes on three different forms for each the three branches AB, BC and CD shown in figure 1:

$$AB: \frac{\mathrm{d}z}{A - \sigma z} = \mathrm{d}\xi,\ BC: \frac{\mathrm{d}z}{A - \delta z} = \mathrm{d}\xi,\ CD: \frac{\mathrm{d}z}{A + \sigma z} = \mathrm{d}\xi \tag{5}$$

From equations $(5)_1$ and $(5)_2$ one can calculate straightforward the slopes α_1 and α_2 of AB and BC branches in the point B:

$$\alpha_1 = \left.\frac{dz}{d\xi}\right|_{\substack{\xi\to\xi_m \\ \text{on } AB}} = A - z_m\sigma,\ \alpha_2 = \left.\frac{dz}{d\xi}\right|_{\substack{\xi\to\xi_m \\ \text{on BC}}} = A - z_m\delta \tag{6}$$

Since the condition $\alpha_1 < \alpha_2$ holds for any physical hysteresis loop, from equation (6) one obtains $\sigma > \delta$. Therefore, the Bouc-Wen model can portray a real hysteretic behavior only for positive values of parameter γ.

Integration of equations (5) on each branch yields three different relationships between the parameters ξ_m, ξ_0, z_m, z_0, measured on the experimental loop, and the Bouc-Wen model parameters A, σ, δ.

A. Integration of equation $(5)_1$ on the branch AB yields

$$\int_{z_0}^{z_m} \frac{dz}{A - \sigma z} = \int_0^{\xi_m} d\xi = \xi_m \tag{7}$$

A1. If $\sigma = 0$ then (7) becomes $\frac{z_m - z_0}{A} = \xi_m$ that implies

$$A = \frac{z_m - z_0}{\xi_m} \tag{8}$$

A2. If $\sigma > 0$, then $A - \sigma z \neq 0$ for $\forall\ z \in [z_0, z_m]$, if and only if the condition $z_m < A/\sigma$ holds.

A3. If $\sigma < 0$, then $A - \sigma z > 0\ \forall\ z \in [z_0, z_m]$.

In both **A2** and **A3** cases one can obtain (for $\sigma \neq 0$ and $z_m < A/|\sigma|$):

$$\frac{A - \sigma z_m}{A - \sigma z_0} = e^{-\sigma\xi_m} \tag{9}$$

B. If equation $(5)_2$ is integrated on BC branch one can find:

$$\int_{z_m}^{0} \frac{dz}{A - \delta z} = \int_{\xi_m}^{\xi_0} d\xi = \xi_0 - \xi_m \tag{10}$$

B1. If $\delta = 0$ then the relation (10) becomes $\frac{z_m}{A} = \xi_m - \xi_0$ that yields

$$A = \frac{z_m}{\xi_m - \xi_0} \tag{11}$$

B2. If $\delta > 0$, then $A - \delta z \neq 0$ for $\forall\, z \in [z_0,\, z_m]$ if the condition $z_m < A/\sigma$ holds ($A/\sigma < A/\delta$)

B3. If $\delta < 0$, then $A - \delta z > 0$ $\forall\, z \in [z_0,\, z_m]$.

In **B2** and **B3** cases, equation (10) implies the relation (for $\delta \neq 0$ and $z_m < A/|\sigma|$)

$$\frac{A - \delta z_m}{A} = e^{-\delta(\xi_m - \xi_0)} \tag{12}$$

It is easily seen that conditions (8) and (11) are not compatible. Therefore, $\sigma = 0$, $\delta = 0$, i.e. $\beta = \gamma = 0$ cannot be a solution such as Bouc-Wen model to portray hysteresis loops.

C. By integration of equation $(5)_3$ on the branch CD, the following relation is obtained:

$$\int_0^{-z_0} \frac{dz}{A + \sigma z} = \int_{\xi_0}^{0} d\xi = -\xi_0 \tag{13}$$

C1. If $\sigma = 0$ then (13) becomes $z_0 / A = \xi_0$ that implies

$$A = \frac{z_0}{\xi_0} \tag{14}$$

By combining the relations (8) and (14), one can derive the condition that must be satisfied by the measured parameters $\xi_m,\, \xi_0,\, z_m,\, z_0$ such as $\sigma = 0$ (i.e. $\beta = -\gamma$) to be an acceptable solution for fitting the Bouc-Wen model to experimental data:

$$\frac{z_m - z_0}{\xi_m} = \frac{z_0}{\xi_0} \tag{15}$$

C2. If $\sigma < 0$ then $A + \sigma z \neq 0$ for $\forall\, z \in [z_0,\, z_m]$ if the condition $z_m < A/|\sigma|$ holds.

C3. If $\sigma > 0$ then $A + \sigma z > 0$ for $\forall\, z \in [z_0,\, z_m]$.

Thus for $\sigma \neq 0$, $z_m < A/|\sigma|$, integrating the equation number (13) yields

$$\frac{A}{A - \sigma z_0} = e^{\sigma \xi_0} \tag{16}$$

By taking into account the relations (9) and (16), one can obtain

$$z_m - z_0 + e^{-\sigma\xi_0}\left[z_0 e^{-\sigma\xi_m} - z_m \right] = 0 \tag{17}$$

which is a transcendent algebraic equation for parameter σ. It can be proved that the equation (19) has always the solution $\sigma = 0$ and at most another one $\sigma \neq 0$. If condition (15) holds then $\sigma = 0$ is the only one solution and the value of parameter A is given by (14).

Otherwise, A is obtained from (16):

$$A = \frac{\sigma z_0}{1 - e^{-\sigma\xi_0}} \tag{18}$$

Next, the equation (12) can be rewritten as

$$A - \delta z_m - A e^{-\delta(\xi_m - \xi_0)} = 0 \tag{19}$$

As in the case of equation (17), one can prove that equation (19) has always the solution $\delta = 0$ and at most another one $\delta \neq 0$. If the value of A, given by (18) also satisfies (11), then $\delta = 0$ is the unique solution of equation (19). Otherwise, the value of δ is found by solving equation (19) in which A is given by (14) or by (18), as condition (15) is fulfilled or not.

Fitting the Bouc-Wen model to symmetric experimental hysteresis loops for n=2

If n =2 then from the relations (2), (3) and (4) one can get the equations on the branches AB, BC and CD:

$$AB: \frac{\mathrm{d}z}{A - \sigma z^2} = \mathrm{d}\xi,\; BC: \frac{\mathrm{d}z}{A - \delta z^2} = \mathrm{d}\xi,\; CD: \frac{\mathrm{d}z}{A - \sigma z^2} = \mathrm{d}\xi \tag{20}$$

Next is applied the same procedure in order to obtain the loop parameters A, β, γ in terms of the given ones, ξ_m, ξ_0, z_m, z_0.

A. Integration of equation $(20)_1$ on the branch AB determine the relation

$$\int_{z_0}^{z_m} \frac{dz}{A - \sigma z^2} = \int_0^{\xi_m} d\xi = \xi_m \tag{21}$$

A1. If $\sigma = 0$ then (21) becomes $\frac{z_m - z_0}{A} = \xi_m$ that implies relation (8).

A2. If $\sigma > 0$, then $A - \sigma z^2 \neq 0$ for $\forall\; z \in [z_0,\, z_m]$ is true if and only if the condition $z_m < \sqrt{\frac{A}{\sigma}}$ holds.

In this case, the equality (21) gives the relation:

$$\frac{\sqrt{\frac{\sigma}{A}}\left(z_m - z_0\right)}{1 - \frac{\sigma}{A} z_m z_0} = \tanh\left(\sqrt{A\sigma}\,\xi_m\right) \tag{22}$$

where is used the notation $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ for the hyperbolic tangent of x.

A3. If $\sigma < 0$, then $A - \sigma z^2 > 0\;\; \forall\, z \in [z_0,\, z_m]$.

By integrating (21) one can obtain:

$$\xi_m \sqrt{A|\sigma|} = \arctan \frac{\sqrt{\frac{|\sigma|}{A}}\left(z_m - z_0\right)}{1 + \frac{|\sigma|}{A} z_m z_0} \tag{23}$$

Hence, for $\sigma = 0$ is obtained the relation (8), while for $\sigma \neq 0$ and $z_m < \sqrt{\frac{A}{|\sigma|}}$ one can get the nonlinear equations (22) (for $\sigma > 0$) and (23) (if $\sigma < 0$).

B. If equation $(20)_2$ is integrated on BC branch then:

$$\int_0^{z_m} \frac{dz}{A - \delta z^2} = \int_{\xi_0}^{\xi_m} d\xi = \xi_m - \xi_0 \tag{24}$$

B1. If $\delta = 0$ then the relation (24), as in n =1 case, becomes the equation (11).

B2. If $\delta > 0$, then $A - \delta z^2 \neq 0$ for $\forall\, z \in [0,\ z_m]$ if the condition $z_m < \sqrt{\frac{A}{\sigma}}$ holds (because $\sqrt{\frac{A}{\sigma}} < \sqrt{\frac{A}{\delta}}$)
Next, by integrating (24) is obtained:

$$\sqrt{\frac{\delta}{A}} z_m = \tanh\left[\sqrt{A\delta}\left(\xi_m - \xi_0\right)\right] \tag{25}$$

B3. If $\delta < 0$, then $A - \delta z^2 > 0$ for $\forall\, z \in [0,\ z_m]$.
In this case the equation (24) yields:

$$\sqrt{A|\delta|}\left(\xi_m - \xi_0\right) = \arctan\left(\sqrt{\frac{|\delta|}{A}} z_m\right) \tag{26}$$

C. By integration of equation $(20)_3$ on the branch CD is obtained:

$$\int_{-z_0}^{0} \frac{dz}{A - \sigma z^2} = \int_{0}^{\xi_0} d\xi = \xi_0 \tag{27}$$

C1. If $\sigma = 0$ then (27) becomes the relation (14).

C2. If $\sigma > 0$ then the condition $A - \sigma z^2 \neq 0$ for $\forall\, z \in [-z_0,\ 0]$ is true if $z_0 < \sqrt{\frac{A}{\sigma}}$ (which is implied by $z_m < \sqrt{\frac{A}{\sigma}}$). Under this hypothesis one can obtain from (27) the equation:

$$\sqrt{\frac{\sigma}{A}} z_0 = \tanh\left[\sqrt{A\sigma}\,\xi_0\right] \tag{28}$$

C3. If $\sigma < 0$, then $A - \sigma z^2 > 0\ \ \forall\, z \in [-z_0,\ 0]$.
In this case, using (27) is derived the relation:

$$\sqrt{A|\sigma|}\,\xi_0 = \arctan\left(\sqrt{\frac{|\sigma|}{A}} z_0\right) \tag{29}$$

As one can see, the parameters A and σ are determined from the relations derived on the branches AB and CD. After that, with A and σ known is obtained δ from integration on BC branch. The equations with A and σ are written in a convenient way if the following notations are introduced ($x,\ y > 0$):

$$\sqrt{A|\sigma|} = x,\ \sqrt{\frac{|\sigma|}{A}} = y \tag{30}$$

Therefore, if x and y are known then A and σ can be computed as

$$|\sigma| = xy,\ A = \frac{x}{y} \tag{31}$$

The algorithm for obtaining A and σ:

1) If the relation (equivalent with (15)) $\frac{z_m - z_0}{\xi_m} - \frac{z_0}{\xi_0} = 0$ holds then $\sigma = 0$ and $A = \frac{z_m - z_0}{\xi_m} = \frac{z_0}{\xi_0}$.

2) If (15) is not true then is considered the system (corresponding to $\sigma > 0$) containing the equations (22) and (28):

$$\begin{cases} \dfrac{(z_m - z_0)y}{1 - y^2 z_m z_0} = \tanh(\xi_m x) \\ z_0 y = \tanh(\xi_0 x) \end{cases} \tag{32}$$

Hence, (32) can be rewritten

$$\begin{cases} y = \dfrac{1}{z_0}\tanh(\xi_0 x) \\ f_+(x) \overset{not}{=} (z_m - z_0)\tanh(\xi_0 x) + [z_m \tanh^2(\xi_0 x) - z_0]\tanh(\xi_m x) = 0 \end{cases} \tag{33}$$

If the equation $f_+(x) = 0$ has a solution $x_\sigma > 0$ then $y_\sigma = \dfrac{1}{z_0}\tanh(\xi_0 x_\sigma)$ and $\sigma = x_\sigma y_\sigma$, $A = \dfrac{x_\sigma}{y_\sigma}$.

3) If (15) is not true and $f_+(x) = 0$ has no solution $x > 0$ (corresponding to $\sigma < 0$) then is considered the system of equations

$$\begin{cases} \xi_m x = \arctan\dfrac{(z_m - z_0)y}{1 + y^2 z_m z_0} \\ \xi_0 x = \arctan(y z_0) \end{cases} \tag{34}$$

By combining the relations in (34) one can derive:

$$\begin{cases} x = \dfrac{1}{\xi_0}\arctan(y z_0) \\ f_-(y) = \xi_m \arctan(z_0 y) - \xi_0 \arctan\dfrac{(z_m - z_0)y}{1 + y^2 z_m z_0} = 0 \end{cases} \tag{35}$$

If the equation $f_-(y) = 0$ has a solution $y_\sigma > 0$ then $x_\sigma = \dfrac{1}{\xi_0}\arctan(z_0 y_\sigma)$ and $\sigma = -x_\sigma y_\sigma$, $A = \dfrac{x_\sigma}{y_\sigma}$.

The next step is to determine the value of δ under the hypothesis that A and σ are computed. The following notation is assumed

$$\sqrt{\frac{\delta}{A}} = y \tag{36}$$

so, if $y > 0$ is given then $|\delta| = Ay^2$. Using this notation the relations (25) and (26) become

$$g_+(y) \overset{not}{=} \tanh\left[Ay(\xi_m - \xi_0)\right] - y z_m = 0 \tag{37}$$

$$g_-(y) \overset{not}{=} \arctan(z_m y) - Ay(\xi_m - \xi_0) = 0 \tag{38}$$

Proposition 1

Consider the equation (37): $g_+(y) = 0$, $y \geq 0$ where A, z_m, ξ_m, $\xi_0 > 0$ and $\xi_0 < \xi_m$. Then

- if $A > \dfrac{z_m}{\xi_m - \xi_0}$ then (37) has an unique positive solution $y_\delta > 0$;

- if $A \le \dfrac{z_m}{\xi_m - \xi_0}$ then (37) has no positive solution.

Proof

It is easy to check that $g_+(0) = 0$ and $\lim\limits_{y\to\infty} g_+(y) = -\infty$.

By computing the derivative of g_+ one can obtain:

$g_+'(y) = A(\xi_m - \xi_0)\left\{1 - \tanh^2\left[A(\xi_m - \xi_0)y\right]\right\} - z_m$, $\forall\ y \ge 0$.

In order to study the monotony of g_+, the critical points are investigated (i. e. the solutions of equation $g_+'(y) = 0$, $y \ge 0$). Then one can write:

$$g_+'(y) = 0 \Leftrightarrow \tanh^2\left[A(\xi_m - \xi_0)y\right] = 1 - \frac{z_m}{A(\xi_m - \xi_0)}.$$

The previous equation has positive solutions if and only if $1 - \dfrac{z_m}{A(\xi_m - \xi_0)} \in (0,\ 1) \Leftrightarrow A > \dfrac{z_m}{\xi_m - \xi_0}$.

If $A \le \dfrac{z_m}{\xi_m - \xi_0}$ then $g_+'(y) < 0$, $\forall\ y > 0$ so $g_+(y) < 0$, $\forall\ y > 0$, therefore (37) has no positive solution.

If $A > \dfrac{z_m}{\xi_m - \xi_0}$ then $g_+'(y) = 0$ has only one positive solution $y_+ = \dfrac{1}{A(\xi_m - \xi_0)}\tanh^{-1}\sqrt{1 - \dfrac{z_m}{A(\xi_m - \xi_0)}}$, where $\tanh^{-1}$ is the inverse of function tanh.

On the other hand $g_+''(y) = A^2(\xi_m - \xi_0)^2\left\{-2\tanh\left[A(\xi_m - \xi_0)y\right]\right\}\left\{1 - \tanh^2\left[A(\xi_m - \xi_0)y\right]\right\}$, $\forall\ y > 0$, then $g_+''(y_+) < 0$, so $y = y_+$ is a maximum point to g_+ on $(0, \infty)$.

As $g_+(0) = 0$, $\lim\limits_{y\to\infty} g_+(y) = -\infty$ and $g_+(y_+) > 0$ one can derive: there is a unique positive value $y_\delta > 0$ such that $g_+(y_\delta) = 0$.

Proposition 2

Consider the equation (38): $g_-(y) = 0$, $y \ge 0$ where A, z_m, ξ_m, $\xi_0 > 0$ and $\xi_0 < \xi_m$. Then

- if $A < \dfrac{z_m}{\xi_m - \xi_0}$ then (38) has an unique positive solution $y_\delta > 0$;
- if $A \ge \dfrac{z_m}{\xi_m - \xi_0}$ then (38) has no positive solution.

Proof

It is easy to prove that $g_-(0) = 0$ and $\lim\limits_{y\to\infty} g_-(y) = -\infty$.

Next is computed the first derivative of g_-: $g_-'(y) = \dfrac{z_m - A(\xi_m - \xi_0)(1 + z_m^2 y^2)}{1 + z_m^2 y^2}$, $\forall\ y \ge 0$.

The equation $g_-'(y) = 0$ is equivalent with: $y^2 = \dfrac{z_m - A(\xi_m - \xi_0)}{A z_m^2(\xi_m - \xi_0)}$, $y \ge 0$. So $g_-'(y) = 0$ has positive solutions if and only if $A < \dfrac{z_m}{\xi_m - \xi_0}$.

If $A \ge \dfrac{z_m}{\xi_m - \xi_0}$ then $g_-'(y) < 0$, $\forall\ y > 0$ so $g_-(y) < 0$, $\forall\ y > 0$, therefore (38) has no positive solution.

If $A<\frac{z_m}{\xi_m-\xi_0}$ then $g_-'(y)=0$ has only one positive solution $y_+=\sqrt{\frac{z_m-A(\xi_m-\xi_0)}{Az_m{}^2(\xi_m-\xi_0)}}$.

On the other hand, $g_-''(y)=-\frac{2z_m{}^3y^2}{(1+z_m{}^2y^2)^2}$<0, $\forall$ y>0, so $g_-''(y_+)<0\Rightarrow y=y_+$ is a maximum point to g_- on (0, ∞). As $g_-(0)=0$, $\lim_{y\to\infty}g_-(y)=-\infty$ and $g_-(y_+)>0$ one can derive: there is a unique positive value $y_\delta>0$ such that $g_-(y_\delta)=0$.

The algorithm for obtaining δ *(* A *and* σ *are known)*:
1) If $\sigma\le 0$ then $\delta<0$ (as $\sigma>\delta$) so is considered only the equation (38) which has a unique solution $y_\delta>0$.Therefore $\delta=-Ay_\delta^2$.
2) If $\sigma>0$ then three cases are possible for δ:
2.1) If the parameter A satisfies the relation (11) ($A=\frac{z_m}{\xi_m-\xi_0}$) then $\delta=0$.
2.2) If $A>\frac{z_m}{\xi_m-\xi_0}$ then only the equation (37) has a positive solution $y_\delta>0$ (which is unique), so $\delta=Ay_\delta^2$.
2.3) If $A<\frac{z_m}{\xi_m-\xi_0}$ then only the equation (38) has an unique positive solution $y_\delta>0$, so $\delta=-Ay_\delta^2$.

Fitting the Bouc-Wen model to asymmetric experimental hysteresis loops

Suppose the experimental hysteretic characteristic is a asymmetric loop $-F_{m2}\le F(x)\le F_{m1}$, obtained for a periodic motion $-x_{m2}\le x(t)\le x_{m1}$, imposed between the mounting ends of the tested element. As before, the loop-axes crossing points are: $A(0,\ f_0)$, $C(x_0,\ 0)$, $D(0,\ -f_0)$ and $E(-x_0,\ 0)$. In this case, the fitting method is developed as a combination of two symmetric cases:

$$-F_{m1}\le F_1(x)\le F_{m1}\quad\text{for}\quad -x_{m1}\le x(t)\le x_{m1},\ \text{and}-F_{m2}\le F_2(x)\le F_{m2}\quad\text{for}\quad -x_{m2}\le x(t)\le x_{m2},\tag{39}$$

With notations similar to (1), the asymmetric hysteresis loop $z(\xi)$ is modeled by

$$\begin{aligned}z(\xi)&=\frac{1}{2}z_1(\xi_1)[1+\mathrm{sign}\xi_1]+\frac{1}{2}z_2(\xi_2)[1-\mathrm{sign}\xi_2],\\ \xi(\tau)&=\frac{1}{2}\{\xi_1(\tau)[1+\mathrm{sign}\xi_1]+\xi_2(\tau)[1-\mathrm{sign}\xi_2]\}\end{aligned}\tag{40}$$

where $z_1(\xi)$ and $z_2(\xi)$, are the solutions of the symmetric Bouc-Wen equations

$$\frac{dz_1}{A_1-|z_1|[\beta_1+\gamma_1\mathrm{sgn}(\xi_1'z_1)]}=d\xi\,,\quad\frac{dz_2}{A_2-|z_2|[\beta_2+\gamma_2\mathrm{sgn}(\xi_2'z_2)]}=d\xi\tag{41}$$

For each of these equations, the loop parameters are determined according to the fitting algorithm presented in the previous section. As the loop-force axis crossing points of both branches have same coordinates, $z(\xi)$ is continuous in these points. By using equations (41), the continuity conditions of its derivative in these points lead to

$$A_1-z_0\sigma_1=A_2-z_0\sigma_2\,,\ \text{where}\ \sigma_1=\beta_1+\gamma_1,\ \sigma_2=\beta_2+\gamma_2\tag{42}$$

As the parameters $A_1,\ \sigma_1,\ A_2,\ \sigma_2$ are uniquely determined such as $z(\xi)$ to have imposed extreme values and axes crossing points, one must take into consideration a trade-off between these requirements and curve smoothness condition (42), such as to minimize a given accuracy cost function. This optimization of Bouc-Wen model fitting to asymmetric experimental hysteresis loops can be approached by iterative or Genetic Algorithms methods.

3. APPLICATION TO EXPERIMENTAL ASYMMETRIC HISTEREZIS LOOPS

The fitting method was applied to identify the differential Bouc-Wen models for $n = 1$ and 2, which the hysteretic behavior of two vibration control devices: SERB-B-194 for earthquake protection of buildings by bracing installation [11], and SERB-B 300C for base isolation of forging hammers [12]. The results presented in figures 2 and 3 prove the efficiency of the proposed method for fitting the considered experimental hysteresis loops by Bouc-Wen model with $n = 1$. As one can see from figures 4 and 5, for $n = 2$, the approximation of experimental loops is not so good as the one obtained for $n = 1$, particularly for SERB-B-194 device. Even though the predicted loop for $n = 2$ and the experimental loop, have same extreme values and axes crossing points, their shape is quite different. This result shows the importance of choosing the right value of the exponent n when fitting a given experimental hysteresis loop by a Bouc-Wen model.

Is worth mentioning these results were obtained by the straight application of the fitting analytical method, without any further accuracy improvement. The model accuracy can be easily improved by conventionally iterative methods or evolutionary (Genetic Algorithms) starting from these parameters values.

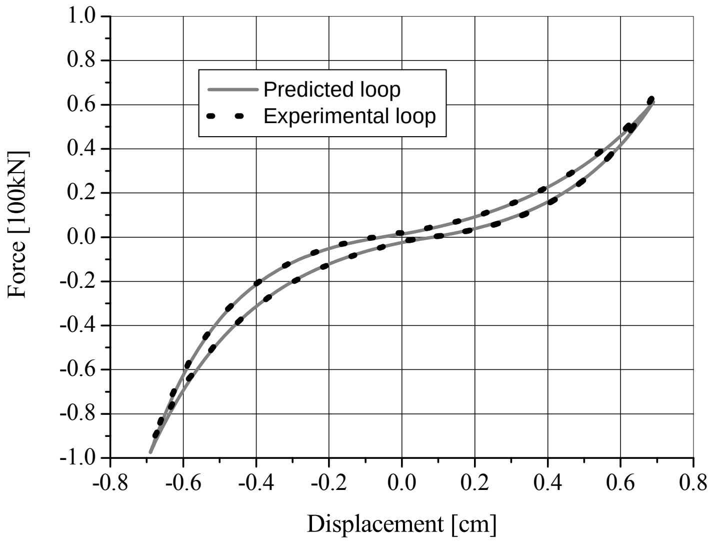

Fig.2. Fitting the hysteretic characteristic of vibration control device SERB-B-194
analytical model (n=1): $A_1 = 0.26$, $\beta_1 = -3.245$, $\gamma_1 = 0.55$; $A_2 = 0.253$, $\beta_2 = -4.035$, $\gamma_2 = 0.62$
experimental data: $\xi_{m1} = \xi_{m2} = 0.69$, $z_{m1} = 0.65$, $z_{m2} = 0.92$, $\xi_0 = 0.07$, $z_0 = 0.02$

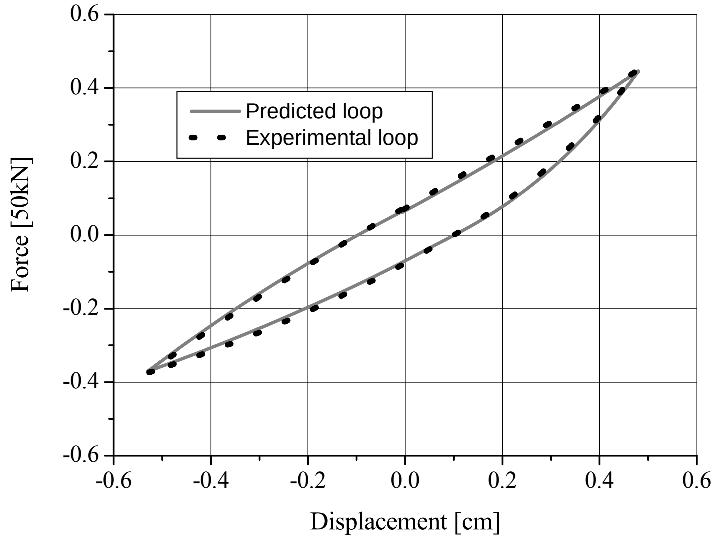

Fig.3. Fitting the hysteretic characteristic of vibration control device SERB-B 300C
analytical model (n=1): $A_1 = 0.686$, $\beta_1 = -1.54$, $\gamma_1 = 1.125$; A_2=0.0.725, $\beta_2 = -0.04$, $\gamma_2 = 0.735$
experimental data: ξ_{m1}=0.48, z_{m1} =0.45, ξ_{m2}=0.53, z_{m2}=0.37, ξ_0 =0.1, $z_0 = 0.07$

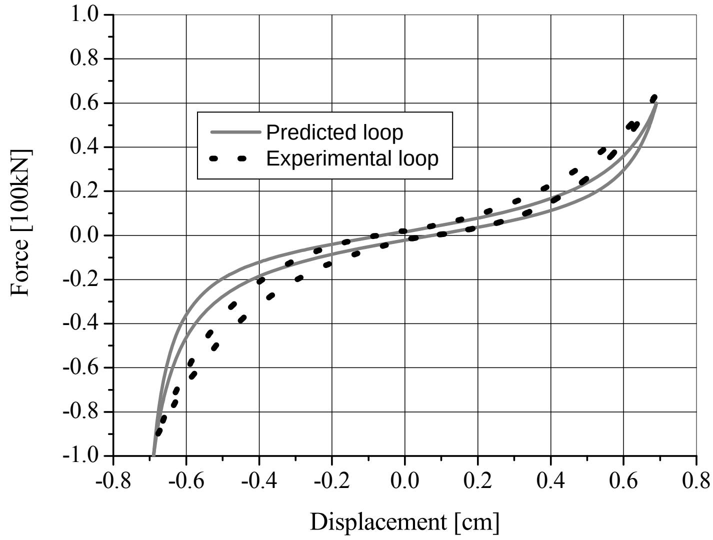

Fig.4. Fitting the hysteretic characteristic of vibration control device SERB-B-194
analytical model (n=2): $A_1 = 0.284$, $\beta_1 = -14$, $\gamma_1 = 3.32$; $A_2 = 0.284$, $\beta_2 = -15.5$, $\gamma_2 = 3.45$
experimental data: $\xi_{m1} = \xi_{m2} = 0.69$, z_{m1} =0. 65, z_{m2}=0.92, ξ_0 =0.07, $z_0 = 0.02$

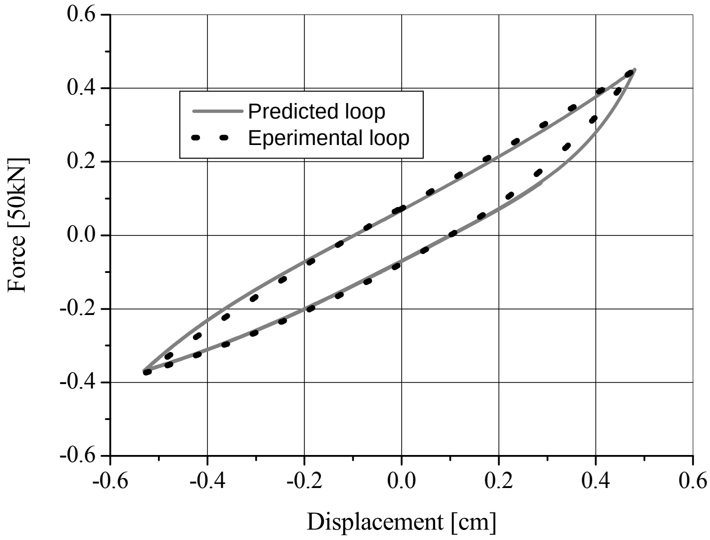

Fig.5. Fitting the hysteretic characteristic of vibration control device SERB-B 300C
analytical model (n=2): $A_1 = 0.7$, $\beta_1 = -6.26$, $\gamma_1 = 5$; $A_2 = 0.7$, $\beta_2 = -0.916$, $\gamma_2 = 3.14$
experimental data: $\xi_{m1}=0.48$, $z_{m1}=0.45$, $\xi_{m2}=0.53$, $z_{m2}=0.37$, $\xi_0=0.1$, $z_0 = 0.07$

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